A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS
OF OPERATORS

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The main objective of this paper is to develop a notion of joint spectrum for complex solvable Lie algebras of operators acting on a Banach space, which generalizes Taylor joint spectrum (T.J.S.) for several commuting operators.

I. Introduction. We briefly recall the definition of Taylor spectrum. Let \( \Lambda(\mathbb{C}^n) \) be the complex exterior algebra on \( n \) generators \( e_1, \ldots, e_n \), with multiplication denoted by \( \Lambda \). Let \( \mathcal{E} \) be a Banach space and \( a = (a_1, \ldots, a_n) \) be a mutually commuting \( n \)-tuple of bounded linear operators on \( E(\text{m.c.o.}) \). Define \( \bigwedge^n_k(E) = \bigwedge^n_k(\mathbb{C}^n) \otimes \mathcal{E} \), and for \( k \geq 1 \), \( D_{k-1} \) by:

\[
D_{k-1} : \bigwedge_1^n(E) \to \bigwedge_{h-1}^n(E)
\]

\[
D_{k-1}(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} x \cdot a_j \otimes \cdots \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}
\]

where \( \hat{\cdot} \) means deletion. Also define \( D_0 = 0 \) for \( k \leq 0 \).

It is easily seen that \( D_k D_{k+1} = 0 \) for all \( k \), that is, \( \{\bigwedge^n_k(E), D_k\}_{k \in \mathbb{Z}} \) is a chain complex, called the Koszul complex associated with \( a \) and \( E \) and denoted by \( R(E, a) \). The \( n \)-tuple \( a \) is said to be invertible or nonsingular on \( E \), if \( R(E, a) \) is exact, i.e., \( \ker D_k = \text{ran} E_{k+1} \) for all \( k \). The Taylor spectrum of \( a \) on \( E \) is \( \text{Sp}(a, E) = \{\lambda \in \mathbb{C}^n : a - \lambda \) is not invertible\}.

Unfortunately, this definition depends very strongly on \( a_1, \ldots, a_n \) and not on the vector subspace of \( L(E) \) generated by then \( = \langle a \rangle \).

As we consider Lie algebras, and then naturally involve geometry, we are interested in a geometrical approach to spectrum which depends on \( L \) rather than on a particular set of operators.

This is done in II. Given a solvable Lie subalgebra of \( L(E), L \), we associate to it a set in \( L^*, \text{Sp}(L, E) \).
This object has the classical properties. \( \text{Sp}(L, E) \) is compact. If \( L' \) is an ideal of \( L \), then \( \text{Sp}(L', E) \) is the projection of \( \text{Sp}(L, E) \) in \( L'^* \). \( \text{Sp}(L, E) \) is non-empty.

Besides, it satisfies other interesting properties.

If \( x \in L^2 \), then \( \text{Sp}(x) = 0 \). If \( L \) is nilpotent, one has the inclusion

\[ \text{Sp}(L, E) \subset \{ f \in [L, L]^+ \mid \forall x \in L, \ |f(x)| \leq \|x\| \} . \]

However the spectral mapping property is ill behaved.

II. The joint spectrum for solvable Lie algebras of operators. First of all, we establish a proposition which will be used in the definition of \( \text{Sp}(L, E) \).

From now on, \( L \) denotes a complex finite dimensional solvable Lie algebra, and \( U(L) \) its enveloping algebra.

Let \( f \) belong to \( L^* \) such that \( f([L, L]) = 0 \), i.e., \( f \) is a character of \( L \). Then \( f \) defines a one dimensional representation of \( L \) denoted by \( \mathbb{C}(f) \). Let \( \varepsilon(f) \) be the augmentation of \( U(L) \) defined by \( f \):

\[
\varepsilon(f): U(L) \to \mathbb{C}(f), \\
\varepsilon(f)(x) = f(x) \quad (x \in L).
\]

Let us consider the pair of spaces and maps \( V(L) = (U(L) \otimes \bigwedge L, \overline{d}_{p-1}) \),

where \( \overline{d}_{p-1} \) is the map defined by:

\[ \overline{d}_{p-1}: U(L) \otimes \bigwedge L \to U(L) \otimes \bigwedge L. \]

If \( p \geq 1 \)

\[
\overline{d}_{p-1}(x_{i_1} \cdots x_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1}(x_{i_k} - f(x_{i_k}))(x_{i_1} \cdots \hat{x}_{i_k} \cdots x_{i_p}) \\
+ \sum_{1 \leq k \leq l \leq p} (-1)^{k+l}<[x_{i_k}, x_{i_l}]x_{i_1} \cdots \hat{x}_{i_k} \cdots \hat{x}_{i_l} x_{i_p}>,
\]

where \( \hat{\cdot} \) means deletion. If \( p \leq 0 \), we also define \( \overline{d}_{p} = 0 \). Then

**Proposition 1.** The pair of spaces and maps \( V(L) \) is a chain complex. Furthermore, with the augmentation \( \varepsilon(f) \), the complex \( V(L) \) is a \( U(L) \)-free resolution of \( \mathbb{C}(f) \) as a left \( U(L) \) module.

We omit the proof of Proposition 1 because it is a straightforward generalization of Theorem 7.1 of [3, XIII, 7].

Let \( L \) be as usual, from now on, \( E \) denotes a Banach space on which \( L \) acts as right continuous operators, i.e., \( L \) is a Lie subalgebra
of $L(E)$ with the opposite product. Then, by [3, XIII, 1], $E$ is a right $U(L)$ module.

If $f$ is a character of $L$, by Proposition 1 and elementary homological algebra, the $q$-homology space of the complex, $(E \otimes \Lambda L, d(f))$ is $\text{Tor}^U_L(E, C(f)) = H_q(L, E \otimes C(f))$.

We now state our definition.

**Definition 1.** Let $L$ and $E$ be as above the set $\{ f \in L^*, f(L^2) = 0 \}$, is the spectrum of $L$ acting on $E$, and is denoted by $\text{Sp}(L, E)$.

By Proposition 1 and Definition 1, it is clear that, if $L$ is a commutative algebra $\text{Sp}(L, E)$ reduces to Taylor joint spectrum.

Let us see an example. Let $(E, \| \|_2)$ be $(\mathbb{C}^2, \| \|_2)$ and $a, b$ the operators

$$a = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$  

It is easily seen that $[b, a] = b$, and then, the vector space $\mathbb{C}(b) \oplus \mathbb{C}(a) = L$ is a solvable Lie subalgebra of $L(\mathbb{C}^2)$.

Using Definition 1, a standard calculation shows that $\text{Sp}(L, E) = \{ f \in (\mathbb{C}^2)^*|f(b) = 0; f(a) = \frac{1}{2}, f(a) = -\frac{3}{2}\}$.

Observe that, $\|a\| = \frac{1}{2}$; however, $\text{Sp}(L, E)$ is not contained in $\{ f \in (\mathbb{C}^2)^*|\forall x \in \mathbb{C}^2 |f(x)| \leq \|x\|\}$.

**III. Fundamental properties of the spectrum.** In this section, we shall see that the most important properties of spectral theory are satisfied by our spectrum.

**Theorem 2.** Let $L$ and $E$ be as usual. Then $\text{Sp}(L, E)$ is a compact set of $L^*$.

**Proof.** Let us consider the family of spaces and maps $(E \otimes \Lambda^i L, d_{i-1}(f)) f \in L^{2^+}$, where $L^{2^+} = \{ f \in L^*|f(L^2) = 0\}$. This family is a parameterized chain complex on $L^{2^+}$. By Taylor [6, 2.1] the set $\{ f \in L^{2^+}|(E \otimes \Lambda^i L, d_{i-1}(f)) \text{ is exact} \} = \text{Sp}(L, E)^c$ is an open set in $L^{2^+}$. Then, $\text{Sp}(L, E)$ is closed in $L^{2^+}$ and hence in $L^*$.

To verify that $\text{Sp}(L, E)$ is a compact set we consider a basis of $L^2$ and we extend it to a basis of $L$, $\{X_i\}_{1 \leq i \leq n}$. If $K = \dim L^2$ and $n = \dim L$ let $L_i$ be the ideal generated by $\{X_j\}_{1 \leq j \leq n, j \neq i}, i \geq K + 1$.

Let $f$ be a character of $L$ and represent it in the dual basis of $\{X_i\}_{1 \leq i \leq n}, \{f_i\}_{1 \leq i \leq n} f = \sum_{i=K+1}^n \xi_i f_i$. For each $i$, there is a positive
number \( r_i \) such that if \( \xi_i \geq r_i \),

\[
\text{Tor}_p^{U(L)}(E, C(f)) = H_p \left( E \otimes \bigwedge^i L, d_{i-1}(f) \right) = 0 \quad \forall p.
\]

To prove our last statement, we shall construct an homotopy operator for the chain complex \((E \otimes \bigwedge^p L, d_{p-1}(f))\) \((f(L^2) = 0)\).

First of all we observe that

\[
E \otimes \bigwedge^p L = \left( E \otimes \bigwedge^p L_i \right) \oplus \left( E \otimes \bigwedge^{p-1} L_i \right) \bigwedge \langle X_i \rangle.
\]

As \( L_i \) is an ideal of \( L \), \( d_{p-1}(E \otimes \bigwedge^p L_i) \subseteq E \otimes \bigwedge^{p-1} L_i \). On the other hand, there is a bounded operator \( L_{p-1} \) such that

\[
d_{p-1}(f)(a \wedge \langle X_i \rangle)
= (d_{p-1}(f)a) \wedge \langle X_i \rangle + (-1)^p L_{p-1}a \quad \left( a \in E \otimes \bigwedge^{p-1} L_i \right).
\]

It is easy to see that, for each \( p \), there is a basis of \( \bigwedge^p L_i, \{V_j^p\} \) \(1 \leq j \leq \dim \bigwedge^p L_i\), such that if we decompose

\[
E \otimes \bigwedge^p L_i = \bigoplus_{1 \leq j \leq \dim \bigwedge^p L_i} E \langle V_j \rangle,
\]

then \( L_p \) has the following form

\[
L_p'' = \left\{ \begin{array}{ll}
\alpha_{i,j}^p & \text{if } i < j, \\
X_i - \xi_i + \alpha_{i,j}^p & \text{if } i = j, \\
0 & \text{if } i > j \text{ where } \alpha_{i,j} \in \mathbb{C}.
\end{array} \right.
\]

Besides, let \( K_p \) be a positive real number such that

\[
\bigcup_{1 \leq j \leq \dim \bigwedge^p L_i} \text{Sp}(X_i + \alpha_{i,j}^p) \subseteq B[0, K_p]
\]

and \( N_i = \max_{0 \leq p \leq n-1} \{K_p\} \). Then, as \( L_p \) has a triangular form, a standard calculation shows that \( L_p \) is a topological isomorphism of Banach spaces if \( \xi_i \geq N_i \).
Outside $B[0, N_i]$ we construct our homotopy operator

$$\text{Sp}: E \otimes \bigwedge^{p} L \to E \otimes \bigwedge^{p+1} L,$$

$$\text{Sp}: E \otimes \bigwedge^{p-1} L_i \wedge \langle X_i \rangle = 0,$$

$$\text{Sp}: E \otimes \bigwedge^{p} L_i \to E \otimes \bigwedge^{p} L_i \wedge \langle X_i \rangle$$

$$\text{Sp} = (-1)^{p+1} L^{-1} \wedge \langle X_i \rangle.$$ 

From the definition of $L_p$, we have the following identity:

$$(-1)^{p+2} S_{p-1} d_{p-1}(f) L_p = d_{p-1}(f) \wedge \langle X_i \rangle.$$ 

The above identity and a standard calculation shows that $\text{Sp}$ in an homotopy operator, i.e., $d_p \text{Sp} + S_p d_{p-1} = I$ and then $\text{Sp}(L, E)$ is a compact set.

**Theorem 3 (Projection property).** Let $L$ and $E$ be as usual, and $I$ an ideal of $L$. Let $\pi$ be the projection map from $L^*$ onto $I^*$, then

$$\text{Sp}(I, E) = \pi(\text{Sp}(L, E)).$$

**Proof.** By [2, 5, 3], there is a Jordan Hölder sequence of $L$ such that $I$ is one of its terms. Then, by means of an induction argument, we can assume $\dim(L/I) = 1$.

Let us consider the connected simply connected complex Lie group $G(L)$ such that its Lie algebra is $L$ [5, LG, V].

Let $\text{Ad}^*$ be the coadjoint representation of $G(L)$ in $L^*$: $\text{Ad}^*(g)f = f \text{Ad}(g^{-1})$, where $g \in G(L)$, $f \in L^*$ and $\text{Ad}$ is the adjoint representation of $G(L)$ in $L$.

Let $f$ belong to $\text{Sp}(I, E)$. Then, as $I$ is an ideal of $L$, by [7, 2.13.4], $\text{Ad}^*(g)f$ belongs to $I^*$; besides, it is a character of $I$. Then, one can restrict the coadjoint action of $G(L)$ to $I^*$. Moreover, $\text{Sp}(I, E)$ is invariant under the coadjoint action of $G(L)$ in $I^*$, i.e.: if $f \in \text{Sp}(I, E)$, $\text{Ad}^*(g)f \in \text{Sp}(I, E)$, $\forall g \in G(L)$.

In order to prove this fact, it is enough to see:

$$(1) \quad \text{Tor}^*(I)(E, C(f)) \cong \text{Tor}^*_*(I)(E, C(h))$$

where $h = \text{Ad}^*(g)f$, $g \in G(L)$.

Let $\Gamma$ be the ring $U(I)$ and $\varphi$ the ring morphism

$$\varphi = U(\text{Ad} g): U(I) \to U(I).$$
Let us consider the augmentation modules \((C(f), E(f))\) and \((C(h), E(h))\).

Then, a standard calculation shows that the hypothesis of [3, VIII, 3.1] are satisfied, which implies (I).

Thus, if \(f \in \text{Sp}(I, E)\), the orbit \(G(L) \cdot f \subseteq \text{Sp}(I, E)\). However, \(\text{Sp}(I, E)\) is a compact set of \(I^*\).

As the only bounded orbits for an action of a complex connected Lie group on a vector space are points; \(G(L) \cdot f = f\).

Let \(\overline{f}\) be an extension of \(f\) to \(L^*\), and consider \(\alpha = G(L) \cdot \overline{f}\), the orbit of \(\overline{f}\) under the coadjoint action of \(G(L)\) in \(L^*\).

As \(G(L) \cdot f = f\), as an analytic manifold

\[(II) \quad \dim \alpha \leq 1.\]

Now suppose \(\overline{f}\) is not a character of \(L\): i.e., \(\overline{f}(L^2) \neq 0\).

Let \(L^\perp\) be the following set: \(L^\perp = \{x \in L | \overline{f}([X, L]) = 0\}\), and let \(n\) be the dimension of \(L\).

As \(I\) is an ideal of dimension \(n - 1\), \(f(I^2) = 0\) and \(f(L^2) \neq 0\), by [2, 5, 3], [1, IV, 4.1] and [4, 1, 1.2.8], we have: \(L^\perp \subseteq I\), and \(\dim L^\perp = n - 2\).

Let us consider the analytic subgroup of \(G(L)\) such that its Lie algebra is \(L^\perp\).

As the Lie algebra of the subgroup \(G(L)_{\overline{f}} = \{g \in GL | \text{Ad}^*(g)\overline{f} = \overline{f}\}\) is \(L^\perp\), the connected component of the identity of \(G(L)_{\overline{f}}\) is \(G(L^\perp)\).

However, by [7, 2.9.1, 2.9.7] \(\alpha = G(L) \cdot \overline{f}\) satisfies the following properties: \(\alpha \cong G(L)/G(L)_{\overline{f}}\), and \(\dim \alpha = \dim G(L) - \dim G(L)_{\overline{f}} = \dim G(L) - \dim G(L^\perp) = \dim L - \dim L^\perp = 2\), which contradicts (II).

Then \(\overline{f}\) is a character of \(L\).

Thus, any extension \(\overline{f}\) of an \(f\) in \(\text{Sp}(I, E)\) is a character of \(L\). However, as in [6], there is a short exact sequence of complexes

\[0 \to \left( \bigwedge^* I \otimes E, d(f) \right) \to \left( \bigwedge^* L \otimes E, d(\overline{f}) \right) \to \left( \bigwedge^* I \otimes E, d(f) \right) \to 0.\]

As \(U(I)\) is a subring with unit of \(U(L)\) and the complex involved in Definition 1 differs from the one of [6] by a constant term, Taylor's argument of [6, 13, 3.1] still applies and then \(\text{Sp}(I, E) = \Pi(\text{Sp}(L, E))\).

As a consequence of Theorem 3 we have
**Theorem 4.** Let $L$ and $E$ be as usual. Then $\text{Sp}(L, E)$ is non-void.

**IV. Some consequences.** In this section we shall see some consequences of the main theorems.

Let $E$ be a Banach space and $L$ a complex finite dimensional solvable Lie algebra acting on $E$ as bounded operators.

One of the well known properties of Taylor spectrum for an $n$-tuple of m.c.o. acting on $E$ is $\text{Sp}(a, E) \subseteq \prod_{i=1}^{n} \mathbb{B}[0, \|a_i\|]$. In the noncommutative case, as we have seen in §II, this property fails.

However, if the Lie algebra is nilpotent, it is still true.

**Proposition 5.** Let $L$ be a nilpotent Lie algebra which acts as bounded operators on a Banach space $E$.

Then, $\text{Sp}(L, E) \subset \{ f \in L^* \mid |f(x)| \leq \|x\|, \ x \in L \}$.

**Proof.** We proceed by induction on $\dim L$. If $\dim L = 1$, we have nothing to verify.

We suppose true the proposition for every nilpotent Lie algebra $L'$ such that $\dim L' < n$.

If $\dim L = n$, by [2, 4, 1], there is a Jordan Hölder series $S = (L_i)_{0 \leq i \leq n}$, such that $[L, L_i] \subseteq L_{i-1}$.

Let $\{X_i\}_{1 \leq i \leq n}$ be a basis of $L$ such that $\{X_i\}_{1 \leq i \leq i}$ generates $L_i$.

Let $L'_{n-1}$ be the vector subspace generated by $\{X_i\}_{1 \leq i \leq n}$. As $[L, L'_{n-1}] \subseteq L_{n-2} \subset L'_{n-1}$, $L'_{n-1}$ is an ideal. Besides, $L_{n-1} + L'_{n-1} = L$.

Then, by means of Theorem 4 and the inductive hypothesis, we complete the inductive argument and the proposition.

Now, we deal with some consequences of the projection property.

**Proposition 6.** Let $L$ and $E$ be as usual.

If $I$ is an ideal contained in $L^2$, then $\text{Sp}(I, E) = 0$. In particular $\text{Sp}(L^2, E) = 0$.

**Proof.** By the projection property, $\text{Sp}(I, E) = \Pi(\text{Sp}(L, E))$, where $\Pi$ is the projection from $L^*$ on $I^*$. However, as $\text{Sp}(L, E)$ is a subset of characters of $L$, $f|_I = 0$, if $I \subseteq L^2$.

**Proposition 7.** Let $L$ and $E$ be as in Proposition 5.

If $\text{Sp}(L, E) = 0$, then $\text{Sp}(x) = 0$ $\forall x \in L$.

**Proof.** By means of an induction argument, the ideals $L_{n-1}, L'_{n-1}$ of Proposition 5 and Theorem 3, we conclude the proof.
PROPOSITION 8. Let $L$ and $E$ be as usual. Then, if $x \in L^2$: $\text{Sp}(x) = 0$.

Proof. First of all, recall that if $L$ is a solvable Lie algebra, $L^2$ is a nilpotent one. Then by Proposition 6 $\text{Sp}(L^2, E) = 0$, and by Proposition 7 $\text{Sp}(x) = 0$ \( \forall x \in L^2 \).

V. Remark about the spectral mapping theorem. Note that the example of §II shows that the projection property fails for subspaces which are not ideals (take $I = \langle x \rangle$). Clearly this implies that the spectral mapping theorem also fails in the noncommutative case.

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