ERRATA: “DENTABILITY, TREES, AND DUNFORD-PETTIS OPERATORS ON $L_1$”

MARIA GIRARDI AND ZHIBAO HU
A Banach space has the complete continuity property if all its bounded subsets are midpoint Bocce dentable. We show that a lemma used in the original proposed proof of this result is false; however, we give a proof to show that the result is indeed true.

1. Introduction. Throughout this paper, \( \mathcal{X} \) denotes an arbitrary Banach space, \( \mathcal{X}^* \) the dual space of \( \mathcal{X} \), \( B(\mathcal{X}) \) the closed unit ball of \( \mathcal{X} \), and \( S(\mathcal{X}) \) the unit sphere of \( \mathcal{X} \). The triple \( (\Omega, \Sigma, \mu) \) refers to the Lebesgue measure space on \([0, 1]\), \( \Sigma^+ \) to the sets in \( \Sigma \) with positive measure, and \( L_1 \) to \( L_1(\Omega, \Sigma, \mu) \). The \( \sigma \)-field generated by a partition \( \pi \) of \([0, 1]\) is \( \sigma(\pi) \). The conditional expectation of \( f \in L_1 \) given a \( \sigma \)-field \( \mathcal{B} \) is \( E(f | \mathcal{B}) \).

A Banach space \( \mathcal{X} \) has the complete continuity property (CCP) if each bounded linear operator from \( L_1 \) into \( \mathcal{X} \) is Dunford-Pettis (i.e. carries weakly convergent sequences onto norm convergent sequences). Since a representable operator is Dunford-Pettis, the CCP is a weakening of the Radon-Nikodym property (RNP). Recall that a Banach space has the RNP if and only if all its bounded subsets are dentable. A subset \( D \) of \( \mathcal{X} \) is dentable if for each \( \epsilon > 0 \) there is \( x \) in \( D \) such that \( x \notin \overline{cD} \{ y \in D : \| x - y \| \geq \epsilon \} \). Midpoint Bocce dentability is a weakening of dentability. The subset \( D \) is midpoint Bocce dentable if for each \( \epsilon > 0 \) there is a finite subset \( F \) of \( D \) such that for each \( x^* \) in \( B(\mathcal{X}^*) \) there is \( x \) in \( F \) satisfying:

\[
\text{if } x = \frac{1}{2}z_1 + \frac{1}{2}z_2 \text{ with } z_i \in D \text{ then } |x^*(x - z_1)| \equiv |x^*(x - z_2)| < \epsilon.
\]

The following theorem is presented in [G1].

**Theorem 1.** \( \mathcal{X} \) has the CCP if all bounded subsets of \( \mathcal{X} \) are midpoint Bocce dentable.
Our purpose in writing this note is to show that Lemma 2.9 in [Gl] (which was used in [Gl] to prove Theorem 1) is false and to provide a proof of the theorem. Lemma 2.9 asserts that if $A$ is in $\Sigma^+$ and $f$ in $L_\infty(\mu)$ is not constant a.e. on $A$, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of $A$ such that

$$\sigma(\bigcup \pi_n) = \Sigma \cap A$$

and for each $n$

$$\mu \left( \bigcup \left\{ E : E \in \pi_n \text{ and } \frac{\int_E f \, d\mu}{\mu(E)} \geq \frac{\int_A f \, d\mu}{\mu(A)} \right\} \right) = \frac{\mu(A)}{2}.$$

Example 2 shows that Lemma 2.9 is false.

**Example 2.** Let $f = 3\chi_{[0, \frac{1}{4})} - \chi_{[\frac{1}{4}, 1]}$. Then $\int_{\Omega} f \, d\mu = 0$. Suppose that $\{\pi_n\}$ is an increasing sequence of positive finite measurable partitions of $[0, 1]$ such that for each $n$

$$\mu \left( \bigcup \left\{ E : E \in \pi_n \text{ and } \frac{\int_E f \, d\mu}{\mu(E)} \geq 0 \right\} \right) = \frac{1}{2}.$$

Then $\sigma(\bigcup \pi_n) \neq \Sigma$.

**Proof.** Consider the martingale $\{f_n\}$ given by

$$f_n(\cdot) = E(f | \sigma(\pi_n)) = \sum_{E \in \pi_n} \frac{\int_E f \, d\mu}{\mu(E)} \chi_E(\cdot).$$

For each $n \in \mathbb{N}$ put

$$P_n = \bigcup \left\{ E : E \in \pi_n \text{ and } \int_E f \, d\mu \geq 0 \right\} \quad \text{and} \quad Q_n = P_n \cap (\frac{1}{4}, 1].$$

Since $\mu(P_n) = \frac{1}{2}$, we have that $\mu(Q_n) \geq \frac{1}{4}$. Thus

$$\int_{\Omega} |f_n - f| \, d\mu \geq \int_{Q_n} |f_n - f| \, d\mu \geq \int_{Q_n} (f_n - f) \, d\mu \geq \int_{Q_n} 1 \, d\mu = \mu(Q_n) \geq \frac{1}{4}.$$

We know that such a martingale $E(f | \sigma(\pi_n))$ converges in $L_1$ norm to $E(f | \sigma(\bigcup \pi_n))$. But $E(f | \Sigma) = f$. Thus $\sigma(\bigcup \pi_n) \neq \Sigma$. \qed

The error in the proof of Lemma 2.9 occurred in assuming that if $A$ is in $\Sigma^+$ and $\{\pi_n\}$ is an increasing sequence of positive measurable partitions of $A$ such that for each $n$ and each $E$ in $\pi_n$ the $\mu(E) \leq \varepsilon_n$ with $\lim_n \varepsilon_n = 0$, then $\sigma(\bigcup \pi_n) = \Sigma \cap A$. This seemingly sound assertion is not true as shown by the following counterexample.
EXAMPLE 3. For \( n \in \mathbb{N} \) and \( 1 \leq i \leq 2^n \), define
\[
E_i^n = \left( \frac{i-1}{2^n+1}, \frac{i}{2^n+1} \right) \cup \left[ \frac{1}{2} + \frac{i-1}{2^n+1}, \frac{1}{2} + \frac{i}{2^n+1} \right]
\]
and
\[
\pi_n = \{ E_i^n : 1 \leq i \leq 2^n \}.
\]
Clearly \( \{\pi_n\} \) is an increasing sequence of positive measurable partitions of \([0, 1]\) such that \( \mu(E) = 2^{-n} \) for each \( n \) and each \( E \in \pi_n \). Let \( f = \chi_{[0, \frac{1}{2}]} \). An easy computation shows that \( E(f | \sigma(\pi_n)) = \frac{1}{2} \chi_{[0,1]} \). We know that such a martingale \( E(f | \sigma(\pi_n)) \) converges in \( L_1 \) norm to \( E(f | \sigma(\bigcup \pi_n)) \). But \( E(f | \Sigma) = f \). Thus \( \sigma(\bigcup \pi_n) \neq \Sigma \). \( \square \)

2. Proof of theorem. Our proof of Theorem 1 uses the following observations. For \( f \) in \( L_1 \) and \( A \) in \( \Sigma \), the average value and the Bocce oscillation of \( f \) on \( A \) respectively are
\[
m_A(f) = \frac{\int_A f \, d\mu}{\mu(A)}
\]
and
\[
\text{Bocce-osc } f|_A = \frac{\int_A |f - m_A(f)| \, d\mu}{\mu(A)}
\]
oberving the convention that \( 0/0 \) is 0.

**Lemma 4.** Fix \( A \) in \( \Sigma \) and \( f \) in \( L_1 \). There is a subset \( E \) of \( A \) with \( 2\mu(E) = \mu(A) \) and
\[
\frac{1}{2} \text{ Bocce-osc } f|_A \leq |m_E(f) - m_A(f)|.
\]
Furthermore, for each subset \( E \) of \( A \) with \( 2\mu(E) = \mu(A) \),
\[
|m_E(f) - m_A(f)| \leq \text{Bocce-osc } f|_A.
\]

**Proof.** Without loss of generality, \( A = \Omega \) and \( \int_{\Omega} f \, d\mu = 0 \) and \( \int_{\Omega} |f| \, d\mu = 1 \). With this normalization, \( \text{Bocce-osc } f|_A = 1 \) and \( |m_E(f) - m_A(f)| = |m_E(f)| \). Let \( P = \{f \geq 0\} \) and \( N = \{f < 0\} \).

The first claim now reads that \( \frac{1}{2} \leq 2 \| \int_E f \, d\mu \| \) for some subset \( E \) of measure one half. Wlog \( \mu(P) \geq \frac{1}{2} \). Partition \( P \) into 2 sets, \( P_1 \) and \( P_2 \), of equal measure such that \( \int_{P_2} f \, d\mu \leq \int_{P_1} f \, d\mu \). Note that
\[
1 = \int_{\Omega} |f| \, d\mu = \int_{P} f \, d\mu + \int_{N} -f \, d\mu = 2 \int_{P} f \, d\mu = 2 \left[ \int_{P_1} f \, d\mu + \int_{P_2} f \, d\mu \right] \leq 4 \int_{P_1} f \, d\mu.
\]
Since $\mu(P_1) \leq \frac{1}{2} \leq \mu(P)$, we can find a set $E$ such that $P_1 \subset E \subset P$ and $\mu(E) = \frac{1}{2}$. For such a set $E$

$$\frac{1}{4} \leq \int_{P_1} f \, d\mu \leq \int_E f \, d\mu,$$

as needed.

Normalized, the second claim reads that for each subset $E$ of measure $\frac{1}{2}$

$$2 \left| \int_E f \, d\mu \right| \leq 1.$$

Fix a subset $E$ of measure $\frac{1}{2}$. Wlog $\int_{E \cap N} -f \, d\mu \leq \int_{E \cap P} f \, d\mu$. So

$$\left| \int_E f \, d\mu \right| = \left| \int_{E \cap P} f \, d\mu + \int_{E \cap N} f \, d\mu \right| \leq \int_{E \cap P} f \, d\mu \leq \int_P |f| \, d\mu = \frac{1}{2},$$

as needed. \hfill \Box

A subset $K$ of $L_1$ satisfies the Bocce criterion if for each $\varepsilon > 0$ and $B$ in $\Sigma^+$ there is a finite collection $\mathcal{F}$ of subsets of $B$ each with positive measure such that for each $f$ in $K$ there is an $A$ in $\mathcal{F}$ satisfying

(*) Bocce-osc $f|_A < \varepsilon$.

Lemma 4 provides an equivalent formulation of the Bocce criterion; namely we can replace condition (*) by the condition

(**) if the subset $E$ of $A$ has half the measure of $A$, then $|m_E(f) - m_A(f)| < \varepsilon$.

We now attack the proof of Theorem 1. Our proof follows mainly the proof in [G1].

Proof of Theorem 1. Let all bounded subsets of $\mathcal{X}$ be midpoint Bocce dentable. Fix a bounded linear operator $T$ from $L_1$ into $\mathcal{X}$. It suffices to show that the subset $T^*(B(\mathcal{X}^*))$ of $L_1$ satisfies the Bocce criterion (this is a necessary and sufficient condition for $T$ to be Dunford-Pettis [G2]). To this end, fix $\varepsilon > 0$ and $B$ in $\Sigma^+$.

Consider the vector measure $F$ from $\Sigma$ into $\mathcal{X}$ given by $F(E) = T(\chi_E)$. For $x^* \in \mathcal{X}^*$

$$m_E(T^*x^*) = \frac{x^*F(E)}{\mu(E)}$$

since $\int_E (T^*x^*) \, d\mu = x^*T(\chi_E) = x^*F(E)$. 


Since the subset \( \{ \frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+ \} \) of \( X \) is bounded, it is midpoint Bocce deniable. Accordingly, there is a finite collection \( \mathcal{F} \) of subsets of \( B \) each in \( \Sigma^+ \) such that for each \( x^* \in B(x^*) \) there is a set \( A \) in \( \mathcal{F} \) such that if
\[
\frac{F(A)}{\mu(A)} = \frac{1}{2} \frac{F(E_1)}{\mu(E_1)} + \frac{1}{2} \frac{F(E_2)}{\mu(E_2)}
\]
for some subsets \( E_i \) of \( B \) with \( E_i \in \Sigma^+ \), then
\[
\left| \frac{x^* F(E_1)}{\mu(E_1)} - \frac{x^* F(A)}{\mu(A)} \right| = \left| \frac{x^* F(E_2)}{\mu(E_2)} - \frac{x^* F(A)}{\mu(A)} \right| < \varepsilon.
\]
Fix \( x^* \in B(\mathcal{X}^*) \) and find the associated \( A \) in \( \mathcal{F} \).

At this point we turn to our new formulation of the Bocce criterion (whereas [G1] used the old formulation and Lemma 2.9).

This \( A \in \mathcal{F} \) satisfies the condition (**). For consider a subset \( E \) of \( A \) with \( \mu(E) = \frac{1}{2} \mu(A) \). Since
\[
\frac{F(A)}{\mu(A)} = \frac{1}{2} \frac{F(E)}{\mu(E)} + \frac{1}{2} \frac{F(A \setminus E)}{\mu(A \setminus E)}
\]
we have that
\[
|m_E(T^*x^*) - m_A(T^*x^*)| = \left| \frac{x^* F(E)}{\mu(E)} - \frac{x^* F(A)}{\mu(A)} \right| < \varepsilon.
\]
Thus \( T^*(B(\mathcal{X}^*)) \) satisfies the Bocce criterion, as needed. \( \square \)

3. Closing comments. A relatively weakly compact subset of \( L_1 \) is relatively norm compact if and only if it satisfies the Bocce criterion [G2]. Thus our new formulation of the Bocce criterion provides another (perhaps at times more useful) method for testing for norm compactness in \( L_1 \).

Fix \( A \) in \( \Sigma^+ \) and \( f \) in \( L_1 \). Put
\[
M_A(f) = \sup \{|m_E(f) - m_A(f)| : E \subset A \text{ and } 2\mu(E) = \mu(A)\}.
\]
This supremum is obtained. For just normalize so that \( A = \Omega \) and \( \int_{\Omega} f \ d\mu = 0 \) and \( \int_{\Omega} |f| \ d\mu = 1 \). As Ralph Howard pointed out, next find disjoint subsets \( E_1 \) and \( E_2 \) of measure \( \frac{1}{2} \) and \( a \in \mathbb{R} \) such that
\[
E_1 \subset [f \leq a] \quad \text{and} \quad E_2 \subset [f \geq a].
\]
Then \( M_A(f) \) will be the larger of \( |m_{E_1}(f)| \) and \( |m_{E_2}(f)| \).

Basically, our Lemma 4 says that
\[
\frac{1}{2} \text{ Bocce-osc } f|_A \leq M_A(f) \leq \text{ Bocce-osc } f|_A.
\]
These bounds are the best possible.
For the second inequality, consider the function defined on \( A \equiv [0, 1] \) by
\[
f = \chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1]}.
\]
Straightforward calculations show that \( m_{[0, \frac{1}{2}]}(f) = 1 \) and that \( \text{Bocce-osc } f |_A = 1 \). Thus
\[
M_A(f) = \text{Bocce-osc } f |_A.
\]

As for the first inequality, consider the family of functions defined on \( A \equiv [0, 1] \) by
\[
f_\delta = \frac{\delta - 1}{\delta} \chi_{[0, \delta)} + \chi_{[\delta, 1]}
\]
for \( 0 < \delta < \frac{1}{2} \). Straightforward calculations show that
\[
M_A(f_\delta) = \frac{1}{2} \left( 1 - \frac{1}{\delta} \right) \text{Bocce-osc } f_\delta |_A.
\]
Actually \( M_A(f) = \frac{1}{2} \text{Bocce-osc } f |_A \) if and only if \( f \) is the zero function on \( A \).

References


Received April 24, 1992.

**University of South Carolina**
**Columbia, SC 29208**

**and**

**Miami University**
**Oxford, OH 45056**
The Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 Mathematics Subject Classification scheme which can be found in the December index volumes of Mathematical Reviews. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Julie Speckart, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $190.00 a year (10 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Copyright © 1993 by Pacific Journal of Mathematics
Strong integral summability and the Stone-Čech compactification of the half-line

JEFF CONNOR and MARY ANNE SWARDSON

The endlich Baer splitting property

THEODORE GERARD FATICONI

The formal group of the Jacobian of an algebraic curve

MARGARET N. FREIJE

Concordances of metrics of positive scalar curvature

Pawel Gajer

Explicit construction of certain split extensions of number fields and constructing cyclic classfields

STANLEY JOSEPH GURAK

Asymptotically free families of random unitaries in symmetric groups

ALEXANDRU MIHAI NICA

On purifiable subgroups and the intersection problem

TAKASHI OKUYAMA

On the incidence cycles of a curve: some geometric interpretations

LUCIANA RAMELLA

On some explicit formulas in the theory of Weil representation

R. RANGA RAO

An analytic family of uniformly bounded representations of a free product of discrete groups

JANUSZ WYSOCZAŃSKI

Errata: “Dentability, trees, and Dunford-Pettis operators on $L_1$”

MARIA GIRARDI and ZHIBAO HU

Errata: “Poincaré cobordism exact sequences and characterisation”

HIMADRI KUMAR MUKERJEE