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ON THE IDEAL STRUCTURE OF POSITIVE, EVENTUALLY COMPACT LINEAR OPERATORS ON BANACH LATTICES

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We study the structure of the algebraic eigenspace corresponding to the spectral radius of a nonnegative reducible linear operator T, having a compact iterate and defined on a Banach lattice E with order continuous norm. The Perron-Frobenius theory is generalized by showing that this algebraic eigenspace is spanned by a basis of eigenelements and generalized eigenelements possessing certain positivity features. A combinatorial characterization of both the *Riesz index* of the spectral radius and the dimension of the algebraic eigenspace is given. These results are made possible by a decomposition of T, in terms of certain closed ideals of E, in a form which directly generalizes the Frobenius normal form of a nonnegative reducible matrix.

I. Introduction. Let E be a Banach lattice, and $\mathbf{T}$, a positive reducible linear operator mapping E into itself and having a compact iterate. Suppose, in addition, that $r(\mathbf{T})$, the spectral radius of $\mathbf{T}$, is positive. The primary purpose of this research is to ascertain, under what conditions on the Banach lattice E, a decomposition of $\mathbf{T}$ is possible, which turns out to be a natural generalization of the Frobenius normal form for reducible matrices. These properties will be seen to generalize those deduced by U. Rothblum [**13**] for the matrix setting, and those by H. D. Victory, Jr. for integral operators on L^p-spaces, $1 \leq p < \infty$, with the underlying measure being σ-finite [**16**, **17**] on a domain set Ω.

We refer the reader to the treatise by H. H. Schaefer [**15**] for an explanation of the notation used in this work and of the lattice concepts of ideals, order convergence and completeness, bands, projection bands, operator-invariant ideals, and (uniform) mean ergodicity of an operator $\mathbf{T}$. By $\mathfrak{L}(E)$, we mean the Banach space of bounded linear endomorphisms of E.

The underlying Banach lattice E will be assumed equipped with an *order continuous norm*. In such Banach lattices, every filter that order converges norm converges. Such lattices are characterized by the fact that every closed ideal is a band [**15**, Theorem 5.14 (Chapter II)]. Since

E is a fortiori order complete, this means that every closed ideal is a projection band by the *Riesz Decomposition Theorem*. The work by T. Ando, W. A. J. Luxemburg, and A. C. Zaanem [**9**, **10**] indicates that any norm closed (and complemented) ideal $\mathfrak{J}$ is a Banach sublattice of E which is also order closed, and hence can be regarded in its own right as a Banach lattice with order continuous norm.

For $\mathbf{T}$ irreducible with a compact iterate, a cyclic-type Frobenius decomposition was carried out by D. Axmann [**2**]. He used very crucially results by H. P. Lotz [7, **8**] refining the assertions of the Niiro-Sawashima Theorem, whereby the peripheral spectrum of any positive operator consists of poles of its resolvent, once its spectral radius is assumed a pole with finite-dimensional residuum. With E having an order continuous norm, Axmann's results can be summarized as:

(A) The Banach lattice E can be decomposed into an order-direct sum of p closed ideals $\{\mathfrak{J}_q : 1 \leq q \leq p\}$ which, for each $n \in \mathbb{N}$, are minimal $\mathbf{T}^{pn}$ ideals and moreover $\mathbf{T}\mathfrak{J}_q \subset \mathfrak{J}_{q+1}$, q mod p;

(B) $\mathbf{P}_{\mathfrak{J}_q}\mathbf{T}^p\mathbf{P}_{\mathfrak{J}_q}$ is irreducible with spectral radius unity, $1 \leq q \leq p$, and there are no other eigenvalues of modulus one;

(C) $\mathbf{T}^r$ is irreducible whenever r is not an integer multiple of p;

(D) Let $x \in E$ and denote $x_q := \mathbf{P}_{\mathfrak{J}_q} x$ as the band component of x in $\mathfrak{J}_q$. If we let f be a normalized eigenelement of $\mathbf{T}$ corresponding to unity, we have that

$$\mathbf{T}f_q = f_{q+1}, \quad q \bmod p. \tag{1.1}$$

We are able to deduce that

$$y := \sum_{q=1}^{p} \exp(-2\pi i(q-1)/p) f_q \tag{1.2}$$

generates the totality of eigenelements associated with the peripheral spectrum of $\mathbf{T}$. It is easy to verify that, in particular, $\mathbf{T}^j y$, $0 \leq j \leq p-2$, is an eigenelement associated with $\exp(\frac{2\pi i}{p}(j+1))$.

In 1985, B. de Pagter [**12**] settled a long-standing conjecture by showing that every positive compact irreducible operator is guaranteed to have a positive spectral radius whenever the underlying Banach lattice has dimension greater than one.

The decomposition of a reducible operator $\mathbf{T}$ in §II will be accomplished by using and refining a bit the techniques, employed by H. P. Lotz [7], to show that an operator, with its spectral radius a pole of the resolvent, is (G)-solvable [**15**, p. 326]. Moreover, some of the analysis will be reminiscent of that in [8] concerning the proof of the fundamental Niiro-Sawashima Theorem. We define in §III *accessibility relations* between the closed ideals of E, constructed in §II, which generalize the graph-theoretic concepts expounded by S. Friedland and H. Schneider [4] and by U. Rothblum [13]. In §IV, we characterize the distinguished eigenvalues of $\mathbf{T}$ (i.e., those positive eigenvalues with an eigenelement in the positive cone E_+). The concluding §V investigates the algebraic eigenspace of $\mathbf{T}$ belonging to its spectral radius.

II. The Frobenius decomposition of T. We now turn to general reducible $\mathbf{T}$ having a compact iterate on a Banach lattice with an order continuous norm. We shall effect a decomposition of E into bands such that $\mathbf{T}$ restricted to each is either irreducible with a positive spectral radius or is quasi-nilpotent. Subsequently, we examine the role our decomposition plays in characterizing the *distinguished eigenvalues* of $\mathbf{T}$ and in describing the positivity structure of the algebraic eigenspace associated with $r(\mathbf{T}) = 1$. An operator in $\mathfrak{L}(E)$ is termed *uniformly mean ergodic* (respectively, *mean ergodic*) if the *Cesàro means* converge in the uniform (respectively, strong) operator topology.

We first consider the case when the spectral radius is a first order pole of the resolvent $\mathbf{R}(\lambda, \mathbf{T}) = (\lambda\mathbf{I} - \mathbf{T})^{-1}$ and label the residuum at $\lambda = 1$ as $\mathbf{Q}$, i.e., in the uniform operator topology,

$$\mathbf{Q} := \lim_{\lambda \downarrow 1} (\lambda - 1)\mathbf{R}(\lambda, \mathbf{T}). \tag{2.1}$$

Let n denote the dimension of the fixed space of $\mathbf{T}$, and observe that $\mathbf{Q}$ is itself a positive linear operator of finite rank satisfying $\mathbf{QT} = \mathbf{Q}$. Define now

$$\mathfrak{J} := \{x \in E : \mathbf{Q}|x| = 0\}, \tag{2.2}$$

and observe that $\mathfrak{J}$ is a closed, operator invariant ideal of E and, a fortiori, a projection band. Accordingly, the Banach lattice E can be expressed as $E = \mathfrak{J} \oplus \mathfrak{J}^\perp$, where $\mathfrak{J}^\perp$ is the kernel of the projection of E onto $\mathfrak{J}$. We let $\mathbf{T}_1 := \mathbf{T}_{\mathfrak{J}^\perp}$ and $\mathbf{T}_2 := \mathbf{T}_{\mathfrak{J}}$ be the induced operators on $\mathfrak{J}^\perp$ and $\mathfrak{J}$ respectively, i.e., $\mathbf{T}_1 := \mathbf{P}_{\mathfrak{J}^\perp}\mathbf{T}\mathbf{P}_{\mathfrak{J}^\perp}$ and $\mathbf{T}_2 := \mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}$ where $\mathbf{P}_{\mathfrak{J}^\perp}$ and $\mathbf{P}_{\mathfrak{J}}$ are respectively the band projections of E onto $\mathfrak{J}^\perp$ (along $\mathfrak{J}$), and onto $\mathfrak{J}$ (along $\mathfrak{J}^\perp$). Relative to this decomposition, we can schematically represent $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & 0 \\ \mathbf{T}_{21} & \mathbf{T}_2 \end{pmatrix}, \tag{2.3}$$

where $\mathbf{T}_1 \in \mathfrak{L}(\mathfrak{J}^\perp)$, $\mathbf{T}_2 \in \mathfrak{L}(\mathfrak{J})$, $\mathbf{T}_{21} := \mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}^\perp} \in \mathfrak{L}(\mathfrak{J}^\perp, \mathfrak{J})$.

We can deduce by positivity of $\mathbf{T}_1$ and $\mathbf{T}_2$ that $r(\mathbf{T}_1) = 1$, with unity a first order pole k with finite-rank residuum of $\mathbf{R}(\lambda, \mathbf{T}_1)$, whereas $r(\mathbf{T}_2) < 1$. This is due to the fact that, for an operator $\mathbf{T}$ so decomposed as in (2.3), the order of $\lambda_0 \in \mathbb{C}$ as a pole of $\mathbf{R}(\lambda, \mathbf{T})$ can be bounded by the orders k_1 and k_2 of λ_0 as poles of $\mathbf{R}(\lambda, \mathbf{T}_1)$ and $\mathbf{R}(\lambda, \mathbf{T}_2)$ respectively, by the inequality

$$\sup(k_1, k_2) \leq k \leq k_1 + k_2 \tag{2.4}$$

[**15**, p. 330].

The (finite-rank) residuum $\mathbf{Q}_1$ of $\mathbf{R}(\lambda, \mathbf{T}_1)$ at $\lambda = 1$ can thus be expressed as

$$\mathbf{Q}_1 := \mathbf{P}_{\mathfrak{J}^\perp}\mathbf{Q}\mathbf{P}_{\mathfrak{J}^\perp} \tag{2.5}$$

and is easily seen to be strictly positive on $\mathfrak{J}^\perp$. Indeed, let $x \in \mathfrak{J}^\perp$, $x > 0$, and consider $\mathbf{Q}_1 x = \mathbf{P}_{\mathfrak{J}^\perp}\mathbf{Q}\mathbf{P}_{\mathfrak{J}^\perp}x = \mathbf{P}_{\mathfrak{J}^\perp}\mathbf{Q}x$. If $\mathbf{Q}_1 x = 0$, then $\mathbf{Q}x \in \mathfrak{J}$ and $\mathbf{Q}^2 x = \mathbf{Q}x = 0$, whence $x \in \mathfrak{J}$, a contradiction. It is well known that the fixed space, $\ker(\mathbf{I} - \mathbf{T}_1)$, in $\mathfrak{J}^\perp$ is a sublattice of $\mathfrak{J}^\perp$, since $\mathbf{T}_1$ as defined is *uniformly mean ergodic* (cf., e.g., Proposition 8.4, p. 188, of [**15**]).

There exists, then, a lattice isomorphism of $\ker(\mathbf{I} - \mathbf{T}_1)$ onto $\mathbb{R}^m$, with $m = \dim\ker(\mathbf{I} - \mathbf{T}_1)$ (cf., e.g., Corollary 1, pp. 69–70 of [**15**]). From such an isomorphism, we can find a basis of $\ker(\mathbf{I} - \mathbf{T}_1)$ residing in $\mathfrak{J}^\perp$ denoted as $\{\mathfrak{b}_1, \dots, \mathfrak{b}_m\}$. Each $\mathfrak{b}_i$, $i = 1, \dots, m$, is positive with $\mathfrak{b}_i \wedge \mathfrak{b}_j = 0$, $i \neq j$. Let $\mathfrak{J}_i$ be the closure of the principal ideal $E_{\mathfrak{b}_i}$ generated by $\mathfrak{b}_i$, that is, $\mathfrak{b}_i$ is a quasi-interior element of $\mathfrak{J}_i$. Since $E_{\mathfrak{b}_i} = \bigcup_{n=1}^\infty n[-\mathfrak{b}_i, \mathfrak{b}_i]$ and $\mathbf{T}_1[-\mathfrak{b}_i, \mathfrak{b}_i] \subset [-\mathfrak{b}_i, \mathfrak{b}_i]$, $\mathbf{T}_1\mathfrak{J}_i \subset \mathfrak{J}_i$. Because $\mathfrak{b}_i \wedge \mathfrak{b}_j = 0$, $1 \leq i \neq j \leq m$, $\mathfrak{J}_i \perp \mathfrak{J}_j$, $1 \leq i \neq j \leq m$. We write

$$\mathfrak{J}^\perp := \ker(\mathbf{I} - \mathbf{T}_1) \oplus \mathfrak{J}_0^\perp = \mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_m \oplus \mathfrak{J}_0^\perp, \tag{2.6}$$

and $\mathbf{T}_{1,i} = \mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i}$, where $\mathbf{P}_{\mathfrak{J}_i}$ is the lattice projection of E onto $\mathfrak{J}_i$. If we let $\mathbf{Q}_{1,i}$ be the corresponding finite-rank residuum of $\mathbf{R}(\lambda, \mathbf{T}_{1,i})$, we observe that $\mathbf{S}_{1,i} := \{\mathbf{T}_{1,i}^n, n \in \mathbb{N}\}$ has nonzero fixed vectors and that $\mathbf{T}_{1,i}$ is uniformly mean ergodic with associated projection $\mathbf{Q}_{1,i}$. Since $\mathbf{Q}_{1,i}$ is strictly positive on $\mathfrak{J}_i$, with range spanned by a quasi-interior element of $\mathfrak{J}_i$, we conclude from [**15**, Proposition 8.5 (Chapter III)] that $\mathbf{T}_{1,i}$ is irreducible on $\mathfrak{J}_i$, $1 \leq i \leq m$.

Suppose now there are two lattice isomorphisms $\mathbf{A}$ and $\mathbf{B}$ from $\ker(\mathbf{I} - \mathbf{T}_1)$ onto $\mathbb{R}^m$. Then $\mathbf{A}\mathbf{B}^{-1}$ is a lattice isomorphism from $\mathbb{R}^m$ into itself, and is representable as a product of a diagonal matrix, with positive diagonal elements, and a permutation matrix [**15**, p. 44]. This implies that the decomposition of $\mathfrak{J}^{\perp}$, and hence the representation of $\mathbf{T}_1$, is unique up to permutations.

We summarize the above discussion as the following proposition.

Proposition II.1. *Let E be a Banach lattice with order continuous norm and $\mathbf{T} \in \mathfrak{L}(E)$ be positive, reducible, eventually compact with spectral radius equal to unity. If $\lambda = 1$ is a simple pole of the resolvent $\mathbf{R}(\lambda, \mathbf{T})$, then we can decompose E into a direct sum of bands of E,*

$$E = \mathfrak{J}^{\perp} \oplus \mathfrak{J} = \mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_m \oplus \mathfrak{J}_0^{\perp} \oplus \mathfrak{J}, \tag{2.7}$$

such that $\mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i}$ is irreducible with spectral radius unity, $1 \le i \le m$, with $\mathbf{P}_{\mathfrak{J}_i}$ denoting the band projection onto $\mathfrak{J}_i$. With respect to this decomposition of E, $\mathbf{T}$ can be represented as

$$\mathbf{T} = \left[\begin{array}{c|ccc|c} \mathbf{T}_{1,0} & 0\ \ 0\ \cdots & \cdots & \cdots\ 0\ \ 0 & 0 \\ \hline \vdots & \mathbf{T}_{1,1} & & \bigcirc & 0 \\ \vdots & & \ddots & & 0 \\ \vdots & & & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & \bigcirc & & \mathbf{T}_{1,m} & 0 \\ \hline & & \mathbf{T}_{2,1} & & \mathbf{T}_2 \end{array}\right]. \tag{2.8}$$

The decomposition of $\mathbf{T}$ is unique up to permutations of $\mathbf{T}_{1,i}$'s, $1 \le i \le m$, where $\mathbf{T}_{1,0} = \mathbf{P}_{\mathfrak{J}_0^{\perp}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_0^{\perp}}$, $\mathbf{T}_{1,i} = \mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i}$, $\mathbf{T}_2 = \mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}$, $\mathbf{T}_{2,1} = \mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}^{\perp}}$ and $r(\mathbf{T}_2) < 1$, $r(\mathbf{T}_{1,0}) < 1$.

We now proceed to treat the general case when $\mathbf{T} \in \mathfrak{L}(E)$, $r(\mathbf{T}) = 1$, with unity a pole of order k possessing a residuum of finite rank. In the following discussion, we shall produce a sequence of ideals $E \supset \mathfrak{J}_n \supset \mathfrak{J}_{n-1} \supset \cdots \supset \mathfrak{J}_1 \supset 0$ which are $\mathbf{T}$-invariant closed ideals such that the operator $\mathbf{T}_k$, induced on $\mathfrak{J}_{k-1}^{\perp} \cap \mathfrak{J}_k$, has unity as a first order pole of its resolvent. The preceding analysis then allows us to effect a further decomposition of $\mathfrak{J}_{k-1}^{\perp} \cap \mathfrak{J}_k$. A decomposition of $\mathbf{T}$ finally results which generalizes the Frobenius normal form of a reducible matrix. At this juncture, we should point out that our arguments are somewhat similar to those used by H. Lotz [7] in showing that a nonnegative operator whose resolvent has unity as a pole is (G)-solvable.

We express $\mathbf{R}(\lambda, \mathbf{T})$ as a Laurent series

$$\mathbf{R}(\lambda, \mathbf{T}) = \sum_{n=-k}^{\infty} (\lambda - 1)^n \mathbf{Q}_n, \tag{2.9}$$

and we observe that

$$\mathbf{Q}_{-k} = (\mathbf{T} - \mathbf{I})^{k-1}\mathbf{P}_k, \qquad \mathbf{Q}_{-k}\mathbf{T} = \mathbf{Q}_{-k}, \tag{2.10}$$

where $\mathbf{P}_k$ is the projection onto $\mathscr{N}(\mathbf{I} - \mathbf{T})^k$ along $\mathscr{R}(\mathbf{I} - \mathbf{T})^k$, with $\mathscr{N}(\cdot)$ and $\mathscr{R}(\cdot)$ respectively denoting the nullity and range of an operator. Define now

$$\mathfrak{S}_k := \{x \in E : Q_{-k}|x| = 0\}, \tag{2.11}$$

and we immediately see that $\mathfrak{S}_k$ is a $\mathbf{T}$-invariant closed ideal, and a fortiori a projection band. Therefore E can be decomposed as

$$E = \mathfrak{J}_k \oplus \mathfrak{S}_k \tag{2.12}$$

where $\mathfrak{J}_k = \mathfrak{S}_k^{\perp}$. Next, we let $\mathbf{T}_1^{(k)}$ be the operator $\mathbf{T}$ induces on $\mathfrak{J}_k$ and denote $\{\mathbf{Q}_{k,n}^{(1)}\}$ as the corresponding coefficients in the Laurent representation of $\mathbf{R}(\lambda, \mathbf{T}_1^{(k)})$.

LEMMA II.1. *The spectral radius is a simple pole of* $\mathbf{R}(\lambda, \mathbf{T}_1^{(k)})$.

Proof. Let $\mathbf{P}_{\mathfrak{J}_k}$ be the canonical band projection of E onto $\mathfrak{J}_k$. We set $\mathbf{Q}_{-r} := (\mathbf{T} - \mathbf{I})^{r-1}\mathbf{Q}$, $1 \le r \le k$, with $\mathbf{Q}$ the residuum of $\mathbf{R}(\lambda, \mathbf{T})$ at $\lambda = 1$. The corresponding coefficient $\mathbf{Q}_{-r}^{(1)}$ in $\mathbf{R}(\lambda, \mathbf{T}_1^{(k)})$ is determined by

$$\mathbf{Q}_{-r}^{(1)} \circ \mathbf{P}_{\mathfrak{J}_k} = \mathbf{P}_{\mathfrak{J}_k} \circ \mathbf{Q}_{-r}. \tag{2.13}$$

Suppose that $\mathbf{R}(\lambda, \mathbf{T}_1^{(k)})$ possesses unity as a pole of order $m_0 > 1$, and thus

$$\mathbf{Q}_{-m_0}^{(1)} \circ \mathbf{P}_{\mathfrak{J}_k} = \mathbf{P}_{\mathfrak{J}_k} \circ \mathbf{Q}_{-m_0} \tag{2.14}$$

is positive. For $x \in E_+$, we see that

$$\mathbf{Q}_{-m_0}x = \mathbf{P}_{\mathfrak{J}_k}(\mathbf{Q}_{-m_0}x) + (\mathbf{I} - \mathbf{P}_{\mathfrak{J}_k})(\mathbf{Q}_{-m_0}x); \tag{2.15}$$

and for some $y \in \mathfrak{S}_k$, $y = (\mathbf{I} - \mathbf{P}_{\mathfrak{J}_k})(\mathbf{Q}_{-m_0}x)$, we have

$$\mathbf{Q}_{-m_0}x - y = \mathbf{P}_{\mathfrak{J}_k}(\mathbf{Q}_{-m_0}x) \ge 0. \tag{2.16}$$

So

$$|\mathbf{Q}_{-m_0}x| \leq \mathbf{Q}_{-m_0}x - y + |y| \tag{2.17}$$

and hence

$$\mathbf{Q}_{-k}|\mathbf{Q}_{-m_0}x| \leq \mathbf{Q}_{-k}\mathbf{Q}_{-m_0}x + \mathbf{Q}_{-k}(|y| - y) = 0 \tag{2.18}$$

for $m_0 > 1$. We conclude $\mathbf{Q}_{-m_0}x \in \mathfrak{S}_k$ for every $x \in E_+$. So $\mathbf{P}_{\mathfrak{J}_k}\mathbf{Q}_{-m_0} = 0$ if $m_0 > 1$ and $\mathbf{Q}^{(1)}_{-m_0} = 0$, $m_0 > 1$. This completes the proof.

Corresponding to the representation of E as

$$E = \mathfrak{J}_k \oplus \mathfrak{S}_k, \tag{2.19}$$

$\mathbf{T}$ can be decomposed as

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}^{(k)}_1 & 0 \\ \mathbf{T}^{(k)}_{2,1} & \mathbf{T}^{(k)}_2 \end{pmatrix} \tag{2.20}$$

where $\mathbf{T}^{(k)}_1 = \mathbf{P}_{\mathfrak{J}_k}\mathbf{T}\mathbf{P}_{\mathfrak{J}_k}$, $\mathbf{T}^{(k)}_2 = \mathbf{P}_{\mathfrak{S}_k}\mathbf{T}\mathbf{P}_{\mathfrak{S}_k}$, $\mathbf{T}^{(k)}_{2,1} = \mathbf{P}_{\mathfrak{S}_k}\mathbf{T}\mathbf{P}_{\mathfrak{J}_k}$ and $r(\mathbf{T}^{(k)}_1) = 1$ is a simple pole of $\mathbf{R}(\lambda, \mathbf{T}^{(k)}_1)$.

We apply the preceding discussion to $\mathfrak{S}_k$. Because $\mathfrak{S}_k$ is a closed ideal of E, then every closed ideal of $\mathfrak{S}_k$ is a projection band of $\mathfrak{S}_k$. Let

$$\mathbf{Q}^{(2)}_{-(k-1)} := \lim_{\lambda \downarrow 1}(\lambda - 1)^{k-1}\mathbf{R}(\lambda, \mathbf{T}^{(k)}_2) \tag{2.21}$$

and define

$$\mathfrak{S}_{k-1} := \{x \in \mathfrak{S}_k : \mathbf{Q}^{(2)}_{-(k-1)}|x| = 0\}. \tag{2.22}$$

Note that $\mathfrak{S}_{k-1}$ is a $\mathbf{T}^{(k)}_2$-ideal of $\mathfrak{S}_k$, and the *Riesz Decomposition Theorem* assures us that $\mathfrak{S}_k$ can be decomposed

$$\mathfrak{S}_k = \mathfrak{S}_{k-1} \oplus \mathfrak{J}_{k-1}. \tag{2.23}$$

Let $\mathbf{T}^{(k-1)}_1 := \mathbf{P}_{\mathfrak{J}_{k-1}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{k-1}}$ $(= \mathbf{P}_{\mathfrak{J}_{k-1}}\mathbf{T}^{(k)}_2\mathbf{P}_{\mathfrak{J}_{k-1}})$, $\mathbf{T}^{(k-1)}_2 := \mathbf{P}_{\mathfrak{S}_{k-1}}\mathbf{T}\mathbf{P}_{\mathfrak{S}_{k-1}}$ $(= \mathbf{P}_{\mathfrak{S}_{k-1}}\mathbf{T}^{(k)}_2\mathbf{P}_{\mathfrak{S}_{k-1}})$, and $\mathbf{T}^{(k-1)}_{2,1} = \mathbf{P}_{\mathfrak{S}_{k-1}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{k-1}}$, where $\mathbf{P}_{\mathfrak{S}_{k-1}}$ and $\mathbf{P}_{\mathfrak{J}_{k-1}}$ are the lattice projections of E onto $\mathfrak{S}_{k-1}$ and $\mathfrak{J}_{k-1}$ respectively. The precise same arguments as used in Lemma II.1 allow us to deduce that $\lambda = 1$ is a simple pole of $\mathbf{R}(\lambda, \mathbf{T}^{(k-1)}_1)$. With respect to the decomposition of $\mathfrak{S}_k = \mathfrak{S}_{k-1} \oplus \mathfrak{J}_{k-1}$, we see that

$$\mathbf{T}^{(k)}_2 = \begin{pmatrix} \mathbf{T}^{(k-1)}_1 & 0 \\ \mathbf{T}^{(k-1)}_{2,1} & \mathbf{T}^{(k-1)}_2 \end{pmatrix}. \tag{2.24}$$

Continuing this procedure k times, we are able to represent E as

$$E = \mathfrak{J}_k \oplus \mathfrak{J}_{k-1} \oplus \cdots \oplus \mathfrak{J}_1 \oplus \mathfrak{S}_1 \tag{2.25}$$

with each $\mathbf{T}_1^{(i)}$, $i = 1, 2, \dots, k$, induced on $\mathfrak{J}_i$ by $\mathbf{T}$ having the property that $\lambda = 1$ is a simple pole of $\mathbf{R}(\lambda, \mathbf{T}_1^{(i)})$, $1 \le i \le k$. Applying Proposition II.1, we can further decompose $\mathfrak{J}_i$ into closed ideals such that the restriction of $\mathbf{T}_1^{(i)}$ to any one is irreducible. These closed ideals of $\mathfrak{J}_i$ are closed ideals of E, and we label these as

$$\{\mathfrak{J}_{i,l} : l = 1, 2, \dots, N(i),\ i = 1, 2, \dots, k\} \tag{2.26}$$

and

$$\{\mathfrak{J}^i_{1,0} : 1 \le i \le k\} \tag{2.27}$$

for which the associated induced operators have spectral radius less than unity. With respect to this decomposition of E, $\mathbf{T}$ can be schematically represented as

(2.28)

$$\mathbf{T} = \left[\begin{array}{c|c|c|c|c}
\begin{matrix} \mathbf{T}^{(k)}_{1,0} & 0 & 0 & \cdot & 0 \\ \vdots & \mathbf{T}^{(k)}_{1,1} & 0 & \cdot\,\cdot & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & 0 \\ \vdots & & & & \mathbf{T}^{(k)}_{1,N(k)} \end{matrix} & & & & \\
\hline
\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix} & \boxed{\begin{matrix} \mathbf{T}^{(k-1)}_{1,0} & 0 & 0 & \cdot\,\cdot & 0 & 0 \\ \vdots & \mathbf{T}^{(k-1)}_{1,1} & 0 & \cdot\,\cdot & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & 0 & & \ddots & & 0 \\ \vdots & & & & & \mathbf{T}^{(k-1)}_{1,N(k-1)} \end{matrix}} & & & 0 \\
\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix} & \ddots & \ddots & & \\
\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix} & & \ddots & \boxed{\begin{matrix} \mathbf{T}^{(1)}_{1,0} & 0 & 0 & \cdots & 0 \\ \vdots & \mathbf{T}^{(1)}_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & & & & \mathbf{T}^{(1)}_{1,N(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}} & 0 \\
\cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots & & \mathbf{T}^{(1)}_2
\end{array}\right]$$

where $r(\mathbf{T}^{(i)}_{1,0}) < 1$, $1 \le i \le k$, $r(\mathbf{T}^{(1)}_2) < 1$, $r(\mathbf{T}^{(j)}_{1,l}) = 1$, $1 \le j \le k$, $1 \le l \le N(k)$, and the dots in the off-diagonal (block) positions denote nonnegative and possibly nonzero operators. The operators occurring in the off-diagonal positions are in general nonzero, as $\mathbf{T}$ is not hypothesized to be *radical-free* (i.e., that the intersection of all the maximal $\mathbf{T}$-ideals is trivial (cf., e.g., [**14**, p. 526; 15, p. 223])).

Let us denote by $\mathfrak{F}$ the family of all bands $\mathfrak{J}_\alpha$ such that: (i) $\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}$ is irreducible, and (ii) $r(\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}) > 0$. The discussion above and preceding Proposition II.1 have shown that this family is nonempty under the hypothesis that unity is a pole of the resolvent $\mathbf{R}(\lambda, \mathbf{T})$ with finite-rank residuum. Any such $\mathfrak{J}_\alpha$ satisfying (i) and (ii) will be called *a principal* $\mathbf{T}$-*band*; and, for brevity, we denote $r(\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha})$ as $\sigma(\mathfrak{J}_\alpha)$. Any principal $\mathbf{T}$-band $\mathfrak{J}_\alpha$ for which $\sigma(\mathfrak{J}_\alpha) = 1$ is called a *basic* $\mathbf{T}$-*band.*

At this juncture, we would like to utilize the compactness properties of $\mathbf{T}$. The most immediate conclusion from such features of $\mathbf{T}$ is that the above constructive procedure terminates after at most a countable number of steps (cf., e.g., Proposition II.2). Another fact, used crucially in §§IV and V, is that the operator $\mathbf{P}_{\mathfrak{P}_1}\mathbf{T}\,\mathbf{P}_{\mathfrak{P}_2}$ itself has a compact iterate where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are any two bands of E. Indeed, if we let l_0 be the power for which $\mathbf{T}^{l_0}$ is compact, we see that $(\mathbf{P}_{\mathfrak{P}_1}\mathbf{T}\mathbf{P}_{\mathfrak{P}_2})^{l_0} \leq \mathbf{T}^{l_0}$, since $\mathbf{P}_{\mathfrak{P}_i}$, $i = 1, 2$, are lattice projections. Then a beautiful result by C. Aliprantis and O. Burkinshaw [**1**, pp. 277–278] assures us that $(\mathbf{P}_{\mathfrak{B}_1}\mathbf{T}\mathbf{P}_{\mathfrak{B}_2})^{2l_0}$ is compact. We note that the power $2l_0$ is independent of the selection of the bands $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

We now turn to defining a partial ordering on the principal $\mathbf{T}$-bands before investigating the cardinality of the family $\mathfrak{F}$. This partial ordering is by means of an *accessibility relation.*

Definition II.1. Let $\mathfrak{B}$ be any band of E. We denote by $\mathfrak{B}_-$ the (norm) closure of the smallest operator-invariant ideal containing $\mathfrak{B}$. For brevity, we say that $\mathfrak{B}_-$ is the $\mathbf{T}$-closure of $\mathfrak{B}$. A band $\mathfrak{B}$ is said to be $\mathbf{T}$-closed if $\mathfrak{B}_- = \mathfrak{B}$.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be any two bands of E. We say that $\mathfrak{B}_1$ has access to $\mathfrak{B}_2$ (or, equivalently, $\mathfrak{B}_2$ has access from $\mathfrak{B}_1$), if the dimension of $\mathfrak{B}_{1^-} \cap \mathfrak{B}_2$ is positive. The following lemma is useful.

Lemma II.2. *Let $\mathfrak{B}_1$ be an arbitrary band of E and $\mathfrak{B}_2$ be a principal* $\mathbf{T}$-*band. Then $\mathfrak{B}_1$ has access to $\mathfrak{B}_2$ if, and only if, $\mathfrak{B}_2 \subset \mathfrak{B}_{1^-}$.*

Proof. The proof of sufficiency is trivial. To show necessity, we suppose $\mathfrak{B}_1$ has access to $\mathfrak{B}_2$. Let $\mathfrak{A} := \mathfrak{B}_{1^-} \cap \mathfrak{B}_2$ with $\dim \mathfrak{A} > 0$, and observe that $\mathfrak{A}$ is a nonempty closed ideal. Then $\mathbf{P}_{\mathfrak{B}_2}\mathbf{T}\mathbf{P}_{\mathfrak{B}_2}\mathfrak{A} \subset \mathfrak{A}$

and thus $\mathfrak{A} = \mathfrak{B}_2$, as $\mathbf{P}_{\mathfrak{B}_2}\mathbf{T}\mathbf{P}_{\mathfrak{B}_2}$ is irreducible. So $\mathfrak{B}_2 \subset \mathfrak{B}_{1^-}$ and the proof is complete.

We next turn to characterizing the **T**-closure of principal **T**-bands; in particular, we define a partial ordering on the principal **T**-bands in terms of the accessibility relation. For principal **T**-bands $\mathfrak{J}_\alpha$, we can describe $\mathfrak{J}_{\alpha^-}$ in the following manner: Let f_α be the fundamental eigenelement to $\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}$, i.e. $r(\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha})f_\alpha = \mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}f_\alpha$, which is a quasi-interior element of $\mathfrak{J}_\alpha$. Form, now, the cyclic semi-group $\mathfrak{S} = \{\mathbf{T}^n : n \in \mathbb{N}\}$. Consider the ideal generated by the orbit $\mathfrak{S}f_\alpha := \{f_\alpha, \mathbf{T}f_\alpha, \mathbf{T}^2 f_\alpha, \dots, \mathbf{T}^n f_\alpha, \dots\}$, labeled as $\mathfrak{I}_\alpha$. Of course, this ideal is precisely the ideal generated by the solid hull $\mathbf{S}(\mathfrak{S}f_\alpha)$ (since any ideal is solid).

Next, recalling that the solid hull of the orbit $\mathfrak{S}f_\alpha$ is characterized by

$$\mathbf{S}(\mathfrak{S}f_\alpha) := \{y \in E : \text{ there exists } n \in \mathbb{N} \text{ such that } |y| \le \mathbf{T}^n f_\alpha\},$$

we see that $\mathfrak{I}_\alpha$ perforce contains all the symmetric order intervals $\{[-\mathbf{T}^n f_\alpha, \mathbf{T}^n f_\alpha], n \in \mathbb{N}\}$, and hence the principal ideals $E_{\mathbf{T}^n f_\alpha}$. The ideal $\mathfrak{I}_\alpha$ then lies within $\mathfrak{J}_{\alpha^-}$, as it must lie in every operator invariant ideal containing $\mathfrak{J}_\alpha$. Moreover, $\mathfrak{I}_\alpha$ is invariant under powers of $\mathbf{T}$, and its closure contains $\mathfrak{J}_\alpha$ itself as f_α is quasi-interior to $\mathfrak{J}_\alpha$. So $\overline{\mathfrak{I}}_\alpha = \mathfrak{J}_{\alpha^-}$.

With these deliberations, we are able to describe $\mathfrak{J}_{\alpha^-}$ in a more succinct fashion. Toward this end, let

$$y = \sum_{n=0}^{\infty} (2^{n+1}\|\mathbf{T}^n f_\alpha\|)^{-1}\mathbf{T}^n f_\alpha; \tag{2.29}$$

we claim that $E_y = \mathfrak{I}_\alpha$. For one thing, it is easy to see that $E_y := \bigcup_{n=0}^{\infty} n[-y, y]$ contains $\mathbf{S}(\mathfrak{S}f_\alpha)$ and thus the ideal $\mathfrak{I}_\alpha$ so generated. Certainly $\mathfrak{I}_\alpha$ contains E_y. So $\mathfrak{I}_\alpha = E_y$ and $\overline{E}_y = \mathfrak{I}_{\alpha_-}$. We easily see that $\mathfrak{J}_{\alpha^-} = \overline{E}_y$ is invariant under $\mathbf{T}$; and Proposition 8.3 of [**15**, pp. 186–187] indicates that if the ideal generated by $\mathfrak{S}f_\alpha$ intersects any other principal **T**-band $\mathfrak{J}_\beta$ in a nontrivial fashion, then $\mathfrak{J}_\beta$ is wholly contained in $\mathfrak{J}_{\alpha^-}$. The irreducibility of $\mathfrak{J}_\alpha$ per se can be exploited in a straightforward manner to show that if $\mathfrak{J}_{\tilde\alpha} \subset \mathfrak{J}_{\alpha^-}$ for which $\mathbf{T}\mathfrak{J}_{\tilde\alpha} \subset \mathfrak{J}_{\tilde\alpha}$, then $\mathfrak{J}_{\tilde\alpha}^{\perp} \cap \mathfrak{J}_{\alpha^-}$ contains $\mathfrak{J}_\alpha$, where $\mathfrak{J}_{\tilde\alpha}$ is a band of $\mathfrak{J}_{\alpha^-}$. Therefore, $\mathfrak{J}_{\tilde\alpha} \subset \mathfrak{J}_{\tilde\alpha}^{\perp} \cap \mathfrak{J}_{\alpha^-}$.

The concept of accessibility enables us to define a partial ordering on the collection of principal **T**-bands. Indeed, if $\mathfrak{J}_\alpha$ and $\mathfrak{J}_\beta$ are any two such bands, we say that

$$\mathfrak{J}_\alpha \leq_{\mathbf{T}} \mathfrak{J}_\beta\,, \tag{2.30}$$

if, and only if, $\mathfrak{J}_{\alpha^-} \supseteq \mathfrak{J}_{\beta^-}$. In the following, we show antisymmetry, as the reflexivity and transitivity properties are trivial. Suppose that

$$\mathfrak{J}_\alpha \leq_{\mathbf{T}} \mathfrak{J}_\beta \quad \text{and} \quad \mathfrak{J}_\beta \leq_{\mathbf{T}} \mathfrak{J}_\alpha. \tag{2.31}$$

Then $\mathfrak{J}_{\alpha^-} = \mathfrak{J}_{\beta^-}$; and any band in $\mathfrak{J}_{\alpha^-}$ and $\mathfrak{J}_{\beta^-}$ has access from $\mathfrak{J}_\alpha$ and $\mathfrak{J}_\beta$ respectively. Thus we can conclude that $\mathfrak{J}_\beta$ must have access from $\mathfrak{J}_\alpha$ and vice versa.

Now, the representation of $\mathfrak{J}_{\alpha^-} = \overline{E}_y$ means that the smallest band will be generated in a natural manner, which contains all the bands of $\mathfrak{J}_{\alpha^-}$ having access both *to* and *from* $\mathfrak{J}_\alpha$. For example, this band, which we label as $\mathfrak{C}$, will contain both $\mathfrak{J}_\alpha$ and $\mathfrak{J}_\beta$. To show that $\mathbf{P}_\mathfrak{C}\mathbf{T}\mathbf{P}_\mathfrak{C}$ is irreducible, we see that, by definition, $\mathfrak{C}^\perp \cap \mathfrak{C}_-$ is a maximal $\mathbf{P}_{\mathfrak{J}_{\alpha^-}}\mathbf{T}$-ideal, since no band therein will have access to $\mathfrak{C}$. By [**15**, Proposition 8.2 (Corollary), p. 186], we can deduce that $\mathbf{P}_\mathfrak{C}\mathbf{T}\mathbf{P}_\mathfrak{C}$ is irreducible, and $\mathfrak{C}_- = \mathfrak{J}_{\alpha^-}$.

We must therefore conclude that $\mathfrak{C}$ is a principal **T**-band. This is easily seen by the following argument. The procedure for describing the Frobenius decomposition of $\mathbf{T}$, when applied to $\mathfrak{J}_{\alpha^-} = \mathfrak{C}_-$, will produce those bands $\mathfrak{J}_\gamma \subset \mathfrak{C}^\perp \cap \mathfrak{C}_-$ such that each $\mathbf{P}_{\mathfrak{J}_\gamma}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\gamma}$ is irreducible, or a quasi-nilpotent Υ_α, and $\mathfrak{C}$ itself will be generated since $\mathbf{P}_\mathfrak{C}\mathbf{T}\mathbf{P}_\mathfrak{C}$ is irreducible. The band $\mathfrak{C}$ is, of necessity, a principal **T**-band, as seen from applying Lemma II.2. But the presence of $\mathfrak{C}$ as a principal **T**-band contradicts the fact that principal **T**-bands are disjoint. So $\mathfrak{J}_\alpha = \mathfrak{J}_\beta$ and antisymmetry is shown.

With respect to $\leq_{\mathbf{T}}$, we say that a principal **T**-band $\mathfrak{I}$ is *minimal* in a **T**-closed band $\mathfrak{J}$ if no other band in $\mathfrak{J}$ has access to $\mathfrak{I}$. We also call $\mathfrak{I}$ an *initial band* in $\mathfrak{J}$. Similarly, we say that a principal **T**-band $\mathfrak{I}$ is *maximal* in a **T**-closed band $\mathfrak{J}$ if no other band in $\mathfrak{J}$ has access from $\mathfrak{I}$. In $\mathfrak{J}_{\alpha^-}$, where $\mathfrak{J}_\alpha$ is a principal **T**-band, we see that $\mathfrak{J}_\alpha$ is minimal; moreover, we see that the band $\mathfrak{J}_\alpha^\perp \cap \mathfrak{J}_{\alpha^-}$ is left invariant by $\mathbf{T}$, if $\dim(\mathfrak{J}_\alpha^\perp \cap \mathfrak{J}_{\alpha^-}) > 0$.

These results allow us to prove the following:

PROPOSITION II.2. *For arbitrary* $\lambda \in (0, 1]$, *there are at most a finite number of principal* **T**-*bands* $\mathfrak{J}_\alpha$ *for which* $r(\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}) > \lambda$.

Proof. We make crucial use of the eventual compactness of $\mathbf{T}$ at this juncture. Suppose there is at least a countable infinity of such bands $\mathfrak{J}_\alpha$, enumerated as $\mathfrak{J}_i$, $i = 1, 2, \ldots$. Let f_i be the nonnegative eigenfunction to $\mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i}$ having $r(\mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i})$ as associated eigenvalue and normalized by $\|\tilde{f}_i\| = 1$, where $\tilde{f}_i$ is the trivial extension of f_i to E. Because $\mathbf{T}^{N_0}$ is compact for some integer N_0, we see that $\{\mathbf{T}^{N_0}\tilde{f}_i\}$ must contain a convergent subsequence. The antisymmetry properties of "$\leq_{\mathbf{T}}$", together with the fact that $\mathbf{T}^{N_0}\tilde{f}_i \geq (\mathbf{P}_{\mathfrak{J}_i}\mathbf{T}\mathbf{P}_{\mathfrak{J}_i})^{N_0} f_i \geq \lambda^{N_0} f_i$, shows that $\|\mathbf{T}^{N_0} f_i - \mathbf{T}^{N_0} f_j\| \geq \lambda^{N_0}$ and hence no subsequence can be Cauchy in E. This completes the proof.

We can reapply our constructive procedure to the operators $\mathbf{T}^{(i)}_{1,0}$, $1 \leq i \leq k$, and to $\mathbf{T}^{(1)}_2$. For example, if $0 < \lambda_1 := r(\mathbf{T}^{(1)}_2)$ ($= r(\mathbf{P}_{\mathfrak{S}_1}\mathbf{T}\mathbf{P}_{\mathfrak{S}_1})$), we can regard $\mathbf{T}^{(1)}_2$ as an eventually compact positive linear operator on $\mathfrak{S}_1$, a Banach lattice with an order continuous norm. We further decompose $\mathfrak{S}_1$ into a direct sum of closed ideals of $\mathfrak{S}_1$ (which are closed ideals of E a fortiori) such that $\mathbf{T}^{(1)}_2$ restricted to these is irreducible with spectral radius λ_1 or has spectral radius less than λ_1.

Similarly, we repeat this procedure as long as $\mathbf{T}$, so restricted to the sublattice of E generated in this manner, possesses a positive spectral radius. After at most countably many steps, we are able to decompose E into closed ideals so that $\mathbf{T}$ restricted to each is either irreducible with positive spectral radius or is quasi-nilpotent.

The main result of this section is as follows:

THEOREM II.1. *Let E be a Banach lattice with order continuous norm, and let $\mathbf{T}$ be a positive, reducible, eventually compact linear operator mapping E into itself. Suppose, in addition, that the spectral radius $r(\mathbf{T})$ of $\mathbf{T}$ is positive and normalized to unity. Then E can be decomposed into a direct sum of bands such that $\mathbf{T}$ restricted to each is either irreducible with positive spectral radius, or is quasi-nilpotent.*

REMARK 1. The principal $\mathbf{T}$-bands correspond to "equivalence classes of communicating states" in the Markov chain context (cf., e.g., [**5**, pp. 59–60]). For a nonnegative, reducible, eventually compact integral operator $\mathbf{K}$ defined on $L^p(\Omega, \mu)$, $1 \leq p < \infty$, with μ a σ-finite measure on Ω, the "significant k-components" determine the principal $\mathbf{K}$-bands [**11**, **16**, **17**].

In finite dimensions, state i is said to have access to state j if there exists an integer m such that $(\mathbf{A}^m)_{ji} > 0$, where $\mathbf{A}$ is the underlying nonnegatve reducible $N \times N$ matrix. The totality of states having access from a given class or state produces the smallest ideal, containing the class or state in question, invariant under $\mathbf{A}$. In the integral operator context, the "k-closure" of a σ-finite subset $C \subset \Omega$, denoted as C_-, is a subset of Ω. Such a subset has the property that functions in $L^p(\Omega, \mu)$, with support thereon, constitute the smallest band in $L^p(\Omega, \mu)$ invariant under $\mathbf{K}$ containing functions with support at most C [**17**, p. 487].

Accessibility between classes J and K in the finite-dimensional setting is defined by requiring that every state in class K has access from every state in class J. For nonnegative reducible integral operators on $L^p(\Omega, \mu)$, $1 \le p < \infty$, we say that a σ-finite set A has access to a σ-finite set B if $\mu(A_- \cap B) > 0$. With these observations, it is clear that our concept of accessibility between bands in this work, in terms of operator-invariant ideals of E, is the appropriate generalization of accessibility as discussed in [**13**] and [**17**]. We are presently in a position to introduce the notions of a chain of bands, length of a chain, etc. These are made precise in the following section.

III. Accessibility relations. In §II, we have shown that there exists at most a countable number of bands $\mathfrak{J}_k$ for which $r(\mathbf{P}_{\mathfrak{J}_k}\mathbf{T}\mathbf{P}_{\mathfrak{J}_k}) > 0$. In the lattice sense, then, E can be decomposed in the natural manner as

$$E = \sum_{k=1}^{\infty} \oplus \mathfrak{J}_k \oplus \Upsilon \tag{3.1}$$

where $\Upsilon := \left(\sum_{k=1}^{\infty} \oplus \mathfrak{J}_k\right)^{\perp}$. On Υ, $\mathbf{T}$ must be quasi-nilpotent for otherwise, we could repeat the procedure in §II on this particular band. The operators $\{\mathbf{P}_{\mathfrak{J}_k}\mathbf{T}\mathbf{P}_{\mathfrak{J}_k} : k \ge 1\}$ are irreducible, and the decomposition of the operator $\mathbf{T}$ is unique up to permutation of the $\mathfrak{J}_k$'s.

DEFINITION III.1. A chain of bands is a collection of bands such that every band in the collection has access to or from every other band. A chain with initial band $\mathfrak{I}$ and final band $\mathfrak{J}$ is called a chain from $\mathfrak{I}$ to $\mathfrak{J}$. The length of a chain is the number of the basic $\mathbf{T}$-bands it contains.

DEFINITION III.2. We say that a band $\mathfrak{I}$ has access to band $\mathfrak{J}$ (or, alternately, band $\mathfrak{J}$ has access from band $\mathfrak{I}$) in n steps, if the length of the longest chain from $\mathfrak{I}$ to $\mathfrak{J}$ is n. The depth of a band $\mathfrak{J}$ is the length of the longest chain in which $\mathfrak{J}$ is initial.

Let $\mathfrak{J}$ be a principal **T**-band, and observe that $\mathfrak{J}_-$ itself is a Banach lattice with order continuous norm. We can decompose $\mathfrak{J}_-$ into a direct sum of bands $\{\mathfrak{J}_\alpha\}$ of $\mathfrak{J}_-$, such that $\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}$ is either irreducible with positive spectral radius or quasi-nilpotent. But, from Proposition 8.3 of [**15**, p. 186], we know that any principal **T**-band which intersects $\mathfrak{J}_-$ is wholly contained in $\mathfrak{J}_-$. This fact, in conjunction with the observation that every band of $\mathfrak{J}_-$ is also a band of E, enables us to deduce that the constructive procedure in Section II when applied to $\mathfrak{J}_-$ decomposes it into a direct sum of principal **T**-bands and quasi-nilpotent bands.

Therefore, in discussing the depth of a principal **T**-band, we enumerate the principal **T**-bands in $\mathfrak{J}_{\alpha^-}$ as $\mathfrak{J}_{\alpha,1}, \mathfrak{J}_{\alpha,2}, \dots, \mathfrak{J}_{\alpha,n}, \dots$ with the following arrangement:

$$\begin{aligned}
(3.2)\quad \mathfrak{A}_0 &= \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \text{ has access from } \mathfrak{J}_\alpha \text{ in zero steps}\}, \\
\mathfrak{A}_1 &= \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \text{ has access from } \mathfrak{J}_\alpha \text{ in exactly one step}\}, \\
&\vdots \\
\mathfrak{A}_{m_-} &:= \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \text{ has access from } \mathfrak{J}_\alpha \text{ in exactly } m_- \text{ steps}\}.
\end{aligned}$$

We let $\Upsilon_\alpha := \Upsilon \cap \mathfrak{J}_{\alpha^-}$, and partition this band into disjoint bands $\Upsilon_{\alpha,0}, \Upsilon_{\alpha,1}, \dots, \Upsilon_{\alpha,m_-}$, each having access from $\mathfrak{J}_\alpha$ as described for the principal **T**-bands. We subdivide $\mathfrak{J}_{\alpha^-}$ in the following manner:

$$\begin{aligned}
(3.3)\qquad \mathfrak{P}_0 &:= \sum \oplus \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \in \mathfrak{A}_0\} \oplus \Upsilon_{\alpha,0} \\
\mathfrak{P}_1 &:= \sum \oplus \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \in \mathfrak{A}_1\} \oplus \Upsilon_{\alpha,1} \\
\mathfrak{P}_2 &:= \sum \oplus \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \in \mathfrak{A}_2\} \oplus \Upsilon_{\alpha,2} \\
&\vdots \\
\mathfrak{P}_{m_-} &:= \sum \oplus \{\mathfrak{J}_{\alpha,i} : \mathfrak{J}_{\alpha,i} \in \mathfrak{A}_{m_-}\} \oplus \Upsilon_{\alpha,m_-}.
\end{aligned}$$

The closing paragraphs of this section will be oriented to showing that $\mathfrak{P}_i$, $i = 1, \dots, m_-$, do in fact constitute bands of E. We give an alternate description of these sets. Toward this end, we list the basic **T**-bands from $\mathfrak{A}_i$ as $\mathfrak{J}^1_{i,j}$, $j = 1, 2, \dots, N(i)$, and those whose **T**-closures contain basic **T**-bands in $\mathfrak{A}_{i+1}$ as $\mathfrak{J}^1_{i,j}$, $j = 1, 2, \dots, r_i$. Then $\mathfrak{P}_i$ can be described more precisely as

$$(3.4)\qquad \mathfrak{P}_0 := \sum_{j=1}^{N(1)} J_{\alpha^-} \cap (\mathfrak{J}^1_{1,j^-})^\perp ,$$

$$(3.5)\qquad \mathfrak{P}_i := \sum_{j=1}^{r_i} \sum_{\{k\,:\,\mathfrak{J}^1_{i+1,k} \subset \mathfrak{J}^1_{i,j^-}\}} [\mathfrak{J}^1_{i,j^-} \cap (\mathfrak{J}^1_{i+1,k^-})^\perp] \oplus \sum_{j=r_i+1}^{N(i)} \mathfrak{J}^1_{i,j^-} , \quad i = 1, 2, 3, \dots, m_- .$$

It is clear that $\mathfrak{P}_i$, $i = 0, 1, \dots, m_-$, constitute bands of E. Moreover $\Upsilon_{\alpha,i}$, $i = 0, 1, 2, \dots, m_-$, are merely the intersection of the right hand sides of (3.4) and (3.5) with Υ. It is clear that a chain of length m_- can be constructed with $\mathfrak{J}_\alpha$ as the initial band, and we conclude that the depth of a principal $\mathbf{T}$-band is well defined.

IV. Results on the distinguished eigenvalues of T. In this section, we show the role of the principal $\mathbf{T}$-bands in characterizing the distinguished eigenvalues of $\mathbf{T}$. We recall that the notion of a *distinguished eigenvalue* of a positive operator was introduced by H. H. Schaefer [**14**]. In this work, however, Schaefer considers radical-free operators on $C(X)$, where $C(X)$ is the Banach algebra of continuous, complex functions defined on the compact Hausdorff space X. (A generalization of this work to general Banach lattices was indicated by Schaefer in [**15**, pp. 223].) His results show that each distinguished eigenvalue of the adjoint $\mathbf{T}^*$ is a simple pole of $\mathbf{R}(\lambda, \mathbf{T}^*)$, and that the geometric eigenspace belonging to differing distinguished eigenvalues are mutually orthogonal. His results are not difficult to understand, since radical-free operators in some cases are completely reducible (i.e., are the direct sum of irreducible operators).

The analysis in this work, on the other hand, characterizes the distinguished eigenvalues of $\mathbf{T}$ itself, in terms of the bands constituting E, as described in Theorem II.1, without the restriction that $\mathbf{T}$ be radical-free. In particular, we show that the dimension of the geometric eigenspace of $\mathbf{T}$ belonging to $r(\mathbf{T})$ is equal to the number of basic $\mathbf{T}$-bands satisfying (4.1) below. We investigate the positivity features in Theorem IV.2.

The next two results are needed for our study of the geometric eigenspace belonging to $r(\mathbf{T})$. Their proofs are straightforward and rely on the fact that $\mathbf{P}_{\mathfrak{J}_\alpha}\mathbf{T}\mathbf{P}_{\mathfrak{J}_\alpha}$ possesses a compact iterate where, here, $\mathfrak{J}_\alpha$ is a principal $\mathbf{T}$-band.

LEMMA IV.1. *Let* $0 < \lambda_0 \leq 1$ *be a distinguished eigenvalue of* $\mathbf{T}$. *Then* $\lambda_0 = \sigma(\mathfrak{J}_\alpha)$ *for some principal* $\mathbf{T}$*-band* $\mathfrak{J}_\alpha$.

PROPOSITION IV.1. *Suppose that* $\mathfrak{J}$ *is a principal* $\mathbf{T}$*-band, with* $\sigma(\mathfrak{J}) \geq \lambda$, $0 < \lambda \leq 1$. *Then for any* $0 < x \in \mathfrak{J}$, $\sum_{n=1}^{\infty} \lambda^{-n}\mathbf{T}^n x$ *diverges.*

The proof of Lemma IV.1 relies on the Frobenius decomposition of the (closure of) the principal ideal generated by a positive eigenelement to λ_0, by using the methods of §II. The proof of Proposition IV.1 proceeds in a straightforward manner by using classical arguments associated with the Neumann series of an operator.

We are now in a position to characterize those T-bands which account for the distinguished eigenvalues of $\mathbf{T}$.

THEOREM IV.1. *Let* $\{\mathfrak{J}_i : i = 1, \ldots, N(\lambda_0)\}$ *be the collection of all principal* $\mathbf{T}$*-bands for which* $\sigma(\mathfrak{J}_i) = \lambda_0$, $0 < \lambda_0 \leq 1$. *Then* λ_0 *itself is a distinguished eigenvalue of* $\mathbf{T}$, *if and only if, there exists a* $\mathfrak{J}_{i_0}$ *for some* i_0, $1 \leq i_0 \leq N(\lambda_0)$, *such that*

$$\mathfrak{J}_{i_0^-} \cap (\mathfrak{J}_{i_0})^{\perp} \subset \overline{\sum_{\sigma(\mathfrak{J})<\lambda_0} \oplus\{\mathfrak{J} : \mathfrak{J}_{i_0} \leq_{\mathbf{T}} \mathfrak{J}\} \cup \Upsilon}\,. \tag{4.1}$$

Proof. Suppose there exists a principal T-band $\mathfrak{J}_{i_0}$ such that (4.1) is true. Observe that $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}$ is irreducible and $\lambda_0 = \sigma(\mathfrak{J}_{i_0})$. Let f_{i_0} be the fundamental eigenelement of $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}$ belonging to λ_0, i.e. $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}} f_{i_0} = \lambda_0 f_{i_0}$, where $0 < f_{i_0} \in \mathfrak{J}_{i_0}$ is a quasi-interior element of $\mathfrak{J}_{i_0}$.

Let $\mathfrak{J}_{i_0^-}$ be the T-closure of $\mathfrak{J}_{i_0}$. In order to show that λ_0 is a distinguished eigenvalue of $\mathbf{T}$, it suffices to show the existence of a nonnegative and nontrivial eigenelement of $\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0^-}}$ belonging to λ_0. Since $\mathfrak{J}_{i_0^-}$ is T-invariant, we have $\mathbf{P}_{\mathfrak{J}_{i_0^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0^-}} = \mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0^-}}$.

We consider the operator equation

$$\mathbf{P}_{\mathfrak{J}_{i_0^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0^-}} g = \lambda_0 g \tag{4.2}$$

and let $\mathbf{P}_{\mathfrak{J}_{i_0}} g := g_{i_0}$ and $\mathbf{P}_{(\mathfrak{J}_{i_0^-} \cap \mathfrak{J}_{i_0}^{\perp})} g = \tilde{g}$. We split up (4.2) into two operator equations, by virtue of the minimality of $\mathfrak{J}_{i_0}$ in $\mathfrak{J}_{i_0^-}$:

$$\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}} g_{i_0} = \lambda_0 g_{i_0} \tag{4.3}$$

and

$$\mathbf{P}_{(\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-})}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}f_{i_0} + \mathbf{P}_{(\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-})}\mathbf{T}\mathbf{P}_{(\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-})}\tilde{g} = \lambda_0\tilde{g}. \tag{4.4}$$

We select g_{i_0} as f_{i_0}, the fundamental eigenelement to $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}$ belonging to λ_0. Moreover, by hypothesis,

$$r(\mathbf{P}_{(\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-})}\mathbf{T}\mathbf{P}_{(\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-})}) < \lambda_0 \tag{4.5}$$

(since otherwise, we could apply the procedure in §II to produce a principal T-band with spectral radius at least λ_0)! Therefore (4.4) is solvable for $\tilde{g}$ via a Neumann series argument. So λ_0 is distinguished.

Conversely, suppose that λ_0 is distinguished. We let x be one of the corresponding nonnegative and nontrivial eigenelements of $\mathbf{T}$ belonging to λ_0, and let $\mathfrak{J}$ be the closure of the principal ideal E_x i.e. $\mathfrak{J} = \overline{\bigcup_{n=1}^{\infty} n[-x, x]}$. Observe that $\mathfrak{J}_- = \mathfrak{J}$ and that x itself is a quasi-interior element of $\mathfrak{J}$. From Theorem II.1, we know that $\mathfrak{J}$ can be decomposed into a direct sum of bands of $\mathfrak{J}$, such that $\mathbf{T}$ restricted to each is either irreducible with positive spectral radius or quasi-nilpotent.

We let $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$ be the collection of all principal $\mathbf{T}$-bands $\mathfrak{J}_\alpha$ in $\mathfrak{J}$ with $\sigma(\mathfrak{J}_\alpha) = \lambda_0$. We claim that at least one of the $\mathfrak{J}_\alpha$ in $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$ is minimal in $\mathfrak{J}$ under $\leq_{\mathbf{T}}$. For suppose *none* of the $\mathfrak{J}_\alpha$ in $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$ is minimal in $\mathbf{\Gamma}$; then there must exist a nontrivial band $\mathfrak{J} \subset \mathfrak{J}$, with $r(\mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}) < \lambda_0$, and x must perforce satisfy

$$\mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}x = \lambda_0\mathbf{P}_{\mathfrak{J}}x, \tag{4.6}$$

$$\mathbf{P}_{\mathfrak{J}^{\perp}\cap\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}x + \mathbf{P}_{\mathfrak{J}^{\perp}\cap\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}^{\perp}\cap\mathfrak{J}}x = \lambda_0\mathbf{P}_{\mathfrak{J}^{\perp}\cap\mathfrak{J}}x. \tag{4.7}$$

Now, observe that $\lambda_0\mathbf{I} - \mathbf{P}_{\mathfrak{J}}\mathbf{T}\mathbf{P}_{\mathfrak{J}}$ is invertible, and hence $\mathbf{P}_{\mathfrak{J}}x = 0$. But $\mathbf{P}_{\mathfrak{J}}$ is positive and continuous on $\mathfrak{J}$ with $\mathbf{P}_{\mathfrak{J}}\mathfrak{J}$ dense in $\mathfrak{J}$. From Proposition 6.4 of [**15**, p. 99], we can conclude that $\mathbf{P}_{\mathfrak{J}}x$ is a quasi-interior point of $\mathfrak{J}$. Now $\mathbf{P}_{\mathfrak{J}}x = 0$, which forces us to conclude $\mathfrak{J} = 0$, a contradiction. Therefore there exists at least one $\mathfrak{J}_{i_0}$ in $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$ such that $\mathfrak{J}_{i_0}$ is minimal in $\mathfrak{J}$ under "$\leq_{\mathbf{T}}$".

We wish to show that *all* principal $\mathbf{T}$-bands in the collection of $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$ are minimal in $\mathfrak{J}$ and hence (4.1) holds for all bands in $\mathbf{\Gamma}_{\lambda_0, \mathfrak{J}}$. We are able to express $\mathbf{P}_{\mathfrak{J}_{i_0}}x = x_1 > 0$, $\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}}x = x_2$ and both are quasi-interior elements of their respective bands. We have that x_1 and x_2 must necessarily solve

$$\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}x_1 = \lambda_0 x_1, \tag{4.8}$$

$$\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}x_1 = (\lambda_0\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}} - \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}})x_2. \tag{4.9}$$

Because x_2 is a quasi-interior element of the closed band $\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}$, we see that it dominates the partial sums of the Neumann series, each of which consists solely of nonnegative terms. The Neumann series must perforce converge and yield a nontrivial and nonnegative element of $\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}$. Two cases arise:

Case I: $r(\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}}) < \lambda_0$. Then the assertion (4.5) is shown.

Case II: $r(\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}}) = \lambda_0$. Then the convergence of the Neumann series indicates that $(\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}})^n \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}} x_1$ does not necessarily possess any component in any principal $\mathbf{T}$-bands with spectral radius λ_0 for *any* n. If we exploit the invariance of $\mathbf{T}$ on $\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}$, we can state more succinctly that for any positive integer n, $\mathbf{T}^n \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}} x_1$ possesses no nontrivial component in any principal $\mathbf{T}$-band with spectral radius λ_0. This can be seen by the following contradiction argument. Let $\mathfrak{J}_r \subset \mathfrak{J}_{i_0^-}$ with $\sigma(\mathfrak{J}_r) = \lambda_0$. Suppose that

$$\mathbf{P}_{\mathfrak{J}_r} \mathbf{T}^{N_0} \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}} \mathbf{T} \mathbf{P}_{\mathfrak{J}_{i_0}} x_1 > 0 \tag{4.10}$$

for some integer N_0. Then $\mathbf{P}_{\mathfrak{J}_r} x_2$ can be seen, after a straightforward manipulation of (4.9), to necessarily solve an equation of the form

$$\mathbf{P}_{\mathfrak{J}_r} x_2 - \lambda_0^{-(N_0+1)} \mathbf{P}_{\mathfrak{J}_r} \mathbf{T}^{N_0+1} \mathbf{P}_{\mathfrak{J}_r} x_2 = y, \tag{4.11}$$

$\mathbf{P}_{\mathfrak{J}_r} y > 0$. Now the element $\mathbf{P}_{\mathfrak{J}_r} x_2 > 0$ dominates the partial sums of the associated Neumann series which must diverge according to Proposition IV.1.

We conclude that there exists no principal $\mathbf{T}$-band with spectral radius λ_0 in $\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}$. The precise same arguments can be utilized to show that *any* principal $\mathbf{T}$-band with spectral radius λ_0 is minimal in $\mathfrak{J}$, with its $\mathbf{T}$-closure satisfying (4.1). This completes the proof of Theorem IV.1.

We next turn to describing the positivity features of the geometric eigenspace of $\mathbf{T}$ associated with $r(\mathbf{T})$. Let $\mathfrak{J}_{i_0}$ be a basic $\mathbf{T}$-band satisfying (4.1) for which $\dim(\mathfrak{J}_{i_0}^{\perp} \cap \mathfrak{J}_{i_0^-}) > 0$. Condition (4.1) is of course equivalent to requiring

$$r(\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}}) < 1 . \tag{4.12}$$

When we represent our eigenelement f_{i_0} as $f_{i_0} = g_{i_0} + \tilde{g}$, $\mathbf{P}_{\mathfrak{J}_{i_0}} f_{i_0} = g_{i_0}$, $\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}} f_{i_0} = \tilde{g}$, via equations (4.2)–(4.4), we see that g_{i_0} is quasi-interior to $\mathfrak{J}_{i_0}$ and $\tilde{g}$ is nonnegative because of (4.12). From the fact that $g_{i_0} \le f_{i_0}$, we have that $\mathfrak{J}_{i_0} \subseteq \overline{E}_{f_{i_0}}$. From the discussion characterizing $\mathfrak{J}_{i_0^-}$ following Lemma II.2, and from the fact that $\mathbf{T}f_{i_0} = f_{i_0}$, we can conclude that

$$\mathfrak{J}_{i_0^-} \subseteq (\overline{E}_{f_{i_0}})_- = \overline{E}_{f_{i_0}} \subseteq \mathfrak{J}_{i_0^-} . \tag{4.13}$$

Hence $\overline{E}_{f_{i_0}} = \mathfrak{J}_{i_0^-}$ and f_{i_0} is quasi-interior to $\mathfrak{J}_{i_0^-}$. We summarize this discussion in a stronger version of Theorem IV.1.

THEOREM IV.2. *Let* $\mathbf{\Gamma} := \{\mathfrak{J}_l : 1 \le l \le N\}$ *be the collection of all basic* $\mathbf{T}$*-bands and* $\mathbf{\Gamma}_1 := \{\mathfrak{J}_l : 1 \le l \le r,\ r(\mathbf{P}_{\mathfrak{J}_l^{\perp}\cap\mathfrak{J}_{l^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_l^{\perp}\cap\mathfrak{J}_{l^-}}) < 1\}$. *Then each positive eigenelement of* $\mathbf{T}$, *residing in the closed ideal* $\mathfrak{J}_{l^-}$, *is, in fact a quasi-interior element of* $\mathfrak{J}_{l^-}$.

V. Results on the algebraic eigenspace to $r(\mathbf{T})$. We recall that the Riesz index of an eigenvalue λ, $|\lambda| \neq 0$, of an operator $\mathbf{T}$ is the smallest number ν such that $\mathscr{N}(\lambda\mathbf{I} - \mathbf{T})^{\nu} = \mathscr{N}(\lambda\mathbf{I} - \mathbf{T})^{\nu+1}$, where $\mathscr{N}(\mathbf{T})$ denotes the null space of an operator $\mathbf{T}$. The subspace of E, $\mathscr{N}(\lambda\mathbf{I} - \mathbf{T})^{\nu}$, is the algebraic eigenspace of $\mathbf{T}$ belonging to λ; its elements are called generalized eigenelements of $\mathbf{T}$ belonging to λ, and its dimension is finite [**18**, pp. 330–344], since $\mathbf{T}$ itself possesses a compact iterate. We say that a generalized eigenelement ψ belonging to λ has index r, if and only if $(\lambda\mathbf{I} - \mathbf{T})^{r}\psi = 0$, but $(\lambda\mathbf{I} - \mathbf{T})^{r-1}\psi \neq 0$. For any $\lambda \neq 0$, ν is well defined and we denote its dependence on λ by writing $\nu(\lambda)$. We shall quote relevant facts from the Riesz-Schauder theory of linear operators with a compact iterate [**18**, Chapter 11] as they are needed in the ensuing discussion. We let $\delta(x, A) := \inf\{\|x - y\|_E,\ y \in A\}$.

We recall from Theorem IV.2 that a nonnegative eigenfunction can be constructed as an element of the band $\mathfrak{J}_{l^-}$, where $\mathfrak{J}_l$ has the property that $r(\mathbf{P}_{\mathfrak{J}_l^{\perp}\cap\mathfrak{J}_{l^-}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_l^{\perp}\cap\mathfrak{J}_{l^-}}) < 1$, and such an eigenelement is quasi-interior to $\mathfrak{J}_{l^-}$. Theorem IV.2 is used very crucially in the proof of the following result about some rather interesting features of the algebraic eigenspace of $\mathbf{T}$ belonging to $r(\mathbf{T})$.

THEOREM V.1. *Let E be a Banach lattice with order continuous norm, and $\mathbf{T}$ be a positive, eventually compact linear operator from E into itself with spectral radius $r(\mathbf{T}) > 0$ normalized to unity and having Riesz index ν_0. Let $\mathbf{\Gamma} = \{\mathfrak{J}_l : 1 \le l \le N\}$ be the collection of all basic $\mathbf{T}$-bands with the following partition:*

$$
\begin{aligned}
(5.1) \quad & \mathbf{\Gamma}_1 := \{\mathfrak{J}_i,\ 1 \le i \le N_1 : \mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-} \cap \mathbf{\Gamma} = \varnothing\}; \\
& \mathbf{\Gamma}_2 := \{\mathfrak{J}_i,\ N_1 + 1 \le i \le N_2 : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma} \subset \mathbf{\Gamma}_1\}; \\
& \mathbf{\Gamma}_3 := \{\mathfrak{J}_i,\ N_2 + 1 \le i \le N_3 : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma} \subset \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_2, \\
& \qquad\qquad (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma}_2 \ne \varnothing\}; \\
& \vdots \\
& \mathbf{\Gamma}_m = \left\{ \mathfrak{J}_i,\ N_{m-1} + 1 \le i \le N : \mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-} \cap \mathbf{\Gamma} \subset \bigcup_{j=0}^{m-1} \mathbf{\Gamma}_i, \right. \\
& \qquad\qquad \left. (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma}_{m-1} \ne \varnothing \right\},
\end{aligned}
$$

where here the notation $\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-} \cap \mathbf{\Gamma}$ indicates those basic $\mathbf{T}$-bands belonging to $\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}$, etc. Then: (1) *a basis for the algebraic eigenspace of $\mathbf{T}$ with respect to $r(\mathbf{T}) = 1$ can be chosen to consist of N generalized eigenelements f_i, such that $\mathbf{P}_{(\mathfrak{J}_{i^-})^{\perp}} f_i = 0$ and $\boldsymbol{\delta}(f_i, E_+) < \varepsilon_i$, ε_i arbitrarily given, $1 \le i \le N$;* (2) *the Riesz index of $r(\mathbf{T}) = 1$ is m;* (3) *there is a generalized eigenelement f such that for $n = 0, 1, \ldots, m-1$, $\boldsymbol{\delta}(\mathbf{P}_{\mathfrak{B}}[(\mathbf{T} - \mathbf{I})^n f], E_+) < \varepsilon_n$, ε_n arbitrarily given, if and only if the band $\mathfrak{B}$ has access from some basic $\mathbf{T}$-band in at least $n + 1$ steps.*

Proof. We choose a basic $\mathbf{T}$-band $\mathfrak{J}_{i_0}$ from $\mathbf{\Gamma}$ and show that: (a) there exists f_{i_0} such that for some integer $k(i_0)$, $(\mathbf{T} - \mathbf{I})^{k(i_0)} f_{i_0} = 0$ where $\mathbf{P}_{\mathfrak{J}_{i_0^-}^{\perp}} f_{i_0} = 0$ and (b) $\boldsymbol{\delta}(f_{i_0}, E_+) < \varepsilon_{i_0}$, ε_{i_0} arbitrarily given.

We let $m_-(i_0)$ be the *depth* of $\mathfrak{J}_{i_0}$ and $(\mathbf{\Gamma}_{i_0})_-$ the collection of elements of $\mathbf{\Gamma}$ consisting of those basic $\mathbf{T}$-bands in $\mathfrak{J}_{i_0^-}$. We partition $(\mathbf{\Gamma}_{i_0})_-$ under the partial ordering "$\le_{\mathbf{T}}$" as in (5.1), namely that

(5.2)

$$\mathbf{\Gamma}_1(i_0) := \{\mathfrak{J}_i,\ 1 \le i \le r_1 : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap (\mathbf{\Gamma}_{i_0})_- = \varnothing\};$$

$$\mathbf{\Gamma}_2(i_0) := \{\mathfrak{J}_i,\ r_1 + 1 \le i \le r_2 : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap (\mathbf{\Gamma}_{i_0})_- \subset \mathbf{\Gamma}_1(i_0)\};$$

$$\mathbf{\Gamma}_3(i_0) := \{\mathfrak{J}_i,\ r_2 + 1 \le i \le r_3 : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap (\mathbf{\Gamma}_{i_0})_- \subset \mathbf{\Gamma}_1(i_0) \cup \mathbf{\Gamma}_2(i_0), (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma}_2(i_0) \neq \varnothing\};$$

$$\vdots$$

$$\mathbf{\Gamma}_{m_-(i_0)-1}(i_0) := \left\{\mathfrak{J}_i,\ r_{m_-(i_0)-2} + 1 \le i \le r_{m_-(i_0)-1} : (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap (\mathbf{\Gamma}_{i_0})_- \subset \bigcup_{j=1}^{m_-(i_0)-2} \mathbf{\Gamma}_j(i_0), (\mathfrak{J}_i^{\perp} \cap \mathfrak{J}_{i^-}) \cap \mathbf{\Gamma}_{m_-(i_0)-2}(i_0) \neq \varnothing \right\}.$$

We observe $\mathfrak{J}_{i_0} \in \mathbf{\Gamma}_{m_-(i_0)}(i_0)$. From the fact that $\mathfrak{J}_{i_0}$ is the minimal band under "$\le_{\mathbf{T}}$" in $\mathfrak{J}_{i_0^-}$, and that $m_-(i_0)$ is the depth of $\mathfrak{J}_{i_0}$, we can conclude that $\mathbf{\Gamma}_{m_-(i_0)}(i_0) = \{\mathfrak{J}_{i_0}\}$. With the partition of $(\mathbf{\Gamma}_{i_0})_-$ as in (5.2), we see that those basic $\mathbf{T}$-bands in $\Gamma_i(i_0)$ which form the final band of a chain of length $m_-(i_0) - i + 1$ with initial band $\mathfrak{J}_{i_0}$ will reside in $\mathfrak{P}_{m_-(i_0)-i+1}(i_0)$, $1 \le i \le m_-(i_0)$, where the bands $\mathfrak{P}_i(i_0)$, $1 \le i \le m_-(i_0)$, are defined analogously to those bands $\mathfrak{P}_i$, $1 \le i \le m_-$, in (3.3). The number of basic $\mathbf{T}$-bands from $\Gamma_i(i_0)$ is positive, and such bands constitute the only minimal bands in $\mathfrak{P}_{m(i_0)-i+1}(i_0)$ with respect to "$\le_{\mathbf{T}}$". We observe that $\mathfrak{P}_0(i_0) = \varnothing$.

On $\mathfrak{J}_{i_0^-}$, we construct our generalized eigenelement f_{i_0} in an inductive manner: first on those bands constituting $\mathfrak{P}_1(i_0)$; then on those bands constituting $\mathfrak{P}_1(i_0) \oplus \mathfrak{P}_2(i_0)$; and at the jth stage, $1 \le j \le m_-(i_0)$, on bands constituting $\mathfrak{P}_1(i_0) \oplus \mathfrak{P}_2(i_0) \oplus \cdots \oplus \mathfrak{P}_j(i_0)$.

From the definition of $\mathfrak{P}_1(i_0)$, and the fact that $\mathfrak{J}_{i_0}$ is the only minimal band in $\mathfrak{P}_1(i_0)$, we want to show that $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}$ has only a simple eigenvalue. Toward this end, we claim that the linear form $f^* \in (\mathfrak{J}_{i_0}^{\circ})^{\perp}$, which coincides on $(\mathfrak{J}_{i_0}^{\circ})^{\perp}$ with the fundamental adjoint linear form to $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}$, is an eigenelement to $(\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)})^*$ belonging to unity. It behooves us to show for this linear form or functional f^* that

(5.3) $$\langle f^*, \mathbf{P}_{\mathfrak{P}_1(i_0)}(\mathbf{I} - \mathbf{T})g\rangle = 0 \quad \text{for every } g \in \mathfrak{P}_1(i_0).$$

Toward this end, we note that

$$(5.4) \qquad \begin{aligned} \langle f^*, \mathbf{P}_{\mathfrak{P}_1(i_0)}(\mathbf{I}-\mathbf{T})g\rangle &= \langle f^*, \mathbf{P}_{\mathfrak{J}_{i_0}}(\mathbf{I}-\mathbf{T})\mathbf{P}_{\mathfrak{J}_{i_0}}g\rangle \\ &+ \langle f^*, \mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}\cap\mathfrak{P}_1(i_0)}(\mathbf{I}-\mathbf{T})\mathbf{P}_{\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}\cap\mathfrak{P}_1(i_0)}g\rangle, \end{aligned}$$

since $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}\cap\mathfrak{P}_1(i_0)\subset\mathfrak{J}_{i_0}^{\perp}\cap\mathfrak{J}_{i_0^-}\cap\mathfrak{P}_1(i_0)$. Thus, we see that

$$(5.5) \qquad \langle f^*, \mathbf{P}_{\mathfrak{P}_1(i_0)}(\mathbf{I}-\mathbf{T})g\rangle = 0 \quad \text{for every } g\in\mathfrak{P}_1(i_0),$$

and our claim is shown.

Now any eigenelement x of $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}$ must have the property that $\mathbf{P}_{\mathfrak{J}_{i_0}}x\neq 0$, and moreover $\mathbf{P}_{\mathfrak{J}_{i_0}}x$ is a scalar multiple of the fundamental eigenelement to $\mathbf{P}_{\mathfrak{J}_{i_0}}\mathbf{T}\mathbf{P}_{\mathfrak{J}_{i_0}}$. Thus, there is no way that f^* can annihilate all but one of the eigenelements, as predicted by the Riesz-Schauder Theory [**18**, pp. 342–344, esp. Theorems 18 and 19], *were* the Riesz index of unity as an eigenvalue of $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}$ greater than one. So one is a simple eigenvalue to $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}$. We can appeal to both Theorems IV.1 and IV.2 to procure a quasi-interior element y_1 of $\mathfrak{P}_1(i_0)$ such that

$$(5.6) \qquad \mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}y_1 = y_1,$$

and $y_1\in\mathfrak{P}_1(i_0)\cap E_+$.

We proceed with our induction argument. Let $\mathbf{L}_i=\sum_{j=1}^{i}\oplus\mathfrak{P}_j(i_0)$ and define the associated operators:

$$(5.7) \qquad \mathbf{R}_i := \mathbf{P}_{\mathbf{L}_i}(\mathbf{T}-\mathbf{I})\mathbf{P}_{\mathbf{L}_i},$$

$$(5.8) \qquad \overline{\mathbf{T}}_i := \mathbf{P}_{\mathfrak{P}_i(i_0)}(\mathbf{T}-\mathbf{I})\mathbf{P}_{\mathfrak{P}_i(i_0)}, \qquad 1\le i\le m_-(i_0).$$

We assume that we have constructed an element y_i such that $(\mathbf{R}_i)^i y_i = 0$ and $\delta(y_i, \mathbf{L}_i\cap E_+) < \varepsilon_{i_0}^{(i)}$, and proceed to construct one with similar properties on $\mathbf{L}_{i+1}$.

Toward this end, we use the accessibility of $\mathbf{L}_i$ to $\mathfrak{P}_{i+1}(i_0)$ to observe that the operator equation

$$(5.9) \qquad \mathbf{R}_{i+1}z = g, \qquad z, g\in\mathbf{L}_{i+1}$$

can be decomposed into the equations

$$(5.10) \qquad \mathbf{R}_i z_1 = g_1, \qquad z_1, g_1\in\mathbf{L}_i,$$

$$(5.11) \qquad \mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathbf{L}_i}z_1 + \overline{\mathbf{T}}_{i+1}z_2 = g_2.$$

Here $\mathbf{P}_{\mathbf{L}_i}z = z_1$, $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}z = z_2$, with g_1 and g_2 given in an analogous manner. We define

$$(5.12) \qquad \mathbf{Q}_i := \mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathbf{L}_i}.$$

In order, then, to procure y_{i+1} for which $(\mathbf{R}_{i+1})^{i+1}y_{i+1} = 0$ we let $\mathbf{P}_{\mathbf{L}_i}y_{i+1} = y_i$, $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}y_{i+1} = \phi_{i+1}$ and attempt to solve for ϕ_{i+1}. Of necessity, y_i and ϕ_{i+1} must solve

$$\mathbf{R}_i^{i+1}y_i = 0, \tag{5.13}$$

$$(\overline{\mathbf{T}}_{i+1})^{i+1}\phi_{i+1} + \sum_{i=1}^{i}(\overline{\mathbf{T}}_{i+1})^l\mathbf{Q}_i(\mathbf{R}_i)^{i-l}y_i = 0. \tag{5.14}$$

We consider the equation for ϕ_{i+1}, but first we make some observations.

Let us suppose there are r_{i+1} basic **T**-bands in $\mathfrak{P}_{i+1}(i_0)$. Because every basic **T**-band is minimal in $\mathfrak{P}_{i+1}(i_0)$ under "$\leq_{\mathbf{T}}$," we can conclude, by an argument similar to that for $\mathfrak{P}_1(i_0)$, that unity is an eigenvalue with Riesz index one to $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}$. Indeed, enumerate the basic **T**-bands in $\mathfrak{P}_{i+1}(i_0)$ as $\mathfrak{J}^{(j)}_{i_0, i+1}$. Theorems IV.1 and IV.2 assure us that there is an eigenelement residing in $\mathfrak{J}^{(j)}_{i_0, i+1}{}^{-} \cap \mathfrak{P}_{i+1}(i_0)$, and is a quasi-interior element of this band. Moreover, we have r_{i+1} adjoint linear eigenelements to $(\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)})^*$, with each residing in the respective $((\mathfrak{J}^{(j)}_{i_0, i+1})^0)^\perp$ and coinciding with the fundamental adjoint linear form to $(\mathbf{P}_{\mathfrak{J}^{(j)}_{i_0, i+1}}\mathbf{T}\mathbf{P}_{\mathfrak{J}^{(j)}_{i_0, i+1}})^*$ associated with one. The proof that unity is an eigenvalue of $\mathbf{P}_{\mathfrak{P}_1(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_1(i_0)}$ with Riesz index one can be applied here also: If the Riesz index of one as an eigenvalue of $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}$ were greater than one, then the basis of generalized eigenelements could not possess the annihilation properties as guaranteed by the Riesz-Schauder theory [**18**, pp. 342–344 (Theorems 18 and 19)]. This is because *any* eigenelement must coincide on at least one of the $\mathfrak{J}^{(j)}_{i_0, i+1}$ with the fundamental eigenelement to $\mathbf{P}_{\mathfrak{J}^{(j)}_{i_0, i+1}}\mathbf{T}\mathbf{P}_{\mathfrak{J}^{(j)}_{i_0, i+1}}$.

We have that the Riesz index of unity as an eigenvalue of $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}$ is precisely one, and we can invoke Theorems IV.1 and IV.2 to obtain an eigenelement to $\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{i+1}(i_0)}$ which is quasi-interior to $\mathfrak{P}_{i+1}(i_0)$. We label this eigenelement f_{i+1}.

So, in (5.14), the range of $(\overline{\mathbf{T}}_{i+1})^j$ is precisely equal to the range of $\overline{\mathbf{T}}_{i+1}$, for any $j = 1, \dots,$ and we conclude that (5.14) is solvable for a real ϕ_{i+1} residing in $\mathfrak{P}_{i+1}(i_0)$. So, for any real number μ, $\mu f_{i+1} + \phi_{i+1}$ solves (5.14). We may express y_{i+1} as

$$y_{i+1} = \begin{cases} f_{i+1} + \mu_{i+1}^{-1}\phi_{i+1} & \text{on } \mathfrak{P}_{i+1}(i_0), \\ \mu_{i+1}^{-1}y_i & \text{on } \mathbf{L}_i, \end{cases} \tag{5.15}$$

and thus our eigenelement $x_{i_0} \in \mathfrak{J}_{i_0^-}$ is precisely

$$x_{i_0} = y_{m_-(i_0)} \in \mathfrak{J}_{i_0^-}. \tag{5.16}$$

It is clear that x_{i_0} can be made as close to a quasi-interior element of $\mathfrak{J}_{i_0^-}$ as wanted by choosing the μ_l's, $l = 1, \ldots, m_-(i_0)$, large enough.

We now proceed to show that the generalized eigenelements, constructed in the manner just described, form a basis. Linear independence follows from the same arguments used by U. Rothblum [**13**, p. 287] for the matrix setting, and the details are omitted. But we must show that this set so constructed exhausts the totality of possible basis elements. Indeed, suppose there is a generalized y which is *not* in the linear span of the x_i's so constructed. Let $\mathscr{T}$ be defined by

$$\mathscr{T} = \bigcap_{i=1}^{N} \mathfrak{J}_{i^-}^{\perp}. \tag{5.17}$$

Then $(\mathbf{P}_{\mathscr{T}}\mathbf{T}\mathbf{P}_{\mathscr{T}} - \mathbf{I})$ is invertible on $\mathscr{T}$ and thus for no n would

$$(\mathbf{P}_{\mathscr{T}}\mathbf{T}\mathbf{P}_{\mathscr{T}} - \mathbf{I})^n \mathbf{P}_{\mathscr{T}} y = 0 \tag{5.18}$$

unless $\mathbf{P}_{\mathscr{T}} y = 0$. So we conclude that $\mathbf{P}_{\mathscr{T}} y = 0$.

It is clear that $\mathscr{T}^{\perp} = \sum_{j=1}^{N} \mathfrak{J}_{j^-}$ is a band invariant under $\mathbf{T}$, with some of the basic $\mathbf{T}$-bands from $\mathbf{\Gamma}$ constituting minimal bands in $\mathscr{T}^{\perp}$. Suppose $\mathscr{T}^{\perp} = \overline{E}_{|y|}$. From Lemma IV.2, we know that $\mathbf{P}_{\mathfrak{J}_i}|y|$ is a quasi-interior element of $\mathfrak{J}_i \in \mathbf{\Gamma}_m$, and hence is a multiple of the quasi-interior element $f_i \in \mathfrak{J}_i$ determined by $\mathbf{P}_{\mathfrak{J}_i}\mathbf{T}P_{\mathfrak{J}_i}f_i = \sigma(\mathfrak{J}_i)f_i$. We may then subtract from y a suitable combination of $x_i \in \mathfrak{J}_{i^-}$, where $\mathfrak{J}_i \in \Gamma_m$, to produce another generalized eigenelement, which we label as y_1. Using the indexing of the basic $\mathbf{T}$-bands from (5.1), we claim that $\mathbf{P}_{\mathfrak{B}} y_1 = 0$, where $\mathfrak{B}$ is the band given by

$$\mathfrak{B} := \bigcap_{i=1}^{N_{m-1}} \mathfrak{J}_{i^-}^{\perp} \cap \mathscr{T}^{\perp}.$$

To see this, we know that by definition of y_1, $\mathbf{P}_{\mathfrak{J}_i} y_1 = 0, i = N_{m-1}+1, \ldots, N$. Therefore, $(\mathbf{P}_{\mathfrak{C}}\mathbf{T}P_{\mathfrak{C}} - \mathbf{I})$ is invertible where $\mathfrak{C} := \mathfrak{B} \cap (\bigcap_{i=N_{m-1}+1}^{N} \mathfrak{J}_i^{\perp})$, and we can also deduce that $\mathbf{P}_{\mathfrak{C}} y_1 = 0$.

From y_1, we can then subtract a suitable linear combination of $x_i \in \mathfrak{J}_{i^-}$, where $\mathfrak{J}_i \in \mathbf{\Gamma}_{m-1}$, to produce a generalized eigenelement y_2. By repeating this process, we ultimately subtract a combination of $x_i \in \mathfrak{J}_{i^-}$, $\mathfrak{J}_i \in \bigcup_{l=2}^{m} \mathbf{\Gamma}_l$, to produce a generalized eigenelement y_{m-1}

such that $\mathbf{P}_{\mathfrak{A}} y_{m-1} = 0$, where $\mathfrak{A}$ is precisely

$$\mathfrak{A} = \bigcap_{i=1}^{N_1} \mathfrak{J}_{i^-}^{\perp} \cap \mathscr{T}^{\perp}.$$

We can likewise conclude that we can subtract from y_{m-1} a suitable linear combination of $x_i \in \mathfrak{J}_{i^-}$, $i = 1, \ldots, N_1$, $\mathfrak{J}_i \in \mathbf{\Gamma}_1$, to produce a generalized eigenelement y_m.

We may assert that $\mathbf{P}_{\mathfrak{A}^\perp} y_m = 0$. Indeed, we have $\mathbf{P}_{\mathfrak{J}_i} y_m = 0$, $i = 1, \ldots, N_1$, by construction. On the band $\mathfrak{D} := \sum_{i=1}^{N_1} \mathfrak{J}_{i^-} \cap \mathfrak{J}_i^{\perp}$, we have $r(\mathbf{P}_{\mathfrak{D}} \mathbf{T} P_{\mathfrak{D}}) < 1$, and $\mathfrak{D}$ is invariant under $\mathbf{T}$ (by virtue of the minimality of each $\mathfrak{J}_i$ in the respective band $\mathfrak{J}_{i^-}$, $i = 1, \ldots, N_1$, $\mathfrak{J}_i \in \mathbf{\Gamma}_1$). So we conclude here too that $\mathbf{P}_{\mathfrak{D}} y_m = 0$ and thus $\mathbf{P}_{\mathfrak{A}^\perp} y_m = 0$. So $y_m = 0$, which contradicts the assumption that y is not in the linear span of $x_i \in \mathfrak{J}_{i^-}$, $\mathfrak{J}_i \in \mathbf{\Gamma}$. The general case where $\mathfrak{J}^{\perp} \neq \overline{E}_{|y|}$ follows easily, and the first part of the theorem is shown.

At this juncture, we see that we have shown the existence of a basis for the algebraic eigenspace of $\mathbf{T}$ associated with unity as an eigenvalue which does not lie in E_+, but whose elements are very close to E_+. In proving part 2 of our theorem, we shall obtain some rather interesting results concerning the positivity features of our basis eigenelements on certain bands of E.

It is enough to show that if $\mathfrak{J}_i$ is a basic $\mathbf{T}$-band with depth $m_-(i)$, and if x_i is a generalized eigenelement for which $\mathbf{P}_{(\mathfrak{J}_{i^-})^\perp} x_i = 0$, and $\delta(x_i, E_+) < \varepsilon_i$, ε_i arbitrarily given, then for every $k = 1, 2, \ldots, m_-(i)$,

(5.19) $\mathbf{P}_B[(\mathbf{T} - \mathbf{I})^l x_i] = 0$, $l \geq k$, for any closed band $B \subset \mathfrak{P}_k(i)$;

whereas

(5.20) $\mathbf{P}_{\mathfrak{P}_k(i)}[(\mathbf{T} - \mathbf{I})^{k-1} x_i]$ is a quasi-interior element of $\mathfrak{P}_k(i)$.

We assume that the bands of $\mathfrak{J}_{i^-}$ have been partitioned in terms of accessibility from $\mathfrak{J}_i$ as was done in the proof of (1).

We show (5.19) and (5.20) by induction on k. For $k = 1$, $\mathbf{P}_{\mathfrak{P}_1(i)} x_i$ is quasi-interior to $\mathfrak{P}_1(i)$ as seen in the proof of (1). In order to verify (5.19), we note that

$$\mathbf{P}_{\mathfrak{P}_1(i)}[(\mathbf{T} - \mathbf{I})^l x_i] = \mathbf{P}_{\mathfrak{P}_1(i)} (\mathbf{T} - \mathbf{I})^l \mathbf{P}_{\mathfrak{P}_1(i)} x_i = 0, \tag{5.21}$$

for every $l \geq 1$, since the Riesz index of one, as an eigenvalue of $\mathbf{P}_{\mathfrak{P}_1(i)} \mathbf{T} \mathbf{P}_{\mathfrak{P}_1(i)}$, is unity.

Suppose now for some integer η_0, $1 \le \eta_0 - 1 < m_-(i)$, (5.19) and (5.20) are true for $k = 1, 2, \dots, \eta_0 - 1$, and consider $k = \eta_0$. Because $(\mathbf{T} - \mathbf{I})^{\nu_0} x_i = 0$, where ν_0 is the Riesz index of one as an eigenvalue of $\mathbf{T}$, and by the induction hypothesis,

$$0 = \mathbf{P}_{\mathbf{L}_{\eta_0-1}}[(\mathbf{T}-\mathbf{I})^j x_i] = \mathbf{R}^j_{\eta_0-1}\mathbf{P}_{\mathbf{L}_{\eta_0-1}} x_i, \tag{5.22}$$

for every $j \ge \eta_0 - 1$, we can conclude that

$$0 = \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\nu_0} x_i] = \overline{\mathbf{T}}^{\nu_0-\eta_0+1}_{\eta_0}\{\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i\}. \tag{5.23}$$

To see this, we note that when utilizing the accessibility of $\mathbf{L}_{\eta_0-1}$ to $\mathfrak{P}_{\eta_0}(i)$,

$$\begin{aligned}
0 &= \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\nu_0}x_i] = (\overline{\mathbf{T}}_{\eta_0})^{\nu_0}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}x_i \\
&\quad + \sum_{l=0}^{\nu_0-1}(\overline{\mathbf{T}}_{\eta_0})^{\nu_0-l-1}\mathbf{Q}_{\eta_0-1}\mathbf{R}^l_{\eta_0-1}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i] \\
&= (\overline{\mathbf{T}}_{\eta_0})^{\nu_0-\eta_0+1}\Bigg\{\overline{\mathbf{T}}^{\eta_0-1}_{\eta_0}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}x_i \\
&\qquad + \sum_{l=0}^{\eta_0-2}(\overline{\mathbf{T}}_{\eta_0})^{\eta_0-l-2}\mathbf{Q}_{\eta_0-1}\mathbf{R}^l_{\eta_0-1}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\Bigg\} \\
&= (\overline{\mathbf{T}}_{\eta_0})^{\nu_0-\eta_0+1}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i].
\end{aligned} \tag{5.24}$$

Because the Riesz index of one as an eigenvalue of $\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}$ is unity, it follows from (5.22), (5.23), and (5.24) that for every $j \ge \eta_0$,

$$\begin{aligned}
0 &= (\overline{\mathbf{T}}_{\eta_0})^{j-\eta_0+1}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i] \\
&= \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^j x_i]
\end{aligned} \tag{5.25}$$

and (5.19) is shown.

We now proceed to show (5.20). By the first equality in (5.25), we have that

$$\begin{aligned}
&\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i] \\
&\quad = (\overline{\mathbf{T}}_{\eta_0})^{\eta_0-1}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}x_i + \sum_{l=0}^{\eta_0-2}(\overline{\mathbf{T}}_{\eta_0})^{\eta_0-l-2}\mathbf{Q}_{\eta_0-1}\mathbf{R}^l_{\eta_0-1}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]
\end{aligned} \tag{5.26}$$

is an eigenelement to $\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}$ associated with one. Now let $\mathfrak{J}_{1,i}, \dots, \mathfrak{J}_{M,i}$ be the M, say, basic T-bands in $\mathfrak{P}_{\eta_0}(i)$. Because the Riesz index of unity to $\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}$ is one, we may apply Theorems IV.1 and IV.2 to deduce the existence of positive eigenelements $\zeta_1, \zeta_2, \ldots, \zeta_M$ to $\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}$ associated with one such that each ζ_i is quasi-interior to $(\mathfrak{J}_{j,i})_- \cap \mathfrak{P}_{\eta_0}(i)$, $1 \le j \le M$; and $\mathscr{N}(\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}(\mathbf{T}-\mathbf{I})\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)})$ is spanned by $\zeta_1, \ldots, \zeta_M$. There exist then real numbers $b_1, \ldots, b_M$ such that

$$\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i] = \sum_{j=1}^{M} b_j \zeta_j. \tag{5.27}$$

We next show that all the b_j's are positive, and then

$$\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-1}x_i]$$

is a quasi-interior element of $\mathfrak{P}_{\eta_0}(i)$. Toward this end, we observe that adjoint eigenelements residing in

$$(\mathfrak{J}^0_{j,i})^\perp, \qquad 1 \le j \le M, \tag{5.28}$$

span the adjoint eigenspace to $\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}$ belonging to unity, as earlier deliberations show, and we denote these respectively by ζ_j^*, $1 \le j \le M$. Moreover

$$\langle \zeta_j^*, \ \zeta_i \rangle = \delta_{ij}. \tag{5.29}$$

Thus,

$$b_j = \langle \zeta_j^*, \ \mathbf{Q}_{\eta_0-1}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i] \rangle. \tag{5.30}$$

In order to deduce that $b_j > 0$, we first note that

$$\langle \zeta_j^*, \mathbf{P}_{\mathfrak{J}_{j,i}}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}y \rangle > 0 \tag{5.31}$$

for every $y > 0$ for which $\mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}y$ is quasi-interior to $\mathfrak{P}_{\eta_0-1}(i)$, since otherwise, the $\mathfrak{J}_{j,i}$'s would be minimal in $\mathfrak{P}_{\eta_0-1}(i)$. By the induction hypothesis,

$$\mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}[\mathbf{R}_{\eta_0-1}^{\eta_0-2}\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i] = \mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-2}x_i] \tag{5.32}$$

is quasi-interior to $\mathfrak{P}_{\eta_0-1}(i)$, and, moreover,

$$\mathbf{P}_{\mathfrak{P}_j(i)}\mathbf{R}_{\eta_0-1}^{\eta_0-2}\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i = \mathbf{P}_{\mathfrak{P}_j(i)}[(\mathbf{T}-\mathbf{I})^{\eta_0-2}x_i] = 0, \qquad j = 1, 2, \ldots, \eta_0 - 2. \tag{5.33}$$

So, we can deduce that

$$
\begin{aligned}
(5.34)\qquad b_j &= \langle \zeta_j^*, \mathbf{Q}_{\eta_0-1}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\rangle \\
&= \langle \zeta_j^*, \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathbf{L}_{\eta_0-1}}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\rangle \\
&= \sum_{l=1}^{\eta_0-1}\langle \zeta_j^*, \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_l(i)}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\rangle \\
&= \langle \zeta_j^*, \mathbf{P}_{\mathfrak{P}_{\eta_0}(i)}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\rangle \\
&= \langle \zeta_j^*, \mathbf{P}_{\mathfrak{J}_{j,i}}\mathbf{T}\mathbf{P}_{\mathfrak{P}_{\eta_0-1}(i)}\mathbf{R}_{\eta_0-1}^{\eta_0-2}[\mathbf{P}_{\mathbf{L}_{\eta_0-1}}x_i]\rangle > 0.
\end{aligned}
$$

So, $b_j > 0$, $1 \le j \le M$, and we can easily see that $\sum_{j=1}^{M} b_j\zeta_j$ is quasi-interior to $\mathfrak{P}_{\eta_0}(i)$. The proof of (2) is complete.

With the partitioning of the basic **T**-bands in (5.1) and (5.2), and with our concept of accessibility between principal **T**-bands as formulated in Definition II.1 and Lemma II.1, we can use the exact same arguments as U. Rothblum [**13**, pp. 290–291] for the matrix case to prove assertion (3). The straightforward details are omitted, and proof of the theorem is complete.

REMARK 2. We can see that the results of Theorem V.1 generalize those by H. D. Victory, Jr. [**17**] for the integral operator setting and the fundamental analysis carried out by U. G. Rothblum [**13**] for the matrix case. Indeed we can briefly summarize our results in this paper by stating that the algebraic eigenspace consists of a basis, whose elements can be chosen arbitrarily close to a quasi-interior element of (the closure of) the respective principal ideal so generated by the eigenelement itself. From (5.16), we see that in each y_{i+1}, $i = 1, 2, \ldots, m_-(i_0)$, $\mu_{i+1}^{-1}\phi_{i+1}$ can be made so small so that the generated x_{i_0} is arbitrarily close to a quasi-interior point of $\mathfrak{J}_{i_0^-}$. This accounts for the presence of sets, with total measure less than some arbitrarily given number, where nonnegativity may be absent in each basis element in the integral operator context [**17**]. In the finite-dimensional context, on the other hand, the positive cone possesses a topological interior, and our treatment produces a basis for the algebraic eigenspace of **T** belonging to $1 = r(\mathbf{T})$, consisting entirely of nonnegative vectors. A recent publication by K.-H. Förster and B. Nagy [**3**, pp. 164–165] provides an example in l^p, $1 \le p < \infty$, which shows that the generalized eigenelements do in general lie outside E_+, thereby indicating that the results of Theorem V.1 are sharp.

After this work was refereed and accepted for publication, we learned of similar results which had been obtained earlier in 1987 by J. Kölsche **[6]** of the Technische Universität Berlin in her Doctoral Dissertation. As far as we can ascertain, her results are still in Thesis form and are unpublished. Our treatment differs slightly from hers in obtaining the Frobenius decomposition of **T** in §II. Our results in Theorem V.1 are stronger, especially assertions (5.19)–(5.20) obtained in the proof of part (2).

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