A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

E. Boasso and Angel Rafael Larotonda
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OF OPERATORS

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The main objective of this paper is to develop a notion of joint spectrum for complex solvable Lie algebras of operators acting on a Banach space, which generalizes Taylor joint spectrum (T.J.S.) for several commuting operators.

I. Introduction. We briefly recall the definition of Taylor spectrum. Let $\bigwedge(C^n)$ be the complex exterior algebra on $n$ generators $e_1, \ldots, e_n$, with multiplication denoted by $\wedge$. Let $E$ be a Banach space and $a = (a_1, \ldots, a_n)$ be a mutually commuting $n$-tuple of bounded linear operators on $E_{(m,c,o.)}$. Define $\bigwedge_k(E) = \bigwedge_k(C^n) \otimes_C E$, and for $k \geq 1$, $D_{k-1}$ by:

$$D_{k-1} : \bigwedge_k(E) \to \bigwedge_{k-1}(E)$$

$$D_{k-1}(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_k})$$

$$= \sum_{j=1}^{k} (-1)^{j+1} x \cdot a_{i_j} \otimes \cdots \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$$

where $\hat{}$ means deletion. Also define $D_k = 0$ for $k \leq 0$.

It is easily seen that $D_kD_{k+1} = 0$ for all $k$, that is, $\{\bigwedge_k(E), D_k\}_{k \in \mathbb{Z}}$ is a chain complex, called the Koszul complex associated with $a$ and $E$ and denoted by $R(E, a)$. The $n$-tuple $a$ is said to be invertible or nonsingular on $E$, if $R(E, a)$ is exact, i.e., $\text{Ker} D_k = \text{ran} E_{k+1}$ for all $k$. The Taylor spectrum of $a$ on $E$ is $\text{Sp}(a, E) = \{\lambda \in \mathbb{C}^n : a - \lambda$ is not invertible}.

Unfortunately, this definition depends very strongly on $a_1, \ldots, a_n$ and not on the vector subspace of $L(E)$ generated by then ($= \langle a \rangle$).

As we consider Lie algebras, and then naturally involve geometry, we are interested in a geometrical approach to spectrum which depends on $L$ rather than on a particular set of operators.

This is done in II. Given a solvable Lie subalgebra of $L(E)$, $L$, we associate to it a set in $L^*$, $\text{Sp}(L, E)$.
This object has the classical properties. \( \text{Sp}(L, E) \) is compact. If \( L' \) is an ideal of \( L \), then \( \text{Sp}(L', E) \) is the projection of \( \text{Sp}(L, E) \) in \( L'^* \). \( \text{Sp}(L, E) \) is non-empty.

Besides, it satisfies other interesting properties.

If \( x \in L^2 \), then \( \text{Sp}(x) = 0 \). If \( L \) is nilpotent, one has the inclusion

\[
\text{Sp}(L, E) \subset \{ f \in [L, L]^1 | \forall x \in L, |f(x)| \leq \|x\| \}.
\]

However the spectral mapping property is ill behaved.

II. The joint spectrum for solvable Lie algebras of operators. First of all, we establish a proposition which will be used in the definition of \( \text{Sp}(L, E) \).

From now on, \( L \) denotes a complex finite dimensional solvable Lie algebra, and \( U(L) \) its enveloping algebra.

Let \( f \) belong to \( L^* \) such that \( f([L, L]) = 0 \), i.e., \( f \) is a character of \( L \). Then \( f \) defines a one dimensional representation of \( L \) denoted by \( C(f) \). Let \( \varepsilon(f) \) be the augmentation of \( U(L) \) defined by \( f \):

\[
\varepsilon(f): U(L) \rightarrow C(f),
\varepsilon(f)(x) = f(x) \quad (x \in L).
\]

Let us consider the pair of spaces and maps \( V(L) = (U(L) \otimes \Lambda L, \overline{d}_{p-1}) \), where \( \overline{d}_{p-1} \) is the map defined by:

\[
\overline{d}_{p-1}: U(L) \otimes \Lambda^p L \rightarrow U(L) \otimes \Lambda^{p-1} L.
\]

If \( p \geq 1 \)

\[
\overline{d}_{p-1}(x_{i_1} \cdots x_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} (x_{i_k} - f(x_{i_k}))x_{i_1}x_{i_2}x_{i_3} \cdots \hat{x}_{i_k} \cdots x_{i_p}
\]

\[
+ \sum_{1 \leq k < l \leq p} (-1)^{k+l} \langle [x_{i_k}, x_{i_l}]x_{i_1}x_{i_2}x_{i_3} \cdots \hat{x}_{i_k} \hat{x}_{i_l}x_{i_p} \rangle
\]

where \( \hat{\cdot} \) means deletion. If \( p \leq 0 \), we also define \( \overline{d}_p = 0 \). Then

**Proposition 1.** The pair of spaces and maps \( V(L) \) is a chain complex. Furthermore, with the augmentation \( \varepsilon(f) \), the complex \( V(L) \) is a \( U(L) \)-free resolution of \( C(f) \) as a left \( U(L) \) module.

We omit the proof of Proposition 1 because it is a straightforward generalization of Theorem 7.1 of [3, XIII, 7].

Let \( L \) be as usual, from now on, \( E \) denotes a Banach space on which \( L \) acts as right continuous operators, i.e., \( L \) is a Lie subalgebra
of $L(E)$ with the opposite product. Then, by [3, XIII, 1], $E$ is a right $U(L)$ module.

If $f$ is a character of $L$, by Proposition 1 and elementary homological algebra, the $q$-homology space of the complex, $(E \otimes \wedge \ell, d(f))$ is $\text{Tot}^q_U(E, \mathcal{C}(f)) = H_q(L, E \otimes \mathcal{C}(f))$.

We now state our definition.

**Definition 1.** Let $L$ and $E$ be as above the set \{\(f \in L^*, f(L^2) = 0|H_*((L, E \otimes \mathcal{C}(f))\) is non-zero\}, is the spectrum of $L$ acting on $E$, and is denoted by $\text{Sp}(L, E)$.

By Proposition 1 and Definition 1, it is clear that, if $L$ is a commutative algebra $\text{Sp}(L, E)$ reduces to Taylor joint spectrum.

Let us see an example. Let $(E, \|\|)$ be $(C^2, \|\|_2)$ and $a, b$ the operators

\[
a = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

It is easily seen that $[b, a] = b$, and then, the vector space $\mathcal{C}(b) \oplus \mathcal{C}(a) = L$ is a solvable Lie subalgebra of $L(C^2)$.

Using Definition 1, a standard calculation shows that $\text{Sp}(L, E) = \{f \in (C^2)^*|f(b) = 0; f(a) = \frac{1}{2}, f(a) = -\frac{3}{2}\}$.

Observe that, $\|a\| = \frac{1}{2}$; however, $\text{Sp}(L, E)$ is not contained in $\{f \in (C^2)^*|\forall x \in C^2 |f(x)| \leq \|x\|\}$.

**III. Fundamental properties of the spectrum.** In this section, we shall see that the most important properties of spectral theory are satisfied by our spectrum.

**Theorem 2.** Let $L$ and $E$ be as usual. Then $\text{Sp}(L, E)$ is a compact set of $L^*$.

**Proof.** Let us consider the family of spaces and maps $(E \otimes \wedge^i L, d_{i-1}(f)) f \in L^{2\pm}$, where $L^{2\pm} = \{f \in L^*|f(L^2) = 0\}$. This family is a parameterized chain complex on $L^{2\pm}$. By Taylor [6, 2.1] the set $\{f \in L^{2\pm}|(E \otimes \wedge^i L, d_{i-1}(f))$ is exact} = $\text{Sp}(L, E)^c$ is an open set in $L^{2\pm}$. Then, $\text{Sp}(L, E)$ is closed in $L^{2\pm}$ and hence in $L^*$.

To verify that $\text{Sp}(L, E)$ is a compact set we consider a basis of $L^2$ and we extend it to a basis of $L$, $\{X_i\}_{1 \leq i \leq n}$. If $K = \text{dim} L^2$ and $n = \text{dim} L$ let $L_i$ be the ideal generated by $\{X_j\}_{1 \leq j \leq n, j \neq i}, i \geq K + 1$.

Let $f$ be a character of $L$ and represent it in the dual basis of $\{X_i\}_{1 \leq i \leq n}$, $\{f_i\}_{1 \leq i \leq n}$ $f = \sum_{i=K+1}^n \xi_i f_i$. For each $i$, there is a positive
number $r_i$ such that if $\xi_i \geq r_i$,

$$\text{Tor}^U_p(E, C(f)) = H_p \left( E \otimes \bigwedge^i L, d_{i-1}(f) \right) = 0 \quad \forall \rho .$$

To prove our last statement, we shall construct an homotopy operator for the chain complex $(E \otimes \bigwedge^p L, d_{p-1}(f))$ ($f(L^2) = 0$).

First of all we observe that

$$E \otimes \bigwedge^p L = \left( E \otimes \bigwedge^p L_i \right) \oplus \left( E \otimes \bigwedge^{p-1} L_i \right) \bigwedge \langle X_i \rangle .$$

As $L_i$ is an ideal of $L$, $d_{p-1}(E \otimes \bigwedge^p L_i) \subseteq E \otimes \bigwedge^{p-1} L_i$. On the other hand, there is a bounded operator $L_{p-1}$ such that

$$d_{p-1}(f)(a \wedge \langle X_i \rangle) = (d_{p-1}(f)a) \wedge \langle X_i \rangle + (-1)^p L_{p-1}a \quad \left(a \in E \otimes \bigwedge^{p-1} L_i \right).$$

It is easy to see that, for each $p$, there is a basis of $\bigwedge^p L_i$, $\{ V_j \}$ $1 \leq j \leq \dim \bigwedge^p L_i$, such that if we decompose

$$E \otimes \bigwedge^p L_i = \bigoplus_{1 \leq j \leq \dim \bigwedge^p L_i} E\langle V_j \rangle ,$$

then $L_p$ has the following form

$$L_{p_{ij}} = \begin{cases} 
\alpha_{ij}^p & i < j, \\
X_i - \xi_i + \alpha_{jj}^p & i = j, \\
0 & i > j \quad \text{where } \alpha_{ij} \in \mathbb{C}.
\end{cases}$$

Besides, let $K_p$ be a positive real number such that

$$\bigcup_{1 \leq j \leq \dim \bigwedge^p L_i} \text{Sp}(X_i + \alpha_{jj}^p) \subseteq B[0, K_p]$$

and $N_i = \max_{0 \leq p \leq n-1} \{ K_p \}$. Then, as $L_p$ has a triangular form, a standard calculation shows that $L_p$ is a topological isomorphism of Banach spaces if $\xi_i \geq N_i$. 
Outside $B[0, N_i]$ we construct our homotopy operator

$$\text{Sp}: E \otimes \bigwedge^{p} L \rightarrow E \otimes \bigwedge^{p+1} L,$$

$$\text{Sp}: E \otimes \bigwedge^{p-1} L_i \wedge \langle X_i \rangle = 0,$$

$$\text{Sp}: E \otimes \bigwedge^{p} L_i \rightarrow E \otimes \bigwedge^{p} L_i \wedge \langle X_i \rangle$$

$$\text{Sp} = (-1)^{p+1} L^{-1}_p \wedge \langle X_i \rangle.$$

From the definition of $L_p$, we have the following identity:

$$(-1)^{p+2} \text{Sp}_{p-1} d_{p-1}(f)L_p = d_{p-1}(f) \wedge \langle X_i \rangle.$$  

The above identity and a standard calculation shows that $\text{Sp}$ in an homotopy operator, i.e., $d_p \text{Sp} + \text{Sp}_{p-1} d_{p-1} = I$ and then $\text{Sp}(L, E)$ is a compact set.

**Theorem 3 (Projection property).** Let $L$ and $E$ be as usual, and $I$ an ideal of $L$. Let $\pi$ be the projection map from $L^*$ onto $I^*$, then

$$\text{Sp}(I, E) = \pi(\text{Sp}(L, E)).$$

**Proof.** By [2, 5, 3], there is a Jordan Hölder sequence of $L$ such that $I$ is one of its terms. Then, by means of an induction argument, we can assume $\dim(L/I) = 1$.

Let us consider the connected simply connected complex Lie group $G(L)$ such that its Lie algebra is $L$ [5, LG, V].

Let $\text{Ad}^*$ be the coadjoint representation of $G(L)$ in $L^*$: $\text{Ad}^*(g)f = f \text{Ad}(g^{-1})$, where $g \in G(L), f \in L^*$ and $\text{Ad}$ is the adjoint representation of $G(L)$ in $L$.

Let $f$ belong to $\text{Sp}(I, E)$. Then, as $I$ is an ideal of $L$, by [7, 2.13.4], $\text{Ad}^*(g)f$ belongs to $I^*$; besides, it is a character of $I$. Then, one can restrict the coadjoint action of $G(L)$ to $I^*$. Moreover, $\text{Sp}(I, E)$ is invariant under the coadjoint action of $G(L)$ in $I^*$, i.e.: if $f \in \text{Sp}(I, E), \text{Ad}^*(g)f \in \text{Sp}(I, E) \forall g \in G(L)$.

In order to prove this fact, it is enough to see:

$$(I) \quad \text{Tor}^*_{U(I)}(E, C(f)) \cong \text{Tor}^*_{U(I)}(E, C(h))$$

where $h = \text{Ad}^*(g)f, g \in G(L)$.

Let $\Gamma$ be the ring $U(I)$ and $\phi$ the ring morphism

$$\phi = U(\text{Ad} g): U(I) \rightarrow U(I).$$
Let us consider the augmentation modules \((C(f), E(f))\) and \((C(h), E(h))\).

Then, a standard calculation shows that the hypothesis of [3, VIII, 3.1] are satisfied, which implies (I).

Thus, if \(f \in \text{Sp}(I, E)\), the orbit \(G(L) \cdot f \subseteq \text{Sp}(I, E)\). However, \(\text{Sp}(I, E)\) is a compact set of \(I^*\).

As the only bounded orbits for an action of a complex connected Lie group on a vector space are points; \(G(L) \cdot f = f\).

Let \(\overline{f}\) be an extension of \(f\) to \(L^*\), and consider \(\alpha = G(L) \cdot \overline{f}\), the orbit of \(\overline{f}\) under the coadjoint action of \(G(L)\) in \(L^*\).

As \(G(L) \cdot f = f\), as an analytic manifold

\[
\dim \alpha \leq 1.
\]

Now suppose \(\overline{f}\) is not a character of \(L\): i.e., \(\overline{f}(L^2) \neq 0\).

Let \(L^\perp\) be the following set: \(L^\perp = \{ x \in L | \overline{f}([X, L]) = 0 \}\), and let \(n\) be the dimension of \(L\).

As \(I\) is an ideal of dimension \(n - 1\), \(f(I^2) = 0\) and \(f(L^2) \neq 0\), by [2, 5, 3], [1, IV, 4.1] and [4, 1, 1.2.8], we have: \(L^\perp \subseteq I\), and \(\dim L^\perp = n - 2\).

Let us consider the analytic subgroup of \(G(L)\) such that its Lie algebra is \(L^\perp\).

As the Lie algebra of the subgroup \(G(L)_\overline{f} = \{ g \in GL | \text{Ad}^*(g) \overline{f} = \overline{f} \}\) is \(L^\perp\), the connected component of the identity of \(G(L)_\overline{f}\) is \(G(L^\perp)\).

However, by [7, 2.9.1, 2.9.7] \(\alpha = G(L) \cdot \overline{f}\) satisfies the following properties: \(\alpha \cong G(L)/G(L)_\overline{f}\), and \(\dim \alpha = \dim G(L) - \dim G(L)_\overline{f} = \dim G(L) - \dim(G(L^\perp)) = \dim L - \dim L^\perp = 2\), which contradicts (II).

Then \(\overline{f}\) is a character of \(L\).

Thus, any extension \(\overline{f}\) of an \(f\) in \(\text{Sp}(I, E)\) is a character of \(L\).

However, as in [6], there is a short exact sequence of complexes

\[
0 \to \left( \bigwedge^* I \otimes E, d(f) \right) \to \left( \bigwedge^* L \otimes E, d(\overline{f}) \right) \to \left( \bigwedge^* I \otimes E, d(f) \right) \to 0.
\]

As \(U(I)\) is a subring with unit of \(U(L)\) and the complex involved in Definition 1 differs from the one of [6] by a constant term, Taylor's argument of [6, 13, 3.1] still applies and then \(\text{Sp}(I, E) = \Pi(\text{Sp}(L, E))\).

As a consequence of Theorem 3 we have
THEOREM 4. Let $L$ and $E$ be as usual. Then $\text{Sp}(L, E)$ is non-void.

IV. Some consequences. In this section we shall see some consequences of the main theorems.

Let $E$ be a Banach space and $L$ a complex finite dimensional solvable Lie algebra acting on $E$ as bounded operators.

One of the well known properties of Taylor spectrum for an $n$-tuple of m.c.o. acting on $E$ is $\text{Sp}(a, E) \subseteq \Pi B[0, \|a_i\|]$. In the noncommutative case, as we have seen in §II, this property fails.

However, if the Lie algebra is nilpotent, it is still true.

PROPOSITION 5. Let $L$ be a nilpotent Lie algebra which acts as bounded operators on a Banach space $E$.

Then, $\text{Sp}(L, E) \subseteq \{f \in L^* | |f(x)| \leq \|x\|, \ x \in L\}$.

Proof. We proceed by induction on $\dim L$. If $\dim L = 1$, we have nothing to verify.

We suppose true the proposition for every nilpotent Lie algebra $L'$ such that $\dim L' < n$.

If $\dim L = n$, by [2, 4, 1], there is a Jordan Hölder series $S = (L_i)_{0 \leq i \leq n}$, such that $[L, L_i] \subseteq L_{i-1}$.

Let $\{X_i\}_{1 \leq i \leq n}$ be a basis of $L$ such that $\{X_j\}_{1 \leq j \leq i}$ generates $L_i$.

Let $L'_{n-1}$ be the vector subspace generated by $\{X_i\}_{1 \leq i \leq n}$. As $[L, L'_{n-1}] \subseteq L_{n-2} \subset L'_{n-1}$, $L'_{n-1}$ is an ideal. Besides, $L_{n-1} + L'_{n-1} = L$.

Then, by means of Theorem 4 and the inductive hypothesis, we complete the inductive argument and the proposition.

Now, we deal with some consequences of the projection property.

PROPOSITION 6. Let $L$ and $E$ be as usual.

If $I$ is an ideal contained in $L^2$, then $\text{Sp}(I, E) = 0$. In particular $\text{Sp}(L^2, E) = 0$.

Proof. By the projection property, $\text{Sp}(I, E) = \Pi(\text{Sp}(L, E))$, where $\Pi$ is the projection from $L^*$ on $I^*$. However, as $\text{Sp}(L, E)$ is a subset of characters of $L$, $f|_I = 0$, if $I \subseteq L^2$.

PROPOSITION 7. Let $L$ and $E$ be as in Proposition 5.

If $\text{Sp}(L, E) = 0$, then $\text{Sp}(x) = 0 \ \forall x \in L$.

Proof. By means of an induction argument, the ideals $L_{n-1}$, $L'_{n-1}$ of Proposition 5 and Theorem 3, we conclude the proof.
PROPOSITION 8. Let $L$ and $E$ be as usual. Then, if $x \in L^2$: $\text{Sp}(x) = 0$.

Proof. First of all, recall that if $L$ is a solvable Lie algebra, $L^2$ is a nilpotent one. Then by Proposition 6 $\text{Sp}(L^2, E) = 0$, and by Proposition 7 $\text{Sp}(x) = 0 \ \forall x \in L^2$.

V. Remark about the spectral mapping theorem. Note that the example of §II shows that the projection property fails for subspaces which are not ideals (take $I = \langle x \rangle$). Clearly this implies that the spectral mapping theorem also fails in the noncommutative case.

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Received September 17, 1990 and in revised form January 22, 1992.

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PACIFIC JOURNAL OF MATHEMATICS
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