SZEGŐ MAPS AND HIGHEST WEIGHT REPRESENTATIONS

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Let $G$ be a connected noncompact simple Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. Assume the space $G/K$ is Hermitian symmetric. We associate to each irreducible representation $\tau$ of $K$ a principal series representation $W(\tau)$ and a $G$-equivariant Szegö-type integral operator $S_\tau$ such that $S_\tau$ maps the $K$-finite vectors in $W(\tau)$ onto an irreducible highest weight $g$-module $L(\tau)$. Of primary concern here are those representations $\tau$ which are reduction points. For such $\tau$, we construct certain systems $D_\tau$ of $G$-equivariant differential operators and then utilize $D_\tau$ to establish the infinitesimal irreducibility of the image of $S_\tau$.

1. Introduction. Let $G$ be a connected noncompact simple Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. Assume the space $G/K$ is Hermitian symmetric. The main purpose of this article is to realize each irreducible highest weight representation of $G$ as the image of a $G$-equivariant quotient map defined on principal series representations. To make this more precise, recall that each irreducible highest weight representation $\pi_\tau$ of $G$ is parametrized by an irreducible unitary representation $\tau$ of $K$. Let $C^\infty(G, \tau)$ denote the space of $\tau$-covariant $C^\infty$-functions on $G$. We associate to $\tau$ a principal series representation $W(\tau)$ and a Szegö map $S_\tau: W(\tau) \to C^\infty(G, \tau)$ having the property that the $K$-finite vectors in $W(\tau)$ are mapped onto an irreducible $g$-module equivalent to the derived action of $\pi_\tau$. In the case of discrete series and limits thereof, this type of result was proved by Knapp and Wallach [16] in the general setting where $G$ is a semisimple equirank Lie group with finite center. The main result here is that the irreducibility of the image of $S_\tau$ persists for all highest weight representations. Moreover, for certain $\tau$ called reduction points, the irreducibility of $\text{Image}(S_\tau)$ is proved by showing this space is annihilated by a system $D_\tau$ of $G$-equivariant differential operators. The system $D_\tau$ somewhat parallels the role of the Schmid operator in the Knapp and Wallach result.

The realization of distinguished representations as irreducible images of quotient maps is a recurring theme in the literature which
appears in somewhat different contexts. For example, Okamoto [19] realizes both discrete series representations and limits thereof in this way when \((G, K)\) is a Hermitian symmetric pair. By inducing from a maximal parabolic, Inoue [13] identifies generalized limits of discrete series representations as Hardy-type spaces and views the Szegő mapping as projection onto the irreducible Hardy space. For semisimple equirank \(G\), both Knapp and Wallach [16] and Blank [1] realize discrete series representations as the image of such quotient mappings. Limits of complementary series are obtained by Gilbert, Kunze, Stanton and Tomas [9] in an analogous way. We refer also to [5], [6] and [8].

The main feature contrasting our results and the results cited above is that we work in the Hermitian symmetric setting with the set of all highest weight representations of \(G\). In particular, unitarity is not assumed. Our results are most significant in the case of exceptional highest weight representations. By this we mean the representations \(\pi_\tau\) such that \(\tau\) lies in the set \(\hat{K}_r\) of reduction points. To explain the notion of reduction point, we briefly recall a standard realization of the highest weight representation space (cf. §5). Define left translation on the space \(C^\infty(G, \tau; \mathfrak{p}_-)^{-}\) of functions on \(G\) which are smooth, vector-valued, annihilated by left invariant vector fields \(r(X), X \in \mathfrak{p}_-\), and which satisfy a transformation property by \(\tau\). Then \(\tau \in \hat{K}_r\) if and only if the subspace \(C^\infty(G, \tau; \mathfrak{p}_-)\) of \(K\)-finite vectors forms a reducible \(g\)-module.

This characterization of \(\hat{K}_r\) reveals why it is natural to focus on those highest weight representations parametrized by \(\tau\) in \(\hat{K}_r\). For given an irreducible unitary representation \(\tau\) of \(K\), Theorem (6.1) [16] gives appropriate inducing parameters \((\sigma, \nu)\) so that the corresponding \(S_\tau\) maps the nonunitary principal series \(W(\sigma, \nu)\) \(G\)-equivariantly and nontrivially into \(C^\infty(G, \tau; \mathfrak{p}_-)\). If \(\tau \notin \hat{K}_r\), then \(C^\infty(G, \tau; \mathfrak{p}_-)\) is irreducible and thus the image of the \(K\)-finite vectors in \(W(\sigma, \nu)\) is an irreducible \(g\)-module. If \(\tau \in \hat{K}_r\) however, the space \(C^\infty(G, \tau; \mathfrak{p}_-)\) is reducible and yet the infinitesimal irreducibility of the image of \(S_\tau\) persists. To prove this result we turn to gradient-type differential operators.

A gradient-type differential operator \(\partial\) is a \(G\)-equivariant homogeneous differential operator \(\partial : C^\infty(G, \tau) \to C^\infty(G, \xi)\) (cf. §7). For \(\tau \in \hat{K}_r\), we construct a finite system \(\mathcal{D}_\tau = \{\partial_1, \ldots, \partial_r\}\) of such operators and show that the subspace of \(C^\infty(G, \tau; \mathfrak{p}_-)\) annihilated by all the \(\partial_i\) is irreducible. In the case where \(\tau\) corresponds to a unitariz-
able highest weight representation, it is shown in [3] that \( D_\tau \) can be chosen to consist of a single differential operator. This distinguished operator, called a covariant differential operator, arises as the dual of a \( g \)-mapping between generalized Verma modules. The irreducibility of the kernel in that case is immediate from duality and a special submodule property resulting from unitarity (cf. [2], [3]). Here we avoid duality arguments and prove the irreducibility of \( \text{Ker}(D_\tau) \cap C_0^\infty(G, \tau; \mathfrak{p}_-) \) directly.

From the above comments it is clear that the proof of the infinitesimal irreducibility of the image of \( S_\tau \) reduces to showing that \( S_\tau \) maps into the kernel of \( D_\tau \). It is natural to attempt to prove \( S_\tau \) maps into \( \text{Ker}(D_\tau) \), \( \tau \in \hat{K} \), by combining an induction argument on the degree of \( \partial_i \) with the calculational techniques found in the proof of (6.1) [16]. This approach forces one to deal with the fact that the kernel function \( s_\tau \) defining \( S_\tau \) is initially expressed in terms of an Iwasawa decomposition while the operator \( D_\tau \) is expressed in terms of a Cartan decomposition. This fact, coupled with the indefinite order of the operators in \( D_\tau \), leads to some computational problems which we find to be intractable.

To circumvent these difficulties, we utilize a particularly useful reformulation of \( s_\tau \) in terms of a factor of automorphy \( J_\tau \). This reformulation is well-suited for our purposes since the factor \( J_\tau \) is defined in terms of the Cartan decomposition. The precise connection between \( s_\tau \) and \( J_\tau \) involves a distinguished point \( b \) in the Bergmann-Shilov boundary and is given explicitly in §4. A connection of this kind had been previously observed by Inoue [13] in the context of "generalized" limits of discrete series. There the image of the Szegö map \( S_\tau \) is viewed as a Hardy space \( H^2 \) and the associated \( G \)-action is the restriction of a holomorphically induced multiplier representation \( T \) to \( H^2 \).

In our context, the multiplier representation \( T \) plays an important role as well. This is because the proof that the image of \( S_\tau \) is annihilated by each \( \partial_i \) in \( D_\tau \) (cf. §8) reduces to showing \( \partial_i s_\tau = 0 \), \( \partial_i \in D_\tau \). Since both \( s_\tau \) and \( T \) are defined in terms of \( J_\tau \), the function \( \partial_i s_\tau \) can be conveniently expressed in terms of the derived action of \( T \). The precise formulation is given in §6.

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and scope of this paper. In particular, our original version dealt only with unitary highest weight representations. The referee suggested that our differential operator techniques may extend to the nonunitary setting as well. We found this to indeed be true and have subsequently dropped the unitarity assumption from the final version.

2. Preliminaries. Let $G$ denote a connected noncompact simple Lie group with finite center. Let $K$ be a maximal compact subgroup and assume $G/K$ is a Hermitian symmetric space. Choose a compact Cartan subgroup $T \subset K$ and let $g_0$, $\mathfrak{k}_0$ and $t_0$ denote the Lie algebras of $G$, $K$ and $T$. Fix a Cartan decomposition $g_0 = \mathfrak{k}_0 \oplus p_0$. We adopt the convention that the removal of the subscript $\circ$ denotes complexification. By our assumptions there exists a decomposition $g = p_+ \oplus t \oplus p_-$ where $p_+$ and $p_-$ are $\text{ad}(\mathfrak{k})$-invariant abelian subalgebras of $p$. The subalgebra $t$ is a Cartan subalgebra of $g$ and we let $\Phi$ be the roots corresponding to $(g, t)$. Let $\Phi_c$ and $\Phi_n$ denote the set of compact and noncompact roots, respectively. We denote the root space corresponding to $\alpha \in \Phi$ by $g_\alpha$ and choose a positive system of roots $\Phi^+$ so that $p_+ = \bigoplus_{\alpha \in \Phi^+_0} g_\alpha$ and $p_- = \bigoplus_{\alpha \in \Phi^-_0} g_\alpha$ where $\Phi^+_0 = \Phi^+ \cap \Phi_0$ and $\Phi^-_0 = (-\Phi^+) \cap \Phi_0$.

If $\theta$ denotes Cartan involution and $B$ denotes the Killing form, then we choose root vectors $E_\alpha \in g_\alpha$, $\alpha \in \Phi$, so that $B(E_\alpha, E_{-\alpha}) = 2/(\alpha, \alpha)$ and $\theta E_\alpha = -E_{-\alpha}$. The bar here is conjugation of $g$ with respect to $g_0$ and $(\cdot, \cdot)$ is the standard inner product on the real space of linear functionals on $t$ taking purely imaginary values on $t_0$. We put $H_0 = [E_\alpha, E_{-\alpha}]$.

Let $r$ be the split rank of $G$. We inductively define a maximal set $\{\gamma_1, \ldots, \gamma_r\}$ of strongly orthogonal positive noncompact roots by setting $\gamma_1 = \text{largest root of } \Phi$ and $\gamma_{j+1} = \text{largest root in } \Phi^+_0$ which is strongly orthogonal to $\gamma_1, \ldots, \gamma_j$. Then $E_{\gamma_j} + E_{-\gamma_j}$ lies in $p_0$ for each $1 \leq j \leq r$ and

\begin{equation}
\mathfrak{a}_0 = \bigoplus_{1 \leq j \leq r} \mathbb{R}(E_{\gamma_j} + E_{-\gamma_j})
\end{equation}

is a maximal abelian subspace of $p_0$. Let $R = R(\mathfrak{a}_0)$ denote the set of restricted roots with respect to $\mathfrak{a}_0$ and let $R^+$ denote the positive roots with respect to the lexicographic ordering induced by the ordered basis $\{E_{\gamma_1} + E_{-\gamma_1}, \ldots, E_{\gamma_r} + E_{-\gamma_r}\}$ of $\mathfrak{a}_0$. If $n_0$ denotes the sum of positive restricted root spaces, then $g_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus n_0$ gives an Iwasawa decomposition of $g_0$ with corresponding decomposition $G = KAN$. 
of \( G \). For \( g \in G \), we write
\[
(2.2) \quad g = \kappa(g)e^{H(g)}n(g)
\]
where \( \kappa(g) \in K \), \( H(g) \in a_\circ \) and \( n(g) \in N \).

We now recall the definition of the principal series representations of \( G \). Let \( M \) denote the centralizer of \( a_\circ \) in \( K \) and let \( \sigma \) be an irreducible unitary representation of \( M \) with representation space \( H_\sigma \). Set \( 2\rho = \sum_{\mu \in R^+} (\dim \mu) \mu \) and let \( \nu \) denote a complex-valued linear function on \( a_\circ \). We induce the representation \( \sigma \otimes e^{\nu} \otimes 1 \) of the minimal parabolic subgroup \( P = MAN \) to a representation of \( G \). Let \( C(\nu, \sigma) \) denote the space of continuous functions \( f: G \to H_\sigma \) satisfying the property
\[
(2.3) \quad f(gman) = e^{-(\nu + \rho)H(a)}\sigma(m)^{-1}f(g).
\]
Let \( W(\nu, \sigma) \) denote the completion of \( C(\nu, \sigma) \) with respect to the norm
\[
(2.4) \quad \|f\|^2 = \int_K |f(k)|^2 \, dk.
\]
The representation \( L = L(\nu, \sigma) \) of \( G \) on \( W(\nu, \sigma) \) is given by left translation: \( (L(g)f)(x) = f(g^{-1}x), \; x \in G \). Then for each \( g \in G \) the operator \( L(g) \) is bounded and for each \( f \in W(\nu, \sigma) \), the mapping \( g \to L(g)f \) is continuous. The parameters are arranged so that \( L \) is unitary if \( \nu \) is imaginary.

3. Quotient maps. We briefly discuss some basic facts about quotient maps which are needed for our purposes. We refer the reader to [8] and [18] for a more comprehensive exposition and some explicit examples. Let \( (\tau, V_\tau) \) (resp. \( (\sigma, H_\sigma) \)) be an irreducible unitary representation of \( K \) (resp. \( M \)) on the complex vector space \( V_\tau \) (resp. \( H_\sigma \)). Let \( \text{Hom}_M(H_\sigma, V_\tau) \) denote the complex space of linear maps \( C: H_\sigma \to V_\tau \) satisfying \( C\sigma(m) = \tau(m)C \) for all \( m \in M \). For \( C \in \text{Hom}_M(H_\sigma, V_\tau) \) we define \( B_C: V_\tau \to W(\nu, \sigma) \) by
\[
(3.1) \quad B_Cv(g) = e^{-(\nu + \rho)H(a)}C^*\tau(\kappa(g))^{-1}v, \quad v \in V_\tau.
\]
Then \( B_C \) is a continuous mapping into \( C(\nu, \sigma) \) which satisfies \( L(k)B_C = B_C\tau(k) \) for all \( k \in K \). The adjoint map \( B_C^*: W(\nu, \sigma) \to V_\tau \) is easily seen to be
\[
(3.2) \quad B_C^*f = \int_K \tau(k)Cf(k) \, dk.
\]
Let \( C(G, \tau) \) denote the space of continuous functions \( f: G \to V_\tau \) satisfying \( f(gk) = \tau(k)^{-1}f(g) \) for all \( g \in G \) and \( k \in K \). Following
Kunze [18], we define the quotient map $S_C: W(\nu, \sigma) \to C(G, \tau)$ by $S_C f(x) = B_C^* L(x^{-1}) f$. For $f \in W(\nu, \sigma)$ and $x \in G$, then

(3.3) \[ S_C f(x) = \int_K \tau(k) C f(xk) \, dk. \]

It is clear that left translation $L(g), \ g \in G,$ leaves the space $C(G, \tau)$ invariant and we have $L(g) S_C = S_C L(g)$. Moreover, the equation $S_C B_C v(1) = B_C^* B_C v$, for $v \in V_\tau$, implies that the $\mathbb{C}$-linear map $C \to S_C$ is injective.

For each $x \in G$ and function $f \in C(\nu, \sigma)$, the function $F_x: K \to H_\sigma$ defined by $F_x(k) = \tau(k) C f(xk)$ is right $M$-invariant and so we may apply the integral formula [15, p. 170]

\[ \int_K F(k) \, dk = \int_K e^{-2\rho H(g^{-1}k)} F(\kappa(g^{-1}k)) \, dk, \quad g \in G, \]

to the function $F_x$ and write $S_C f(x)$ in (3.3) as

\[
S_C f(x) = \int_K e^{-2\rho H(g^{-1}k)} F_x(\kappa(g^{-1}k)) \, dk \\
= \int_K e^{-2\rho H(g^{-1}k)} \tau(\kappa(g^{-1}k)) C f(x\kappa(g^{-1}k)) \, dk \\
= \int_K e^{-2\rho H(g^{-1}k)} \tau(\kappa(g^{-1}k)) \\
\times e^{-(\nu + \rho) H(x\kappa(g^{-1}k))} C f(x\kappa(g^{-1}k)) \, dk.
\]

If we let $x = g$ and observe that $\kappa(g\kappa(g^{-1}k)) = k$ and $H(g\kappa(g^{-1}k)) = -H(g^{-1}k)$ we obtain

(3.4) \[ (S_C f)(g) = \int_K e^{-(\rho - \nu) H(g^{-1}k)} \tau(\kappa(g^{-1}k)) C f(k) \, dk. \]

Define $s_C: G \to \text{Hom}(H_\sigma, V_\tau)$ by $s_C(g) = e^{-(\rho - \nu) H(g^{-1})} \tau(\kappa(g^{-1})) C$ for $g \in G$. Then

(3.5) \[ (S_C f)(g) = \int_K s_C(k^{-1}g) f(k) \, dk. \]

For a fixed representation $(\sigma, H_\sigma)$ of $M$, let $\sigma_\nu$ denote the representation of $P = MAN$ on $H_\sigma$ given by

(3.6) \[ \sigma_\nu(man) = e^{(\nu - \rho) H(a)} \sigma(m). \]

Let $K(\nu, \sigma, \tau)$ denote the space of functions $s: G \to \text{Hom}(H_\sigma, V_\tau)$ satisfying the transforming property

(3.7) \[ s(pgk) = \tau(k)^{-1} s(g) \sigma_\nu(p)^{-1} \]

for all $p \in P, \ g \in G$ and $k \in K$. We call $K(\nu, \sigma, \tau)$ the space of kernel functions associated to $(\nu, \sigma, \tau)$. 
(3.8) **Proposition.** The map $\Gamma: \text{Hom}_M(H_\sigma, V_\tau) \to K(\nu, \sigma, \tau)$ given by $\Gamma(C) = s_C$ is an isomorphism of $\text{Hom}_M(H_\sigma, V_\tau)$ onto $K(\nu, \sigma, \tau)$.

**Proof.** We first show that $s_C$ satisfies (3.7). Using the Iwasawa decomposition $G = ANK$, write $g \in G$ as $g = e^{H_1(g)} n_1(g) \kappa_1(g)$ where $H_1(g) \in a_\omega$, $n_1(g) \in N$ and $\kappa_1(g) \in K$. One has

$$g = (g^{-1})^{-1} = (\kappa(g^{-1})e^{H(g^{-1})}n(g^{-1}))^{-1}$$

$$= n(g^{-1})^{-1}e^{-H(g)}\kappa(g^{-1})^{-1}$$

$$= e^{-H(g^{-1})}(e^{H(g)}n(g^{-1})^{-1}e^{-H(g^{-1})})\kappa(g^{-1})^{-1}.$$

Since $A$ normalizes $N$, the two Iwasawa decompositions are related by the formulas $H_1(g) = -H(g^{-1}), n_1(g) = e^{H_1(g)}n(g^{-1})e^{-H(g^{-1})}$ and $\kappa_1(g) = \kappa(g^{-1})^{-1}$. Hence $s_C(g) = e^{(\rho-\nu)H_1(g)}\tau(\kappa_1(g))^{-1}C$. Since $MA$ normalizes $N$ we have $\kappa_1(mang) = m\kappa_1(g)$ and $e^{H_1(mang)} = ae^{H_1(g)} = e^{H(a)}e^{H_1(g)}$ for all $man \in MAN$ and $g \in G$. Then

$$s_C(mangk) = \tau(k^{-1})s_C(mang)$$

$$= \tau(k)^{-1}e^{(\rho-\nu)H_1(mang)}\tau(\kappa_1(mang))^{-1}C$$

$$= \tau(k)^{-1}e^{(\rho-\nu)H(a)}e^{(\rho-\nu)H_1(g)}\tau(\kappa_1(g))^{-1}\tau(m)^{-1}C$$

$$= \tau(k)^{-1}s_C(g)\sigma_\nu(man)^{-1}.$$ 

This shows that $\Gamma$ maps into $K(\nu, \sigma, \tau)$. If $s \in K(\nu, \sigma, \tau)$, then (3.7) implies $s(1) = s(m^{-1}m) = \tau(m)^{-1}s(1)\sigma(m)$ so that $s(1) \in \text{Hom}_M(H_\sigma, H_\tau)$. Moreover, $\Gamma(s(1))(ank) = \tau(k)^{-1}e^{(\rho-\nu)H(a)}s(1) = s(ank)$ and since $G = ANK$, it follows that $\Gamma$ is onto. Finally, the equation $\Gamma(C)(1) = C$ implies $\Gamma$ is injective and so the proof is complete.

Let $\lambda = \lambda(\tau)$ be the highest weight of $(\tau, V_\tau)$ and let $\phi_\lambda \in V_\tau$ denote a nonzero highest weight vector. We define a representation $(\sigma, H_\sigma)$ of $M$ as follows:

$$(3.9) \quad \left\{ \begin{array}{l} H_\sigma = \text{span}_C\{\tau(m)\phi_\lambda: m \in M\} \quad \text{and} \\ \sigma \text{ is the restriction of } \tau \text{ to } M \text{ on } H_\sigma. \end{array} \right.$$ 

Since we are in the Hermitian symmetric setting we know by [17] and Proposition 5.5 [16; p. 176] that $(\sigma, H_\sigma)$ is an irreducible representation of $M$. 

Let $W(τ)$ denote the nonunitary principal series representation with $σ = σ(τ)$ chosen as in (3.9) and $ν = ν(τ)$ chosen to satisfy

(3.10) \[ (ν - ρ)(E_{γ_j} + E_{−γ_j}) = \frac{2(λ, γ_j)}{(γ_j, γ_j)}, \quad 1 \leq j \leq r. \]

The inclusion map $C_τ: H_σ → V_τ$ is clearly a nonzero element of $\text{Hom}_M( H_σ, V_τ)$ and we let $S_τ$ (resp. $s_τ$) denote the associated quotient map (resp. kernel function). We call $S_τ$ the Szegő map associated to $τ$. Note that since the map $C → SC$ is injective, $S_τ$ is not the zero map.

4. Factors of automorphy and kernel functions. We begin by recalling some standard facts on covering groups. The details that are omitted here may be found in [12] and [20]. Since $G$ has finite center it covers a group $G°$ which admits a faithful matrix representation. (One may take for example $G° = G/Z$, where $Z$ is the center of $G$. Then $G°$ is isomorphic to the adjoint group $\text{Ad}(G)$.) Let $γ: G → G°$ denote the covering map and $C$ the kernel of $γ$. Then $C$ is contained in the center of $G$. The image $K°$, $A°$ and $N°$ of $K$, $A$ and $N$ is an Iwasawa decomposition of $G°$ and $A$ and $N$ are diffeomorphic to $A°$ and $N°$, respectively. Furthermore, since the center of $G$ is contained in the center of $K$, the restriction $γ_K: K → K°$ is a covering map with kernel $C$. Let $K_C$, $K°_C$ and $G_C°$ denote the complexifications of $K°$, $K$ and $G°$, respectively. Since $K°$, $K$ and $G°$ admit faithful finite dimensional representations, we can and do identify each group with its image in the complexification. Furthermore, we identify $K_C°$ with the connected component of $G_C°$ whose Lie algebra is $t$ and note that $γ_K: K → K_C°$ extends to a covering map, again denoted by $γ_K$, of $K_C$ onto $K_C°$. The kernel of this extension is also $C$. By identifying $G°$ (resp. $K_C°$) with the quotient $G/C$ (resp. $K_C/C$), we assume the differential of $γ$ (resp. $γ_K$) at the identity element 1 of $G$ (resp. $K_C$) to be the identity map. In particular, if $exp: g_o → G$ (resp. $exp_o: g_o → G°$) denotes the exponential map, then $γ(exp(X)) = exp_o(X)$ for all $X ∈ g_o$.

Recall the Harish-Chandra realization [10] of $G/K$ as a bounded domain in $p_+$. If $P_± = exp_o(p_±)$ in $G_C°$, then the map $p_+ × K_C° × p_− → G_C°$ defined by $(a, k, b) → exp_o(a)k exp_o(b)$ is a holomorphic diffeomorphism onto a dense open set $Ω = P_+K_C°P_−$ of $G_C°$. We uniquely write each $x ∈ Ω$ as the product

(4.1) \[ x = π_+(x)π_o(x)π_-(x), \quad \text{where } π_±(x) ∈ P_± \text{ and } π_o(x) ∈ K_C°. \]
One knows $G^\circ \subset \Omega$ and the map $\zeta: \Omega \to p_+$ defined by $\zeta(x) = \log(\pi^+(jt))$ induces a holomorphic diffeomorphism of $G^\circ/K^\circ = G/K$ onto $\zeta(G^\circ)$. The set $D = \zeta(G^\circ)$ is a bounded domain in $p_+$ and we identify $D$ with $G/K$. In particular, the coset $K$ in $G/K$ is identified with the origin 0 in $p_+$.

For each $(g, z) \in G_C^\circ \times p_+$ for which $g \exp_0(z) \in \Omega$ we set

\begin{align}
(4.2) \quad g \cdot z &= \log \pi^+(g \exp_0(z)) \quad \text{and} \\
(4.3) \quad j_0(g, z) &= \pi_0(g \exp_0(z)).
\end{align}

It is known that if $g \in G^\circ$ and $z \in \overline{D}$ then $g \exp(z) \in \Omega$. Consequently, (4.2) defines a $G^\circ$-action on $\overline{D}$ (the closure of $D$ in $p_+$). This action lifts to $G$ through the covering map and we denote it in the same way: $g \cdot z = \gamma(g) \cdot z$, $g \in G$ and $z \in \overline{D}$. The map $j_0$ satisfies

\begin{align}
(4.4) \quad j_0(k, z) &= k \quad \text{for all } (k, z) \in K^{\circ}_C \times p_+,
(4.5) \quad j_0(p, z) &= 1 \quad \text{for all } (p, z) \in P_+ \times p_+,
(4.6) \quad j_0(g_1 g_2, z) &= j_0(g_1, g_2 \cdot z) j_0(g_2, z),
\end{align}

where (4.6) holds for all $g_1, g_2 \in G_C^\circ$ and $z \in p_+$ for which $g_1 g_2 \exp(z)$ and $g_2 \exp(z)$ both lie in $\Omega$.

\begin{equation}
(4.7) \quad \text{PROPOSITION.} \quad \text{The map } j_0: G^\circ \times \overline{D} \to K^{\circ}_C \text{ lifts uniquely to a continuous map } j: G \times \overline{D} \to K^{\circ}_C \text{ satisfying}
\end{equation}

\begin{enumerate}
\item $\gamma_K(j(g, z)) = j_0(\gamma(g), z)$, for all $(g, z) \in G \times \overline{D}$,
\item $j(k, z) = k$, for all $(k, z) \in K \times \overline{D}$,
\item $j(g_1 g_2, z) = j(g_1, g_2 \cdot z) j(g_2, z)$, for all $g_1, g_2 \in G$, $z \in \overline{D}$,
\item $j$ is $C^\infty$ on $G$ and holomorphic on $D$.
\end{enumerate}

\begin{proof}
A norm $|\cdot|$ can be defined on $p_+$ so that $D = \{X \in p_+: |X| < 1\}$ (cf. [14]). Since $D$ is connected and simply connected, so is $\overline{D}$. Since $AN$ is simply connected, the map-lifting theorem states that there is a unique continuous map $j: AN \times \overline{D} \to K^{\circ}_C$ satisfying $j(1, 0) = 1$ and $\gamma_K(j(an, z)) = j_0(\gamma(an), z)$. Now extend $j$ to $G \times \overline{D}$ by $j(kan, z) = kj(an, z)$. Such an extension is clearly continuous. Moreover,

$$
\gamma_K(j(kan, z)) = \gamma_K(k) \gamma_K(j(an, z)) = \gamma_K(k) j_0(\gamma(an), z) = j_0(\gamma_K(k) \gamma(an), z) = j_0(\gamma(kan), z).
$$

Thus $j$ satisfies (1).

By (4.5) \( j_0(1, z) = 1 \) for all \( z \in \overline{D} \) so that by (1), \( z \rightarrow j(1, z) \) is a continuous \( C \)-valued function on \( \overline{D} \) and hence a constant function. Since \( j(1, 0) = 1 \), we have \( j(1, z) = 1 \) for all \( z \in \overline{D} \). This implies (2).

Let \( g, h \in G \) and \( z \in \overline{D} \). By (4.6) and (1) we have \( j(gh, z) = j(g, h \cdot z)j(h, z)F(g, h, z) \), where \( F \) is a continuous \( C \)-valued function on \( G \times G \times \overline{D} \). Since \( G \) and \( \overline{D} \) are connected, it follows that \( F \) is constant. But \( F(1, 1, 0) = j(1, 0)^{-1} = 1 \) so that (3) follows.

Property (4) follows from the fact that \( j_0 \) is \( C^\infty \) on \( G^o \) and holomorphic on \( D \). Finally, a continuous function \( j: G \times \overline{D} \rightarrow K_\mathbb{C} \) satisfying (1)-(4) is unique. This completes the proof.

We define the Cayley transform \( c \in G^o_\mathbb{C} \) to be

\[
(4.8) \quad c = \exp_0 \left( \sum_{i=1}^{r} \pi \frac{\pi}{4} (E_{\gamma_i} - E_{-\gamma_i}) \right).
\]

One easily finds

\[
\exp \left( -\text{ad} \left( \frac{\pi}{4} (E_{\gamma_i} - E_{-\gamma_i}) \right) \right) (E_{\gamma_j} + E_{-\gamma_j}) = \begin{cases} -H_{\gamma_i}, & \text{if } j = i, \\ E_{\gamma_j} + E_{-\gamma_j}, & \text{for } j \neq i. \end{cases}
\]

Consequently we have

\[
(4.9) \quad \text{Ad}(c^{-1})(E_{\gamma_j} + E_{-\gamma_j}) = -H_{\gamma_j}, \quad 1 \leq i \leq r.
\]

Let \( t^- = \bigoplus_{i=1}^{r} \mathbb{R}H_{\gamma_i} \). Then \( t^- \subset t \) and \( it^- \subset t_0 \). Recall the Iwasawa decomposition \( g_o = t_o \oplus a_o \oplus n_o \) chosen in §2.

(4.10) Lemma. We have

(i) \( \text{Ad}(c^{-1})a_o = t^- \) and

(ii) \( \text{Ad}(c^{-1})n_o \subset \sum_{\alpha \in \Phi^+} g_{-\alpha} \).

Proof. Part (i) follows from (4.9). To prove (ii), let \( h_o^- \) denote the subspace of \( t_o \) annihilated by \( \{\gamma_j : 1 \leq j \leq r\} \). Then clearly \( h_o^- \subset m_o \) and \( t_o = h_o^- \oplus it^- \). It follows that \( h_o^- \) is maximal abelian in \( m_o \). The abelian subalgebra \( h_o = h_o^- \oplus ia_o \) is a Cartan subalgebra of the compact form \( u_o = \mathfrak{t}_o \oplus i\mathfrak{p}_o \) and we form the root space decomposition of \( g = u = h \oplus \sum_{\delta \in \Delta(g, h)} u_\delta \) with respect to \( h \). By the definition of \( h_o^- \), \( \text{Ad}(c^{-1}) \) is the identity on \( h_o^- \). From (i) we have \( \text{Ad}(c^{-1})(h_o^- \oplus ia_o) = h_o^- \oplus it^- = t_o \). If \( \text{Ad}(c)^t \) denotes the transpose of \( \text{Ad}(c) \), then it
follows $\text{Ad}(c)^t \colon \Delta(g, h) \to \Phi$ is a bijection and $\text{Ad}(c^{-1})u_{\delta} = g_{\text{Ad}(c)^t \delta}$ for all $\delta \in \Delta(g, h)$.

Let $w_\mu$ denote the root space corresponding to the restricted root $\mu$. Since $n_\circ = \bigoplus_{\mu > 0} w_\mu$, it suffices to show that

$$w_\mu \subset \text{Ad}(c) \left( \sum_{\alpha \in \Phi^+} g_{-\alpha} \right)$$

for each positive $\mu$. The restricted roots $R$ are the nonzero restrictions of the roots $\Delta(g, h)$ to $a_\circ$. For $\mu \in R$, let $\Delta(\mu) = \{ \delta \in \Delta(g, h) : \delta \text{ restricted to } a_\circ \text{ is } \mu \}$ and let $U_\mu = \bigoplus_{\delta \in \Delta(\mu)} u_\delta$. Since $w_\mu = g_\circ \cap U_\mu$, it suffices to show $u_\delta \subset \text{Ad}(c)( \sum_{\alpha \in \Phi^+} g_{-\alpha} )$ for all $\delta \in \Delta(\mu)$ with $\mu > 0$. Since $u_\delta = \text{Ad}(c)(g_{\text{Ad}(c)^t \delta})$, we need only show the root $\text{Ad}(c)^t \delta$ lies in $\Phi^-$, or equivalently, that $\text{Ad}(c)^t \delta$ cannot lie in $\Phi^+$. First note that since $\mu$ is positive with respect to the lexicographic ordering induced by $\{E_{\gamma_i} + E_{-\gamma_i}, \ldots, E_{\gamma_r} + E_{-\gamma_r}\}$, if $i \in \{1, \ldots, r\}$ is the smallest integer for which $\mu(E_{\gamma_i} + E_{-\gamma_i}) \neq 0$ then $\mu(E_{\gamma_i} + E_{-\gamma_i}) > 0$. Consequently, for this integer $i$, $(\text{Ad}(c)^t \delta)(H_{\gamma_i}) = \delta(\text{Ad}(c)H_{\gamma_i}) = -\delta(E_{\gamma_i} + E_{-\gamma_i}) = -\mu(E_{\gamma_i} + E_{-\gamma_i}) < 0$ by (4.9). Let $\pi : t' \to (t^-)^t$ denote restriction to $t^-$. From Lemma 16 [10; p. 588] one knows that

$$\pi(\Phi^+) = \{0\} \cup \left\{ \frac{1}{2} \pi(\gamma_j) : 1 \leq j \leq r \right\}$$

$$\cup \left\{ \frac{1}{2} \pi(\gamma_i \pm \gamma_j) : 1 \leq i < j \leq r \right\} \cup \{\pi(\gamma_j) : 1 \leq j \leq r\}.$$

(See also [13; p. 82].) In particular, the nonzero elements $\pi(\alpha), \alpha \in \Phi^+$, have the property that if $i \in \{1, \ldots, r\}$ is the smallest integer for which $\pi(\alpha)(H_{\gamma_i}) \neq 0$, then $\pi(\alpha)(H_{\gamma_i}) > 0$. Since $\pi(\text{Ad}(c)^t \delta) \neq 0$, it follows that $\text{Ad}(c)^t \delta$ cannot lie in $\Phi^+$. This completes the proof of (ii).

Let $T^o_C$ and $B^o_C$ (resp. $T_C$ and $B_C$) be the connected subgroups of $K^o_C$ (resp. $K_C$) whose Lie algebras are $t$ and $\bigoplus_{\alpha \in \Phi^+} g_{-\alpha}$. We then have from (4.10) the following result.

(4.11) Corollary. (1) $c^{-1}A^o c \subset T^o_C$ and (2) $c^{-1}N^o c \subset B^o_C P_-$.

The strong orthogonality of the $\gamma_j$ along with an explicit calculation
in $\text{SL}(2, \mathbb{C})$ gives

$$(4.12) \; \exp \left( \sum_{i=1}^{r} t(E_{\gamma_i} - E_{-\gamma_i}) \right)$$

$$= \exp \left( \tan(t) \sum_{i=1}^{r} E_{\gamma_i} \right) \exp \left( -\ln(\cos(t)) \sum_{i=1}^{r} H_{\gamma_i} \right)$$

$$\times \exp \left( -\tan(t) \sum_{i=1}^{r} E_{-\gamma_i} \right),$$

for $t \in \mathbb{R}$. In particular if $t = \frac{\pi}{4}$ then $c \in \Omega = P_+K_0^0P_-$ and $c \cdot 0 = \log \pi_+(c) = \sum_{i=1}^{r} E_{\gamma_i}$. Consequently we have $G^o c = K^o A^o N^o c = K^o c (-A^o c)(c^{-1} N^o c) \subset K^o c K_0^0 P_-$. Since $c \in \Omega$, we conclude $G^o c \subset \Omega$. Thus if $b = \sum_{i=1}^{r} E_{\gamma_i}$, then $g \cdot b$ is defined for each $g \in G^o$ and by extension for each $g \in G$.

(4.13) **Lemma.** The stability subgroup $G_b$ of $b = \sum_{i=1}^{r} E_{\gamma_i}$ in $G$ contains $MAN$.

**Proof.** To show $an \cdot b = b$ for $an \in AN$, it suffices to show that $c^{-1} an \cdot 0 = 0$ for $an \in A^o N^o$. But this follows from (4.11) and the fact that $K_0^0 P_- \cdot 0 = 0$. To show $m \cdot b = b$ we first observe that $k \cdot z = \text{Ad}(k)z$ for all $k \in K$ and $z \in p_+$. Since $p_+$ and $p_-$ are $\text{Ad}(K)$-invariant, $M \subset K$ and $(b + \bar{b}) \in a_0$, we have $b + \bar{b} = \text{Ad}(m)(b + \bar{b}) = \text{Ad}(m)(b) + \text{Ad}(m)(\bar{b})$ so that $m \cdot b = \text{Ad}(m)(b) = b$. This completes the proof.

(4.14) **Remarks.** (1) In general, $MAN$ is properly contained in $G_b$. (2) By the cocycle property (4.7.3) and by (4.13) the function $g \rightarrow j(g, b)$, $g \in G_b$, is a homomorphism.

Throughout the remainder of the paper we assume that $\tau$ is an irreducible unitary representation of $K$ on the space $V_\tau$ with highest weight $\lambda$ and we fix a nonzero highest weight vector $\phi_\lambda$. The representation $\tau$ extends to a holomorphic representation of $K_C$ on $V_\tau$. We define the factor of automorphy $J_\tau$ by $J_\tau(g, z) = \tau(j(g, z))$ for $(g, z) \in G \times D$. By (4.14.2) the map $\omega: G_b \rightarrow GL(V_\tau)$ given by $\omega(g) = J_\tau(g, b)^{s-1}$, $g \in G_b$, defines a representation of $G_b$ and hence of $MAN$.

Choose the representation $\sigma = \sigma(\tau)$ as in (3.9) and the linear functional $\nu = \nu(\tau)$ satisfying (3.10), i.e. $(\nu - \rho)(E_{\gamma_j} + E_{-\gamma_j}) = 2(\lambda, \gamma_j)/(\gamma_j, \gamma_j) = \lambda(H_{\gamma_j})$ for $1 \leq j \leq r$. Keeping in mind these
choices for \( \sigma \) and \( \nu \), recall the definition (3.6) of the corresponding irreducible representation \( \sigma_\nu \) of \( MAN \) on \( H_\sigma \).

(4.15) **Lemma.** For each \( g \in MAN \), the operator \( \omega(g) \) leaves the subspace \( H_\sigma \) invariant. Moreover, the restriction of \( \omega(g) \), \( g \in MAN \), to \( H_\sigma \) is \( \sigma_\nu(g) \).

**Proof.** Recall that \( H_\sigma \) is the span of \( \tau(m)\phi_\lambda \), \( m \in M \). Since \( M \) commutes with \( A \) and normalizes \( N \), the lemma will follow if we show \( \omega(g)\phi_\lambda = \sigma_\nu(g)\phi_\lambda \) for all \( g \) in \( M, A, \) and \( N \). Now for \( m \in M \), \( \omega(m)\psi = \sigma_\nu(m)\psi \) for \( \psi \in H_\sigma \) by (4.7.2).

We now show \( \omega(a)\phi_\lambda = \sigma_\nu(a)\phi_\lambda \). Let \( a \in A^o \). Then by (4.6) and (4.13), one has \( j_\circ(c^{-1}ac, 0) = j_\circ(c^{-1}, c \cdot 0)j_\circ(a, b)j_\circ(c, 0) \). By (4.11) and (4.12), \( j_\circ(c^{-1}ac, 0) \) and \( j_\circ(c, 0) \) are in \( T_\circ \). Since \( j_\circ(c^{-1}, c \cdot 0) = j_\circ(c, 0)^{-1} \) and \( T_\circ \) is abelian, it follows that \( j_\circ(a, b) = j_\circ(c^{-1}ac, 0) = c^{-1}ac \). If \( a = \exp(\sum a_i(E_\gamma_i + E_{-\gamma_i})) \) \( (a_i \in \mathbb{R}) \), then \( j_\circ(\gamma(a), b) = c^{-1}\gamma(a)c = \exp(\sum -a_iH_\gamma_i) \) in \( T_\circ \) by (4.9). From this it follows that \( j(a, b) = \exp(\sum -a_iH_\gamma_i) \). We now obtain

\[
\omega(a)\phi_\lambda = J_\tau(a, b)^{-1}\phi_\lambda
\]

\[
= \tau \left( \exp \left( - \sum a_iH_\gamma_i \right) \right)^{-1} \phi_\lambda
\]

\[
= \tau \left( \exp \left( \sum a_iH_\gamma_i \right) \right) \phi_\lambda
\]

\[
= e^{\sum a_i(H_\gamma_i)} \phi_\lambda
\]

\[
= e^{\sum a_i(\nu-\rho)(E_\gamma_i + E_{-\gamma_i})} \phi_\lambda \quad \text{by (3.10)}
\]

\[
= e^{(\nu-\rho)H(a)} \phi_\lambda
\]

\[
= \sigma_\nu(a)\phi_\lambda.
\]

We now show \( \omega(n)\phi_\lambda = \sigma_\nu(n)\phi_\lambda \) for all \( n \in N \). As in the preceding paragraph, we have \( j_\circ(n, b) = j_\circ(c, 0)j_\circ(c^{-1}nc, 0)j_\circ(c, 0)^{-1} \), for \( n \in N^o \). By (4.11), \( c^{-1}Nc \in \mathbb{C}P_- \) so that \( j_\circ(c^{-1}nc, 0) \) in \( \mathbb{B}_\circ \). Since \( j_\circ(c, 0) \in T_\circ \) and \( T_\circ \) normalizes \( \mathbb{B}_\circ \) it follows that \( j_\circ(n, b) \) in \( \mathbb{B}_\circ \). Therefore, for \( n \in N \), \( j(n, b) \in \gamma_K^{-1}(\mathbb{B}_\circ) \) by (4.7.1). On the other hand, \( j(n, b) \) is in the connected component of the identity of \( \gamma_K^{-1}(\mathbb{B}_\circ) \) because \( j(1, b) = 1 \), \( N \) is connected and \( n \to j(n, b) \) is continuous. Thus \( j(n, b) \in \mathbb{B}_\circ \). But then \( \omega(n)\phi_\lambda = J_\tau(n, b)^{-1}\phi_\lambda = \tau(j(n, b))^{*-1}\phi_\lambda = \phi_\lambda \) since \( \phi_\lambda \) is a highest weight vector. However \( \sigma_\nu(n)\phi_\lambda = \phi_\lambda \) for \( n \in N \). This completes the proof.
(4.16) **Proposition.** Let $C_\tau : H_\sigma \to V_\tau$ denote inclusion. Then $s_\tau(g) = J_\tau(g^{-1}, b)^{-1} C_\tau$ for all $g \in G$.

**Proof.** Let $J(g) = J_\tau(g^{-1}, b)^{-1} C_\tau$. By (3.8) it suffices to show that $J$ satisfies transforming property (3.7) and $J(1) = s_\tau(1)$. We have

$$J(man) = J_\tau(g^{-1}(man)^{-1}, b)^{-1} C_\tau$$

$$= J_\tau(g^{-1}, (man)^{-1} \cdot b)^{-1} J_\tau((man)^{-1}, b)^{-1} C_\tau \quad \text{by (4.7.3)}$$

$$= J_\tau(g^{-1}, b)^{-1} \omega((man)^{-1}) C_\tau \quad \text{by (4.13)}$$

$$= J(g) \sigma_v(man)^{-1} \quad \text{by (4.15)}$$

for all $g \in G$ and $man \in MAN$. By (4.7.2) we also have $J(gk) = J_\tau(k^{-1}g^{-1}, b)^{-1} C_\tau = \tau(k^{-1})^{-1} J_\tau(g^{-1}, b)^{-1} C_\tau = \tau(k^{-1}) J(g)$ for all $g \in G$ and $k \in K$. Finally, observe that $J(1) = C_\tau = s_\tau(1)$.

### 5. Holomorphic-type functions on $G$. The purpose of this section is to show that the Szegö operator $S_\tau$ defined in §3 takes its values in a standard realization of a highest weight representation space.

With $\tau$ as in §4, let $C^\infty(G, \tau)$ denote the subspace of $C^\infty$ functions in $C(G, \tau)$. On $C^\infty(G, \tau)$ we define left-invariant vector fields $r(X)$ for $X \in \mathfrak{g}_0$ by $(r(X)f)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}$ where $g \in G$. For $X = X_1 + iX_2$ with $X_j \in \mathfrak{g}_0$, $j = 1, 2$, we set $r(X) = r(X_1) + ir(X_2)$. Put

$$C^\infty(G, \tau ; \mathfrak{p}_- ) = \{ f \in C^\infty(G, \tau) : r(X)f = 0 \text{ for all } X \in \mathfrak{p}_- \}. $$

Clearly $C^\infty(G, \tau ; \mathfrak{p}_- )$ is $L(G)$-invariant. We call $C^\infty(G, \tau ; \mathfrak{p}_- )$ the space of holomorphic-type functions on $G$ determined by $\tau$.

Note that from (4.7.3) and (4.7.2) one has

$$J_\tau(g^{-1}k, b)^{-1} C_\tau = J_\tau(g^{-1}, k \cdot b)^{-1} J_\tau(k, b)^{-1} C_\tau$$

$$= J_\tau(g^{-1}, k \cdot b)^{-1} \tau(k) C_\tau .$$

It then follows from (3.5) and (4.16) that the Szegö map $S_\tau$ may be written in the form

$$S_\tau f(g) = \int_K J_\tau(g^{-1}, k \cdot b)^{-1} \tau(k) C_\tau f(k) \, dk ,$$

$$f \in W(\tau), \ g \in G.$$

(5.3) **Proposition.** The Szegö operator $S_\tau$ maps $W(\tau)$ into $C^\infty(G, \tau ; \mathfrak{p}_- )$. 
Proof. Since the kernel function $s_\tau$ is $C^\infty$, it follows that $S_\tau$ maps $W(\tau)$ into $C^\infty(G, \tau)$. By $G$-equivariance, it suffices to show that $(r(X)S_\tau f)(1) = 0$ for all $f \in W(\tau)$ and $X \in \mathfrak{p}_-$. However, from (5.2) it suffices to show $r(X)J_\tau(g, z)^{s-1}|_{g=1} = 0$ for all $z \in \overline{D}$. By the assumptions of §4, the differentials of $\gamma$ and $\gamma_K$ are the identity maps. Since $\gamma_K(j(g, z)) = j_\circ(\gamma(g), z)$, for all $(g, z) \in G \times \overline{D}$, it follows that for each $z \in \overline{D}$, $r(X)j(g, z)|_{g=1} = r(X)j_\circ(\gamma(g), z)|_{g=1}$ for all $X \in \mathfrak{g}$. But for each $z \in \overline{D}$, there exists a neighborhood $\mathcal{N}_z$ of the identity in $G_\mathbb{C}$ on which $g \rightarrow j_\circ(g, z)$ is holomorphic. For $Y \in \mathfrak{p}_+$ we consequently have $r(Y)j_\circ(g, z)|_{g=1} = \frac{d}{dt}j_\circ(\exp(tY), z)|_{t=0} = 0$ since $j_\circ(p, z)$ is the identity for all $(p, z) \in \mathcal{P}_+ \times \mathfrak{p}_+$ by (4.5). Thus for $X \in \mathfrak{p}_-$ we find $r(X)J_\tau(g, z)^{s-1}|_{g=1} = (r(X)J_\tau(g, z)^{-1})^*|_{g=1} = 0$. This completes the proof.

(5.4) RemarKs. (1) The proof of (5.3) given here depends on the results thus far developed and reflects the special nature of the Hermitian symmetric setting. Proposition (5.3) is however an immediate consequence of the more general Theorem (6.1) [16]. One only needs to check that parameters may be chosen in a consistent way.

(2) It is shown in [21] that the choice of imbedding parameters of irreducible highest weight $(\mathfrak{g}, \mathfrak{t})$-modules in principal series is unique. Thus the parameters chosen in §3 for the Szegö map $S_\tau$ are the only parameters possible.

6. Holomorphically induced multiplier representations. In order to apply certain $G$-equivariant differential operators to the kernel function $s_\tau$, we first need to compute the action of left-invariant vector fields by elements of $\mathfrak{p}_+$ on the factor of automorphy $J_\tau$. The factor $J_\tau$ also appears as a multiplier in the definition of the holomorphic representation $T$ of $G$ induced by $\tau$. This section shows that the desired $\mathfrak{p}_+$-action can be conveniently expressed in terms of the derived action of $T$.

We begin by reviewing some basic facts concerning holomorphically induced multiplier representations. We refer the reader to [10] and [15] for more details.

Let $(\tau, V_\tau)$ be an irreducible unitary representation of $K$ and let $O(D, V_\tau)$ denote the space of $V_\tau$-valued holomorphic functions on $D$. The associated multiplier representation $T$ of $G$ on $O(D, V_\tau)$ is defined by the formula

\begin{equation}
(T(g)F)(\zeta) = J_\tau(g^{-1}, \zeta)^{-1}F(g^{-1} \cdot \zeta),
\end{equation}
where $F \in O(D, V_{\tau}), g \in G$ and $\zeta \in D$. The $K$-finite vectors in $O(D, V_{\tau})$ are the polynomial functions. Since $D$ is open in $p_+$, we may identify the $K$-finite vectors with the $g$-module $V_{\tau}$ of $V_{\tau}$-valued polynomials on $p_+$. We will again denote the derived representation of $g$ on $V_{\tau}$ by $T$. To give the $g$-action explicitly, we first define the differential operator $\delta(x), x \in p_+$, as follows: for $f \in V_{\tau}$ and $x \in p_+$ set

\begin{equation}
\delta(x)f(z) = \frac{d}{dt} f(z + tx)|_{t=0}, \quad z \in p_+.
\end{equation}

Then for $f \in V_{\tau}$ one has

\begin{align}
T(x)f(z) &= -\delta(x)f(z), \quad x \in p_+, \\
T(x)f(z) &= d\tau(x)f(z) - \delta([x, z])f(z), \quad x \in \mathfrak{k}, \\
T(x)f(z) &= d\tau([x, z])f(z) - \frac{1}{2}\delta([x, z], z)f(z), \quad x \in p_-. 
\end{align}

Let $V_{\tau}^j$ denote the space of homogeneous polynomials of degree $j$ in $V_{\tau}$. Then it is clear from (6.3) that

\begin{align}
T(x) : V_{\tau}^j &\to V_{\tau}^{j-1}, \quad x \in p_+, \\
T(x) : V_{\tau}^j &\to V_{\tau}^j, \quad x \in \mathfrak{k}, \\
T(x) : V_{\tau}^j &\to V_{\tau}^{j+1}, \quad x \in p_-, 
\end{align}

for all integers $j \geq 0$. (Here it is understood that $V_{\tau}^{-1} = 0$.)

For $\phi \in V_{\tau}$, let $1_\phi \in V_{\tau}^0$ denote the constant function with value $\phi$. Recall from §4 that $\lambda$ (resp. $\phi_\lambda$) denotes the highest weight (resp. highest weight vector) for $(\tau, V_{\tau})$. Let $L(\tau)$ denote the $g$-module generated by the constant functions $V_{\tau}^0$. Then (6.4) implies every nontrivial submodule of $V_{\tau}$ contains $L(\tau)$ and thus $L(\tau)$ is irreducible. Clearly $L(\tau)$ is a highest weight module with highest weight vector $1_{\phi_\lambda}$ and highest weight $\lambda$.

We now define a Hermitian product on $V_{\tau}$ which makes the $T(K)$-action unitary. Normalize Lebesgue measure $dv$ on $p_+$ so that

\[ \int_{p_+} e^{-|v|^2} dv = 1 \]

where $|\cdot|$ is the norm given by the standard Hermitian inner product on $g$. Put $dm(v) = e^{-|v|^2} dv$ and set

\begin{equation}
(f, g) = \int_{p_+} (f(v), g(v)) dm(v), \quad f, g \in V_{\tau}.
\end{equation}
Note that with this normalization, the inner product on the space of constant functions $V^0_\tau$ agrees with the inner product on $V_\tau$. Also note that if $f \in V_\tau$, then from (6.1) it follows that

\[(6.6) \quad T(k)f(z) = \tau(k)f(\text{Ad}(k^{-1})z), \quad k \in K, \; z \in p_+ .\]

Hence, $T(k)$, $k \in K$, is unitary with respect to the inner product in (6.5) and each subspace $V^k_\tau$ is invariant under $T(K)$. Finally, observe that for $k \in K$, $T(k)$ is equivalent to $\tau(k)$ on $V^0_\tau$, i.e.

\[(6.7) \quad T(k)1_\phi = 1_{\tau(k)\phi}, \quad k \in K.\]

Let $\varepsilon$ denote the conjugate linear antiautomorphism of $U(g)$ induced by the map $X \to -X$ on the real form $g_\Re$ of $g$. Equipping $S(p_{\pm}) = U(p_{\pm}) \subset U(g)$ with the usual grading, we let $S_j(p_{\pm})$ denote the homogeneous elements of degree $j$. Since $\varepsilon(p_{\pm}) = p_{\mp}$, note that $\varepsilon(S_j(p_{\pm})) = S_j(p_{\mp})$ for all $j \geq 0$.

\[(6.8) \quad \text{LEMMA. Let } \phi, \psi \in V_\tau \text{ and } Y, Z \in S_j(p_{\mp}) \text{ for } j \geq 1. \text{ Then}
\[(1_\phi, T(Y)T(\varepsilon(Z))1_\psi) = (T(Z)T(\varepsilon(Y))1_\phi, 1_\psi).
\]

\textbf{Proof.} The proof proceeds by induction on $j$. For brevity, we suppress the use of $T$. Let $Y, Z \in S_1(p_{\mp})$. Then since $p_+$ annihilates $V^0_\tau$, we have

\[(1_\phi, Y\varepsilon(Z)1_\psi) = (1_\phi, ([Y, \varepsilon(Z)] + \varepsilon(Z)Y)1_\psi)
= (1_\phi, [Y, \varepsilon(Z)]1_\psi)
= (\varepsilon([Y, \varepsilon(Z)])1_\phi, 1_\psi) \quad \text{since } [Y, \varepsilon(Z)] \in k
= ([Z, \varepsilon(Y)]1_\phi, 1_\psi)
= (Z\varepsilon(Y)1_\phi, 1_\psi).
\]

Now suppose $j > 1$. By linearity, we may assume both $Z$ and $Y$ are monomials in $S_j(p_{\mp})$. Write $Y = Y'y$ where $y \in p_+$ and $Y' \in S_{j-1}(p_{\mp})$. We have

\[(1_\phi, Y\varepsilon(Z)1_\psi) = (1_\phi, Y'(y\varepsilon(Z) - \varepsilon(Z)y)1_\psi)
= (1_\phi, Y'[y, \varepsilon(Z)]1_\psi).
\]

It is readily checked that $[y, \varepsilon(Z)] = t + \sum s_ik_i$ for $t, s_i \in S_{j-1}(p_-)$
and \( k_i \in \mathfrak{g} \). Since \( \epsilon(S_{j-1}(p_-)) \subset S_{j-1}(p_+) \), induction gives
\[
(1_\phi, Y\epsilon(Z)1_\psi) = (1_\phi, Y't1_\psi) + \sum_i (1_\phi, Y's_i 1_{\tau(k_i)\psi}) \quad \text{by (6.7)}
\]
\[
= (1_\phi, Y'\epsilon(\epsilon(t))1_\psi) + \sum_i (1_\phi, Y'\epsilon(\epsilon(s_i))1_{\tau(k_i)\psi})
\]
\[
= (\epsilon(t)\epsilon(Y')1_\phi, 1_\psi) + \sum_i (\epsilon(s_i)\epsilon(Y')1_\phi, 1_{\tau(k_i)\psi})
\]
\[
= (\epsilon(t)\epsilon(Y')1_\phi, 1_\psi) + \sum_i (\epsilon(k_i)\epsilon(s_i)\epsilon(Y')1_\phi, 1_\psi)
\]
\[
= \left( \epsilon \left( t + \sum_i s_i k_i \right) \epsilon(Y')1_\phi, 1_\psi \right)
\]
\[
= ([Z, \epsilon(y)]\epsilon(Y')1_\phi, 1_\psi)
\]
\[
= (Z\epsilon(y)\epsilon(Y')1_\phi, 1_\psi) - (\epsilon(y)Z\epsilon(Y')1_\phi, 1_\psi).
\]

By (6.4) we know \( Z\epsilon(Y')1_\phi = 0 \). Since \( \epsilon(y)\epsilon(Y') = \epsilon(Y) \), the proof is complete.

(6.9) Lemma. Let \( \phi, \psi \in V_\tau \). Then for all integers \( j \geq 1 \),
\[
(\phi, (T(Z)^j 1_\psi)(y)) = ((T(Y)^j 1_\phi)(z), \psi) \quad \text{for all } z, y \in p_+.
\]

Proof. Set
\[
f_1(z, y) = (\phi, (T(Z)^j 1_\psi)(y)) \quad \text{and} \quad f_2(z, y) = ((T(Y)^j 1_\phi)(z), \psi).
\]

For \( i = 1, 2 \), note that \( f_i(z, y) \) is a homogeneous polynomial in \( z \) of degree \( j \) and a homogeneous polynomial of degree \( j \) in \( Y \). It suffices to show therefore that all \( j \)th order partial derivatives of \( f_1 \) and \( f_2 \) agree. Choose \( z_1, \ldots, z_j, y_1, \ldots, y_j \) in \( p_+ \) and set \( d = \prod_{i=1}^j \delta(z_i) \prod_{k=1}^j \delta(y_k) \) where \( \delta(z_i) \) (resp. \( \delta(y_k) \)) differentiates with respect to the \( z \) (resp. \( y \)) variable. Using (6.3) and the linearity of \( T \) we find
\[
df_1 = (-1)^j j!(\phi, T(Y)T(Z)1_\psi(0)) \quad \text{and} \quad df_2 = (-1)^j j!(T(Z)T(Y)1_\phi(0), \psi),
\]
where \( T(Y) = \prod_{k=1}^j T(y_k) \) and \( T(Z) = \prod_{k=1}^j T(z_k) \). Consequently, \( df_1 = j!(1_\phi, T(Y)T(\epsilon(Z))1_\psi) \) and \( df_2 = j!(T(Z)T(\epsilon(Y))1_\phi, 1_\psi) \). By (6.8), we conclude \( df_1 = df_2 \). Since \( z_1, \ldots, z_j, y_1, \ldots, y_j \) are arbitrary in \( p_+ \) it follows that \( f_1 = f_2 \). This completes the proof.
(6.10) Proposition. If $\phi \in V_\tau$ and $y \in \overline{D}$, then
\[ r(z)^j J_\tau(g^{-1}, y)^{*-1} \phi|_{g=1} = (T(\bar{y})^j 1_\phi)(z) \]
for all $z \in p_+$ and integers $j \geq 1$.

Proof. Let $\phi, \psi \in V_\tau$. Then for $z \in p_+$ and $y \in \overline{D}$ we have
\[
(r(z)^j J_\tau(g^{-1}, y)^{*-1} \phi|_{g=1}, \psi) = r(z)^j (J_\tau(g^{-1}, y)^{*-1} \phi, \psi)|_{g=1}
\]
\[
= r(z)^j (\phi, J_\tau(g^{-1}, y)^{-1} \psi)|_{g=1}
\]
\[
= (\phi, r(\bar{z})^j J_\tau(g^{-1}, y)^{-1} \psi)|_{g=1}
\]
\[
= (\phi, T(\bar{z})^j 1_\psi(y))
\]
\[
= ((T(\bar{y})^j 1_\phi)(z), \psi) \quad \text{by (6.9)}.\]

Since $\psi \in V_\tau$ is arbitrary, the proof of the proposition is complete.

We conclude this section with a brief discussion of the $G$-isomorphism of the spaces $O(D, V_\tau)$ and $C^\infty(G, \tau; p_-)$. In particular, we give a characterization of the unique irreducible $g$-module in $C^\infty(G, \tau; p_-)$ which is used in connection with the differential operators defined in §7.

For $f \in O(D, V_\tau)$, define the function $\Theta F$ on $G$ by the formula
\[
\Theta F(g) = J_\tau(g, 0)^{-1} F(g \cdot 0).
\]
It is readily checked that $\Theta F \in C^\infty(G, \tau; p_-)$ and $\Theta T(g) = L(g) \Theta$.

The mapping $\Theta$ is invertible with inverse given by
\[
\Theta^{-1} f(\zeta) = J_\tau(g, 0) f(g), \quad \zeta \in D,
\]
where $g \in G$ satisfies $g \cdot 0 = \zeta$.

Let $C_0^\infty(G, \tau; p_-)$ denote the $K$-finite vectors in $C^\infty(G, \tau; p_-)$. Then $\Theta(V_\tau) = C_0^\infty(G, \tau; p_-)$ and we put $L(\tau)^\sim = \Theta(L(\tau))$.

(6.11) Proposition. An element $f \in C_0^\infty(G, \tau; p_-)$ lies in $L(\tau)^\sim$ if and only if the function $z \rightarrow (r(z)^j f)(1)$, $z \in p_+$, lies in $L(\tau)$ for each $j \geq 0$.

Proof. Let $F \in V_\tau$ and $z \in p_+$. By the proof of (5.3) one has
\[
(r(z)^j \Theta F)(1) = (d^j/dt^j) F(\exp(tz) \cdot 0)|_{t=0}
\]
\[
= (d^j/dt^j) F(tz)|_{t=0} = \nabla^j F(0)(z, \ldots, z)
\]
where $\nabla^j F(0)$ denotes the $j$th derivative of $F$ at $0$. By Taylor's theorem, $F(z) = \sum_{j=0}^{\infty} \frac{\nabla^j F(0)(z, \ldots, z)}{j!} = \sum_{j=0}^{\infty} (r(z)^j \Theta F)(1)/j!$. Setting $f = \Theta F$ gives

$$
\Theta^{-1} f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} (r(z)^j f)(1), \quad \text{for } f \in C_c^\infty(G, \tau; p_-).
$$

Since $\Theta^{-1}(L(\tau)\sim) = L(\tau)$ and $L(\tau)$ is graded, the proposition now follows.

7. Gradient-type differential operators. Gradient-type differential operators are distinguished differential operators which intertwine the $G$-actions on spaces of $K$-covariant functions on $G$ (cf. [3], [11]). First order gradient-type operators were used in connection with certain unitary representations of $SU(p, q)$ [4]. For our purposes we require systems of such operators. These systems may be viewed as $G$-equivariant analogues of the systems given in (2.9) of [2]. Here we associate to certain representations $\tau$ of $K$ a finite system $\mathcal{D}_\tau$ of gradient-type operators and show that the $K$-finite vectors in the kernel of this system is an irreducible highest weight $g$-module. In the context of unitarity, it was shown in [3] that $\mathcal{D}_\tau$ can be chosen to consist of a single operator.

Let $\hat{K}$ denote the set of irreducible unitary representations of $K$. For $(\tau, V_\tau)$ in $\hat{K}$, let $p_+$ act trivially on the $t$-module $V_\tau$ and form the generalized Verma module $\mathcal{N}(\tau) = U(g) \otimes_{U(t \oplus p_-)} V_\tau$. Identifying $p_-$ with the dual of $p_+$ gives a $t$-isomorphism $\mathcal{N}(\tau) \to V_\tau$. Denote the $g$-structure on $V_\tau$ arising from $\mathcal{N}(\tau)$ by juxtaposition. The $g$-module $\mathcal{N}(\tau)$ contains a unique maximal submodule; let $M_\tau$ denote its image in $V_\tau$. Define the subset $\hat{K}_r$ of $\hat{K}$ by

$$
\hat{K}_r = \{ \tau \in \hat{K} : M_\tau \neq 0 \}.
$$

Since $\hat{K}_r$ corresponds to the representations of $K$ for which $\mathcal{N}(\tau)$ is reducible, we call $\hat{K}_r$ the set of reduction points. This terminology is taken from the Enright, Howe and Wallach classification of unitary highest weight modules (cf. [7]). Those $\tau \in \hat{K}_r$ for which $L(\tau)$ is unitarizable appear in that classification as either the endpoint of a half-line or as isolated equally spaced points.

The $g$-structure on $V_\tau$ arising from $\mathcal{N}(\tau)$ is the conjugate dual of $T$ given in (6.3). To make this terminology more precise, we recall the inner product (6.5) on $V_\tau$. The two $g$-structures on $V_\tau$ are then related by the formula
where \( \varepsilon \) is the anti-automorphism given in §6. It follows from (7.2) that the orthogonal complement of a \( g \)-module is a \( T(g) \)-module and vice versa. In particular, note that \( L(\tau) = M^{\perp}_\tau \) is properly contained in \( V_\tau \) if and only if \( \tau \in \hat{K}_r \).

Using the Killing form, we identify the dual of \( p_- \) with \( p_+ \). Let \( \{X_i\} \) denote a basis of \( p_- \) and let \( \{X'_i\} \subset p_+ \) denote the dual basis. Following [3], we define the operator \( \nabla : C^\infty(G, \tau) \to C^\infty(G, \tau \otimes \text{Ad}|_{p_-}) \) by the formula

\[
\nabla f(g) = \sum_i r(X'_i)f(g) \otimes X_i.
\]

Then \( \nabla \) is easily seen to be independent of the basis \( \{X_i\} \) and clearly intertwines the left \( G \)-actions. For each integer \( j \geq 1 \), \( \nabla^j f(g) \) is a symmetric \( j \)-tensor and thus may be regarded as homogeneous polynomial of degree \( j \) on \( p_+ \). One readily checks the formula

\[
(\nabla^j f(g))(z) = r(z)^j f(g), \quad g \in G, z \in p_+.
\]

Let \( W \subset V_\tau \) denote an irreducible \( T(K) \)-space and let \( \tau_W \) denote the restriction of \( T(K) \) to \( W \). Since the space of homogeneous polynomials of a fixed degree is \( T(K) \)-invariant, there exists a unique integer \( j \geq 0 \) such that \( W \subset V^j_\tau \). Let \( P_W \) denote orthogonal projection of \( V^j_\tau \) onto \( W \). We then have

\[
\tau_W(k)P_W = P_W T(k), \quad k \in K.
\]

We refer to the differential operator \( \partial_W : C^\infty(G, \tau) \to C^\infty(G, \tau_W) \) defined by

\[
\partial_W f(g, \cdot) = P_W (r(\cdot)^j f(g))
\]
as a gradient-type differential operator.

We henceforth assume that \( \tau \in \hat{K}_r \). Since \( V_\tau \), viewed as a \( S(p_-) \)-module, is Noetherian, it follows that the submodule \( M_\tau \) is finitely generated over \( S(p_-) \). Let \( \{m_i : 1 \leq i \leq r\} \) be generators of \( M_\tau \) over \( S(p_-) \) and let \( W \) denote the \( K \)-invariant space generated by \( \{m_i\} \). Decompose \( W \) into \( K \)-irreducible spaces \( W_1, \ldots, W_s \) and put \( \partial_i = \partial_{W_i} \). We call \( \mathcal{D_\tau} = \{\partial_1, \ldots, \partial_s\} \) a system of gradient-type differential operators associated to \( \tau \). We remark that the notation for a system \( \mathcal{D_\tau} \) does not reflect its dependence on the choice of generators \( \{m_i\} \). However, the next result shows that the kernel of such a system \( \mathcal{D_\tau} \) in \( C^\infty(G, \tau; p_-) \) is in fact independent of the generators.
(7.6) Proposition. Let $\tau \in \hat{K}_r$ and let $\mathcal{D}_\tau$ be any system of gradient-type differential operators associated to $\tau$. Then the kernel of $\mathcal{D}_\tau$ in $C^\infty_\circ(G, \tau; p_-)$ is $L(\tau)^\sim$.

Proof. Let $\mathcal{D}_\tau$ be any system of gradient-type differential operators associated to $\tau$. Suppose $f \in \Ker(\mathcal{D}_\tau) \cap C^\infty_\circ(G, \tau; p_-)$. Then $\partial_i f = 0$ for all $i = 1, \ldots, s$. With the notation of the preceding paragraph, let $j_i$ denote the positive integer for which $W_i \subset \nabla^j_i$. We then have $(r(\cdot)^j f(g), W_i) = 0$, $1 \leq i \leq s$. Since $\nabla^j_i$ is orthogonal to $\nabla^j_m$ for $j \neq m$, we have $(r(\cdot)^j f(g), W_i) = 0$, $1 \leq i \leq s$ and $j \geq 0$. In particular, since $\{m_1, \ldots, m_r\} \subset \bigoplus_{i=1}^s W_i$ we have

$$
(7.7) \quad (r(\cdot)^j f(g), m_i) = 0, \quad 1 \leq i \leq r, j \geq 0.
$$

Now let $y \in p_+$. By (6.2) and (6.3) one has

$$
(7.8) \quad r(y)r(z)^j f(g) = \frac{-1}{j+1} T(y)(r(z)^{j+1} f(g)), \quad z \in p_+.
$$

Let $x \in S_k(p_-)$. Since $\deg(m_i) > 0$, if $k \geq j$ then $(r(\cdot)^j f(g)|xm_i) = 0$. If $k < j$, then we have

$$(r(\cdot)^j f(g)|xm_i) = (T(\epsilon(x))(r(\cdot)^j f(g)), m_i) \quad \text{by (7.2)}$$

$$= C(r(\epsilon(x))r(\cdot)^{j-k} f(g), m_i), \quad C \in \mathbb{R}, \quad \text{by (7.8)}$$

$$= C(r(\epsilon(x))(r(\cdot)^{j-k} f(g), m_i) = 0 \quad \text{by (7.7)}.$$

Now $M_\tau$ is the linear span of elements of the form $xm_i$ and $L(\tau) = M_\tau^\perp$. It follows that for each $j \geq 0$ and $g \in G$, the function $z \rightarrow r(z)^j f(g)$, $z \in p_+$, lies in $L(\tau)$. Letting $g = 1$, we conclude by (6.11) that $f \in L(\tau)^\sim$. Thus $\Ker(\mathcal{D}_\tau) \cap C^\infty_\circ(G, \tau; p_-) \subset L(\tau)^\sim$.

Since $L(\tau)^\sim$ is irreducible, it remains to show that the $g$-module $\Ker(\mathcal{D}_\tau) \cap C^\infty_\circ(G, \tau; p_-)$ is nontrivial. Let $B_\tau: V_\tau \rightarrow W(\tau)$ denote the map defined in §3 which corresponds to the inclusion map $C_\tau: H_\sigma \rightarrow V_\tau$. The vector $S_\tau B_\tau \phi_\lambda$ is clearly a nonzero $K$-finite vector. The explicit calculation in the proof of (8.1) shows in particular that $S_\tau B_\tau \phi_\lambda \in \Ker(\mathcal{D}_\tau)$. Consequently, the space $\Ker(\mathcal{D}_\tau) \cap C^\infty_\circ(G, \tau; p_-)$ equals $L(\tau)^\sim$.

8. Irreducibility of the image of $S_\tau$. In §7 we let $\hat{K}_r$ denote the irreducible unitary representations of $K$ such that the irreducible $g$-module $L(\tau)$ generated by the constant polynomials $\nabla^\circ_\tau$ is properly contained in $\nabla_\tau$. The relevancy of the set $\hat{K}_r$ of reduction points is apparent for the following reason. If $\tau$ is not a reduction point, then the subspace $C^\infty_\circ(G, \tau; p_-)$ of $K$-finite vectors is canonically isomor-
phic to $L(\tau)$. Since the (nonzero) image of $S_\tau$ lies in $C^\infty(G, \tau; p_-)$ by (5.3), it follows from the $G$-invariance of $S_\tau$ that the space of $K$-finite vectors in the image of $S_\tau$ is $C_0^\infty(G, \tau; p_-)$. Thus the $K$-finite vectors in the image of $S_\tau$ form an irreducible highest weight space with highest weight $\lambda = \lambda(\tau)$. The main theorem here extends this result to the case where $\tau$ is a reduction point. The proof of this result utilizes the system $D_\tau$ defined in section seven.

Recall from §3 that $W(\tau)$ denotes the nonunitary principal series representation with imbedding parameters $\sigma$ and $\nu$ chosen according to (3.9) and (3.10). As in §7, we let $D_\tau = \{\partial_1, \ldots, \partial_3\}$ be any system associated to $\tau \in \tilde{K}_r$.

(8.1) **Theorem.** The Szegö operator $S_\tau$ associated to $\tau \in \tilde{K}_r$ maps $W(\tau)$ into the kernel of $D_\tau$ and the $K$-finite vectors in $W(\tau)$ are mapped $g$-equivariantly onto an irreducible highest weight module with highest weight $\lambda = \lambda(\tau)$.

**Proof.** By (5.3), $S_\tau$ maps into $C_0^\infty(G, \tau; p_-)$. We now show that each $\partial_i \in D_\tau$ annihilates the image of $S_\tau$. Let $\partial_i \in D_\tau$. With the notation of §7, we have $\partial_i f(g, \cdot) = P_i(\tau(\cdot)\cdot^i f(g))$ where $P_i: \mathcal{V}_i^i \to W_i$ denotes projection. Since $\partial_i$ and $S_\tau$ are both $G$-equivariant, it suffices to show that $(\partial_i S_\tau f)(1) = 0$ for all $f$ in $W(\tau)$.

Let $c_g: G \to G$ denote conjugation by $g: c_g(x) = gxg^{-1}$. Then for $k \in K$ and integers $j \geq 1$, one has

(8.2) \[ r(z)^j(f \circ c_k) = (r(\operatorname{Ad}(k)z)^j f) \circ c_k, \quad f \in C^\infty(G). \]

For $f$ in $W(\tau)$ we compute

\[
(\partial_i S_\tau f)(1) = \int_K \partial_i(s_\tau(k^{-1}g))|_{g=1} f(k) \, dk \\
= \int_K \partial_i(s_\tau(k^{-1}g k k^{-1}))|_{g=1} f(k) \, dk \\
= \int_K \partial_i(\tau(k)(s_\tau \circ c_k^{-1}))(g)|_{g=1} f(k) \, dk \quad \text{by (3.7)} \\
= \int_K P_i \tau(k)(r(\cdot)^i)(s_\tau \circ c_k^{-1}))(g)|_{g=1} f(k) \, dk \\
= \int_K P_i \tau(k)(r(\operatorname{Ad}(k^{-1}) \cdot)^i, s_\tau)(g)|_{g=1} f(k) \, dk, \quad \text{by (8.2)} \\
= \int_K P_i(T(k)(r(\cdot)^i, s_\tau)(g))|_{g=1} f(k) \, dk, \quad \text{by (6.6)} \\
= \int_K \tau_i(k) \partial_i(s_\tau(g))|_{g=1} f(k) \, dk \quad \text{by (7.4)},
\]
where \( \tau_i(k) \) denotes the restriction of \( T(k) \) to \( W_i \). It suffices to show that for each \( \phi \in H_\sigma \) we have \( \partial_i(s_\tau(g)\phi)|_{g=1} = 0 \). But by Propositions (4.16) and (6.10) we find

\[
\partial_i(s_\tau(g)\phi)|_{g=1} = \partial_i(J_\tau(g^{-1}, b)^*1\phi)|_{g=1} = P_i(T(b)^i, 1\phi).
\]

Since \( T(b)^i, 1\phi \in L(\tau) \) and \( P_i \) projects onto \( W_i \subset M_\tau = L(\tau)\perp \), we conclude \( P_i(T(b)^i, 1\phi) = 0 \) for all \( \phi \in H_\sigma \). This proves the image of \( S_\tau \) lies in \( \text{Ker}(\mathcal{D}_\tau) \). By (7.6), \( S_\tau \) maps the space of \( K \)-finite vectors in \( W(\tau) \) to a nonzero \( g \)-module in \( \text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; p_-) = L(\tau)\perp \). Since \( L(\tau)\perp \) is irreducible, the theorem now follows.

**References**


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