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ROBERT FITZGERALD

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ROBERT W. FITZGERALD

For a separable odd degree field extension K/F the kernel of a Scharlau transfer of Witt rings $s_* : WK \rightarrow WF$ is a WF -module. We compute the prime ideals attached to $\ker s_*$ and deduce that WK is not a projective WF -module if an ordering on F extends uniquely to K . An example shows WK may be a free WF -module if F is real and no ordering extends uniquely. For non-real, non-rigid F we show that K/F Galois and WK noetherian implies WK is not a projective WF -module.

If K/F is a finite extension of fields (characteristic not 2) then each non-trivial linear functional $s : K \rightarrow F$ induces a Scharlau transfer $s_* : WK \rightarrow WF$ on the Witt rings. When $K = F(\sqrt{d})$ the kernel and image of s_* are well known. We restrict our attention to separable odd degree extensions, where s_* is surjective but little is known of $\ker s_*$. The map induced by inclusion $r_* : WF \rightarrow WK$ is injective and we view WF as a subring of WK . Then WK and $\ker s_*$ are WF -modules and our approach is module theoretic.

WF need not be noetherian and $\ker s_*$ need not be finitely generated over WF . So the usual theory of prime ideals associated to modules must be replaced by the notion of attached primes (in the sense of Dutton). We show no $P(\alpha, p)$ is attached to $\ker s_*$, $P(\alpha)$ is attached iff α has more than one extension to K and IF is attached iff $W_t K \cap \ker s_* \neq 0$. As a consequence, $WK = WF$ iff each ordering on F extends uniquely to K and $W_t K \cap \ker s_* = 0$. Another consequence is that WK is finitely generated over WF if F has only finitely many orderings and IF is not attached to $\ker s_*$.

The main result deduced from the work on attached primes is that WK is not a projective WF -module if some ordering on F extends uniquely to K . WK may be projective, however, if F is real and no ordering extends uniquely. We present an example where K/F is Galois, F is real, both WK and WF noetherian rings and WK is a free WF -module. When F is non-real and non-rigid we show the same conditions (K/F Galois, WK and WF noetherian) implies WK is not a free WF -module. Weaker results hold under fewer restrictions on K/F .

The first section gives basic results and several examples. The last section concerns the possible values of $[G(K) : G(F)]$ when this is finite (here $G(E) = E^*/E'^2$). Two sample results: If K/F is Galois and $[K : F] = p$ a prime then p divides $[G(K) : G(F)] - 1$. If K/F has a real normal closure then $[K : F] \leq [G(K) : G(F)]$.

$\text{Hom}(K, F)'$ denotes the non-trivial linear functionals $s : K \rightarrow F$. The set of orderings on a field E is denoted X_E . If $\alpha \in X_F$ then $X(\alpha) = \{\beta \in X_K \mid \beta|F = \alpha\}$. For $\alpha \in X_F$ and an odd prime p we write $P(\alpha, p)$ for $\{r \in WF \mid \text{sgn}_\alpha r \equiv 0 \pmod{p}\}$ and $P(\alpha) = \{r \in WF \mid \text{sgn}_\alpha r = 0\}$. These ideals, with $IF = \{r \in WF \mid \dim r \equiv 0 \pmod{2}\}$, are the prime ideals of WF .

$W_t F$ denotes the torsion part of WF . The height of F , $h(F)$, is the least positive k such that $2^k \cdot W_t F = 0$ (or infinity if no such k exists). If R_1 and R_2 are Witt rings then the fiber product $R_1 \square R_2 = \{(r_1, r_2) \mid r_i \in R_i, \dim r_1 \equiv \dim r_2 \pmod{2}\}$ is again a Witt ring. If C is a group of exponent two then the group ring $R_1[C]$ is again a Witt ring.

1. Basic facts.

DEFINITION. (i) $m(K/F) = \bigcap \ker s_*$, over all $s \in \text{Hom}(K, F)'$.
(ii) $M(K/F) = \sum \ker s_*$, over all $s \in \text{Hom}(K, F)'$.

LEMMA 1.1. Let $s \in \text{Hom}(K, F)'$.

- (1) $\ker s_*$ is a WF -submodule of WK .
- (2) If $t \in \text{Hom}(K, F)'$ then $\ker s_* = \langle z \rangle \ker t_*$ for some $z \in K'$.
- (3) $m(K/F) = [\ker s_* : WK]$ is an ideal of WK .
- (4) $M(K/F)$ is the ideal generated by $\ker s_*$.
- (5) There exists $t \in \text{Hom}(K, F)'$ with $t_*(1) = \langle 1 \rangle$.
- (6) If $s_*(1) = \langle 1 \rangle$ then $WK \approx WF \oplus \ker s_*$.
- (7) If $s_*(1) = \langle 1 \rangle$ then $\ker s_*$ is generated (over WF) by $\{\langle x \rangle - s_*\langle x \rangle \mid x \in K'\}$.

Proof. (1) s_* is additive and if $\phi \in \ker s_*$ and $r \in R$ then $s_*(r\phi) = rs_*(\phi) = 0$. Thus $\ker s_*$ is a WF -submodule of WK .

(2) There exists $z \in K'$ such that $s(x) = t(zx)$ for all $x \in K$. Then $s_*(\phi) = t_*(\langle z \rangle \phi)$ for all $\phi \in WK$ and so $\ker s_* = \langle z \rangle \ker t_*$.

(3) Let $\phi \in m(K/F)$ and $z \in K'$. Define $t(x)$ to be $s(zx)$ for all $x \in K$. Then $\phi \in \ker t_* = \langle z \rangle \ker s_*$. Since z was arbitrary, we have $\phi \in [\ker s_* : WK]$. Conversely, if $\phi \in [\ker s_* : WK]$ then for every $z \in K'$, $\langle z \rangle \phi \in \ker s_*$ and $\phi \in \langle z \rangle \ker s_* = \ker t_*$, for some

$t \in \text{Hom}(K/F)'$. Thus $\phi \in m(K/F)$. Clearly $[\ker s_* : WK]$ is an ideal.

(4) $M(K/F) = \sum \ker t_* = \sum \langle z \rangle \ker s_*$ is the ideal generated by $\ker s_*$.

(5) We may write $K = F(x)$ since K is separable over F . Take $t \in \text{Hom}(K, F)'$ with $t(1) = 1$ and $t(x) = \dots = t(x^{n-1}) = 0$ ($n = [K : F]$). Then $t_*\langle 1 \rangle = \langle 1 \rangle$ by [15, II 5.8].

(6) If $s_*\langle 1 \rangle = \langle 1 \rangle$ then the exact sequence $0 \rightarrow \ker s_* \rightarrow WK \rightarrow WF \rightarrow 0$ splits. This also proves (7). □

There are few examples of Witt rings under odd degree extensions in the literature. We present several to illustrate the range of possible $m(K/F)$ and $M(K/F)$.

EXAMPLES. (1) The definitions of $m(K/F)$ and $M(K/F)$ make sense for any finite extension $F \subset K$. Consider $K = F(\sqrt{d})$ and define $s : K \rightarrow F$ by $s(1) = 0$, $s(\sqrt{d}) = 1$. Then $\ker s_* = r_*(WF)$. Since $\langle 1 \rangle \in \ker s_*$ we have $M(K/F) = WK$. Also, $m(K/F) = \text{ann}_{WF}(\text{ann}_{WF}\langle 1, -d \rangle) \otimes K$ by [5, 2.12]. Note that if WF is Gorenstein (e.g., a group ring extension of a Witt ring of local type) then $\text{ann}_{WF}(\text{ann}_{WF}\langle 1, -d \rangle) = (\langle 1, -d \rangle)$ and hence $m(K/F) = 0$ (cf. [9]).

(2) Let $F = \mathbf{Q}_2$ and $K = \mathbf{Q}_2(e)$ where e is a root of $x^3 + 2$. Then K/K^2 may be represented by the group generated by $\langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle \alpha \rangle, \langle \beta \rangle$ where $\alpha = 2 + e^2$ and $\beta = 1 + e^2$. Define $s : K \rightarrow F$ by $s(1) = 1$, $s(e) = 0$ and $s(e^2) = 0$. Then $s_*\langle 1 \rangle = \langle 1 \rangle$, $s_*\langle \alpha \rangle = \langle 3 \rangle$, $s_*\langle \beta \rangle = \langle 5 \rangle$ and $s_*\langle \alpha\beta \rangle = \langle 2 \rangle\langle 1, -7, -14 \rangle \simeq \langle 2 \rangle\langle 1, 1, 2 \rangle \simeq \langle 1, 1, 1 \rangle$ (see [15, p. 188]). Set $\rho = 4 \cdot \langle 1 \rangle$ and $\chi = 3 \cdot \langle 1 \rangle$.

We verify that $m(K/F) = 0$. Let $\phi = r_1 + r_2\langle \alpha \rangle + r_3\langle \beta \rangle + r_4\langle \alpha\beta \rangle \in m(K/F)$ with $r_i \in WF$. From $s_*\phi = 0$, $s_*\langle \alpha \rangle\phi = 0$ and $s_*\langle \beta \rangle\phi = 0$ we obtain:

$$\begin{aligned} r_1 + \langle 3 \rangle r_2 + \langle 5 \rangle r_3 + \chi r_4 &= 0, \\ \rho r_3 + \rho r_4 &= 0, \\ \rho r_2 + \rho r_4 &= 0. \end{aligned}$$

The last two equations imply $\dim r_2 \equiv \dim r_3 \equiv \dim r_4 \pmod{2}$. The first equation yields $\phi = \langle \alpha, -3 \rangle r_2 + \langle \beta, -5 \rangle r_3 + (\langle \alpha\beta \rangle - \chi) r_4$. When all r_i ($2 \leq i \leq 4$) are even dimensional then $\phi \in I^2K$. When all r_i are odd dimensional then $d(\phi) = 1$ and again $\phi \in I^2K$. But $I^2K = \{0, \rho\}$ and $s_*(\rho) = \rho \neq 0$. Thus $\phi = 0$.

Lastly, $M(K/F) = (\langle 1, -3\alpha \rangle, \langle 1, -5\beta \rangle)$. Namely, $M(K/F)$ is

generated by $\langle 1, -3\alpha \rangle$, $\langle 1, -5\beta \rangle$ and $\chi - \langle \alpha\beta \rangle$. Now $\rho \in \langle 1, -3\alpha \rangle IK$ and $\chi - \langle \alpha\beta \rangle = \rho - \langle 1, \alpha\beta \rangle = \rho - \langle 15 \rangle (\langle 1, -3\alpha \rangle + \langle 3\alpha \rangle \langle 1, -5\beta \rangle)$.

(3) Let $F = \mathbf{C}(x)$. It is easy to see $t^3 + xt + x$ is irreducible over F . Let α be a root and let $K = F(\alpha)$. Pick $s \in \text{Hom}(K, F)^*$ with $s_*(1) = \langle 1 \rangle$. Now for all $u \in K$, $s_*(u) = \langle N_{K/F}(u) \rangle + \phi$, for some $\phi \in I^2K = 0$. We are using here that K is a C_1 -field for every finite extension [15, II 15.2]. So $s_*(u) = \langle N_{K/F}(u) \rangle$, and s_* is a ring homomorphism. Thus $m(K/F) = \ker s_* = M(K/F) = \{ \langle 1, -u \rangle \mid N_{K/F}(u) = 1 \}$.

This is the only example (of the three) for which $m(K/F) \neq 0$. To verify this it is enough to show $-x\alpha \notin K^2$ as $N_{K/F}(-x\alpha) \in F^2$. But if $-x\alpha = (a+b\alpha+c\alpha^2)^2$ then $b = a^2/2cx$ and $(a/c)^4 + 8(a/c)x^2 = 4x^3$. However $t^4 + 8x^2t - 4x^3$ has no roots in F .

(4) In §3 an extension $F \subset K$ will be constructed with $WF \approx \mathbf{Z}$ and $WK \approx \mathbf{Z}^3$. Here $\dot{F}/\dot{F}^2 = \{\pm 1\}$ and $\dot{K}/\dot{K}^2 = \{\pm 1, \pm\alpha, \pm\beta, \pm\alpha\beta\}$. Here α corresponds to $(1, -1, -1) \in \mathbf{Z}^3$ and β corresponds to $(-1, 1, -1)$. There is, by a later result (1.4), an $s \in \text{Hom}(K, F)^*$ with $s_*(1) = \langle 1 \rangle$, $s_*(\alpha) = \langle 1 \rangle$, $s_*(\beta) = \langle 1 \rangle$ and $s_*(\alpha\beta) = -3\langle 1 \rangle$. Thus $\ker s_*$ is generated by $\langle 1, -\alpha \rangle$, $\langle 1, -\beta \rangle$, $\langle 1, 1, 1, \alpha\beta \rangle$. Using $\langle 1, \alpha, \beta, \alpha\beta \rangle = 0$ it is straightforward to show $m(K/F) = 0$ and $M(K/F) = (\langle 1, -\alpha \rangle, \langle 1, -\beta \rangle)$.

For any field E let $G(E) = E^*/E^{*2}$. Set $U = \{ \langle x \rangle \in G(K) \mid N_{K/F}(x) \in \dot{F}^2 \}$.

LEMMA 1.2. $G(K) \approx U \times G(F)$.

Proof. The sequence $1 \rightarrow U \rightarrow G(K) \rightarrow G(F) \rightarrow 1$ is exact and splits since for $a \in F^*$ we have $N_{K/F}(a) = a^m$ where $m = [K:F]$ is odd and so $N_{K/F}(a) \in a\dot{F}^2$. \square

LEMMA 1.3. If $s_*(1) = \langle 1 \rangle$ and $\dim(s_*(x))_{an} = 1$ for some $x \in K^*$ then $s_*(x) = \langle N_{K/F}(x) \rangle$.

Proof. Suppose $[K:F] = 2k+1$. Then $s_*(1) \simeq k \cdot \langle 1, -1 \rangle + \langle 1 \rangle$ so that $\det(s_*(1)) = (-1)^k$. Hence $\det(s_*(x)) = (-1)^k N_{K/F}(x)$ [15, II 5.12] and so $s_*(x) = \langle N_{K/F}(x) \rangle$. \square

PROPOSITION 1.4. Let $s \in \text{Hom}(K, F)^*$ with $s_*(1) = \langle 1 \rangle$. Set $L(s) = \{ \langle y \rangle \in G(K) \mid N_{K/F}(y) \in F^2 \text{ and } s_*(y) = \langle 1 \rangle \}$. Then:

- (1) $\{ \langle 1, -y \rangle \mid y \in L(s) \} \subset \ker s_*$, and
- (2) $L(s)L(s) = U$.

Proof. (1) is clear as is the inclusion $L(s)L(s) \subset U$. Suppose then that $\beta \in U$ and set $E = F(\beta)$. Define $v: E \rightarrow F$ by $v(1) = 1$ and $v(\beta^i) = 0$, $1 \leq i < [E:F]$. Then $v_*\langle 1 \rangle = \langle 1 \rangle$ and $v_*\langle \beta \rangle = \langle 1 \rangle$ [15, II 5.8] (note $N_{E/F}(\beta) = 1$ as $1 = N_{K/F}(\beta) = N_{E/F}(N_{K/E}(\beta)) = N_{E/F}(\beta)$, modulo squares). Pick any $u \in \text{Hom}(K, E)^*$ with $u_*\langle 1 \rangle = \langle 1 \rangle$. Then $(vu)_*\langle 1 \rangle = \langle 1 \rangle$ and $(vu)_*\langle \beta \rangle = v_*(u_*\langle \beta \rangle) = v_*\langle \beta \rangle = \langle 1 \rangle$, as $\beta \in E$. Thus $\{1, \beta\} \subset L(vu)$. Now there exists $z \in K^*$ with $vu(x) = s(zx)$ for all $x \in K$. Note $\langle 1 \rangle = (vu)_*\langle 1 \rangle = s_*(z)$ so that $N_{K/F}(z) \in F^2$ by (1.3). Also $zL(vu) = L(s)$. Thus $z, z\beta \in L(s)$ and $\beta \in L(s)L(s)$. \square

PROPOSITION 1.5. $m(K/F) \subset W_1K$, the torsion ideal of WK .

Proof. If $x \in K^*$ and $\phi \in m(K/F)$ then $\text{tr}_*(\langle x \rangle \phi) = 0$ where tr is the trace map $\text{tr}_{K/F}$. Let $Q \in X_K$ and let $P = Q \cap F$. Since $X(P)$ is finite, we may find a Pfister form p and integer m with $\text{sgn}_Q(p) = 2^m$ and $\text{sgn}_{Q'}(p) = 0$ for $Q' \in X(P) - \{Q\}$. Then by [15, III 4.5]:

$$0 = \text{sgn}_p \text{tr}_*(p\phi) = \sum_{Q' \in X(P)} \text{sgn}_{Q'}(p\phi) = 2^m \text{sgn}_Q(\phi).$$

Thus $\text{sgn}_Q(\phi) = 0$ and as Q was arbitrary, we have $\phi \in W_1K$. \square

PROPOSITION 1.6. Suppose $s \in \text{Hom}(K, F)^*$ satisfies $s_*\langle 1 \rangle = \langle 1 \rangle$. Let $m = [K:F]$ and set $k = (m-1)/2$ and $n = m - (-1)^k$. Let $J \subset WK$ be the ideal generated by $\{\langle 1, -y \rangle | y \in U\}$. Then:

- (1) $M(K/F) = J + (\{\langle 1 \rangle - s_*\langle y \rangle | y \in U\})$.
- (2) If K/F is Galois then $n \cdot \langle 1 \rangle \in M(K/F)$.
- (3) If K/F is Galois then $M(K/F) = J$.

Proof. (1) $J \subset M(K/F)$ by (1.4). If $N_{K/F}(y) \in F^2$ then $\langle y \rangle - s_*\langle y \rangle \in \ker s_* \subset M(K/F)$ and $\langle 1 \rangle - s_*\langle y \rangle = \langle 1, -y \rangle + \langle y \rangle - s_*\langle y \rangle \in M(K/F)$. Conversely, $M(K/F)$ is generated by $\ker s_*$, by (1.1), which is generated by $\langle y \rangle + s_*\langle y \rangle$, for $y \in U$. And $\langle y \rangle - s_*\langle y \rangle = -\langle 1, -y \rangle + (\langle 1 \rangle - s_*\langle y \rangle) \in J + (\{\langle 1 \rangle - s_*\langle y \rangle | y \in U\})$.

(2) Let $G = \text{Gal}(K/F)$. Let $\text{tr} = \text{tr}_{K/F}: K \rightarrow F$. There exists $z_0 \in K^*$ with $\text{tr}_*\langle z_0 \rangle = s_*\langle 1 \rangle = \langle 1 \rangle$. So $(-1)^k = \det \text{tr}_*\langle z_0 \rangle = (\det \text{tr}_*\langle 1 \rangle)N_{K/F}(z_0) = N_{K/F}(z_0)$, as $\text{tr}_*\langle 1 \rangle = m\langle 1 \rangle$. Set $z = (-1)^k z_0$. Then $N_{K/F}(z) \in F^2$ and $\text{tr}_*\langle z \rangle = \langle (-1)^k \rangle$. Thus $\langle (-1)^k \rangle = \sum_G \langle g(z) \rangle$ and $\sum_G \langle 1, -g(z) \rangle = |G|\langle 1 \rangle - \langle (-1)^k \rangle = n\langle 1 \rangle \in J \subset M(K/F)$.

(3) If $N_{K/F}(y) = 1$ then we need to show $\langle 1 \rangle - s_* \langle y \rangle \in J$. Pick z_0 and $z = (-1)^k z_0$ as in (2). Then $\langle 1 \rangle - s_* \langle y \rangle = \langle 1 \rangle - \text{tr}_* \langle y z_0 \rangle = \langle 1 \rangle - \sum_G \langle g(y z_0) \rangle = \langle 1 \rangle - (-1)^k \sum \langle g(y z) \rangle = \langle 1 \rangle + (-1)^k \sum \langle 1, -g(y z) \rangle - (-1)^k m \langle 1 \rangle$. As $N_{K/F}(y z) = 1$ we have each $\langle 1, -g(y z) \rangle \in J$. Also $(1 - (-1)^k m) \langle 1 \rangle \in J$ by the proof of (2) and so $\langle 1 \rangle - s_* \langle y \rangle \in J$. \square

COROLLARY 1.7. *Suppose K/F is Galois and $s_* : WK \rightarrow WF$ is a ring homomorphism. Let $m = [K : F]$ and $k = (m - 1)/2$. Then $(m - (-1)^k) \langle 1 \rangle = 0$. In particular, F is non-real.*

Proof. Here $(m - (-1)^k) \langle 1 \rangle \in M(K/F) = \ker s_*$, using (1.6). Yet $s_* \langle 1 \rangle = \langle 1 \rangle$, so that $(m - (-1)^k) \langle 1 \rangle = 0$. \square

COROLLARY 1.8. *Suppose K/F is Galois. Let $m = [K : F]$, $k = (m - 1)/2$ and $n = m - (-1)^k$. Let 2^a be the largest 2-power dividing n . If $|X_K| < \infty$ and the height $h(K)$ is finite then $2^a \in M(K/F)$.*

Proof. Write $n = 2^a \cdot b$, where b is odd. If K is non-real then $b \langle 1 \rangle$ is a unit in WK and so $2^a \in M(K/F)$ by (1.6)(2). Suppose then that K is real. Let $Q \in X_K$. We Claim $U \not\subset \text{pc}(Q)$, the positive cone of Q . Namely, suppose $U \subset \text{pc}(Q)$. Then $\text{pc}(Q) = U \cdot \text{pc}(P)$ where $P = Q \cap F$. If $S \in X(P) - \{Q\}$ (and such an S exists as $|X(P)| = [K : F]$) then $\text{pc}(S) = g(\text{pc}(Q))$ for some $g \in \text{Gal}(K/F)$. But $g(U) = U$ and g fixes F so that $\text{pc}(S) = g(U \cdot \text{pc}(P)) = U \cdot \text{pc}(P) = \text{pc}(Q)$, a contradiction.

The Claim shows that the only prime ideal to contain $M(K/F) = (\{ \langle 1, -y \rangle \mid y \in U \})$ is IF . By primary decomposition [8, 2.3], $M(K/F)$ is IF -primary. Since no power of b is in $M(K/F) \subset IF$ we have $2^a \in M(K/F)$. \square

2. Attached primes. For modules M over non-noetherian rings R there are several notions of associated primes (cf. [10]). We will use three:

$$\text{Ass}(M) = \{P \in \text{Spec}(R) \mid P = \text{ann}_R(m), \text{ some } m \in M\}$$

$$\text{Asf}(M) = \{P \in \text{Spec}(R) \mid P \text{ minimal over some } \text{ann}_R(m)\}$$

$$\text{Att}(M) = \{P \in \text{Spec}(R) \mid \text{for all f.g. ideals } I \subset P, \text{ there exists } m \in M \text{ with } I \subset \text{ann}_R(m) \subset P\}$$

$\text{Ass}(M)$ is given by the usual definition of associated primes in the noetherian case. $\text{Asf}(M)$ is denoted by $\text{Ass}_f(M)$ in [10] and $\text{Att}(M)$

is denoted by $sK(M)$ there. Primes in $\text{Att}(M)$ are called *primes attached to M* (following Dutton [3]).

LEMMA 2.1. *Let R be a commutative ring and M an R -module.*

(1) $\text{Ass}(M) \subset \text{Asf}(M) \subset \text{Att}(M)$, with equality if R is noetherian.

(2) $\text{Asf}(M) \neq 0$ iff $M \neq 0$.

(3) If $s, t \in \text{Hom}(K, F)$ then $\mathcal{A}(\ker s_*) = \mathcal{A}(\ker t_*)$ for $\mathcal{A} = \text{Ass}, \text{Asf}$ and Att .

Proof. (1) and (2) are clear cf. [10, p. 346]. For (3) note that $\ker s_* = \langle z \rangle \ker t_*$ for some $z \in K$ by (1.1) and $\text{ann}_{WF}(\langle z \rangle m) = \text{ann}_{WF}(m)$. \square

We remark that equality in (2.1)(1) can fail at either place for non-noetherian R , cf. [10].

LEMMA 2.2. *Let M be a WF -submodule of WK . No $P(\alpha, p)$ is attached to M (where $\alpha \in X_F$, p an odd prime).*

Proof. WK contains no odd dimensional zero-divisors, hence $pm \neq 0$ for all $0 \neq m \in M$. Thus if $\text{ann}_{WF}(m) \subset P(\alpha, p)$ then $m \neq 0$ and $(p) \not\subset \text{ann}_{WF}(m)$. So $P(\alpha, p) \notin \text{Att}(M)$. \square

PROPOSITION 2.3. *Let M be a WF -submodule of WK . The following are equivalent:*

(1) $M \cap W_t K \neq 0$.

(2) $IF \in \text{Att}(M)$.

(3) $IF \in \text{Asf}(M)$.

(4) $zd(M) = IF$.

Proof. (1) \rightarrow (2). By [3, Cor. to Prop. 6], $zd(M) = \bigcup_{P \in \text{Att}(M)} P$. If $M \cap W_t K \neq 0$ then $2^k \in zd(M)$ for some k and so $2^k \in P$, for some prime P attached to M . But then $P = IF$.

(2) \rightarrow (4). By (2.2) we have that $\text{Att}(M)$ consists of some $P(\alpha)$ and possibly IF . Thus every $P \in \text{Att}(M)$ is contained in IF . If $IF \in \text{Att}(M)$ then $IF = \bigcup_{\text{Att}(M)} P = zd(M)$.

(4) \rightarrow (1) is clear as then $2 \in zd(M)$. (3) \rightarrow (2) is clear by (2.1). For (1) \rightarrow (3) note that we have $2^k m = 0$ for some $m \in M$. IF is minimal over $2^k \langle 1 \rangle$ so that $IF \in \text{Asf}(M)$. \square

COROLLARY 2.4. *Let M be a WF -submodule of WK . Then $\text{Asf}(M) = \text{Att}(M)$.*

Proof. We need only show $\text{Att}(M) \subset \text{Asf}(M)$ by (2.1). Let $P \in \text{Att}(M)$. P is not any $P(\alpha, p)$ by (2.2) and if $P = IF$ then $P \in \text{Asf}(M)$ by (2.3). So suppose $P = P(\alpha)$ for some $\alpha \in X_F$. Then for some $m \in M$ $\text{ann}_{WF}(m) \subset P(\alpha)$ and clearly $P(\alpha)$ is minimal over $\text{ann}_{WF}(m)$. Thus again $P \in \text{Asf}(M)$. \square

THEOREM 2.5. *Let $s \in \text{Hom}(K, F)$ and let $\alpha \in X_F$. Then $P(\alpha)$ is attached to $\ker s_*$ iff $|X(\alpha)| > 1$.*

Proof. Suppose first that $|X(\alpha)| > 1$. Let $\beta, \gamma \in X(\alpha)$ be distinct and choose $e \in K$ with $e >_\beta 0$ and $e <_\gamma 0$. We may assume $s_*(1) = \langle 1 \rangle$ by (1.1) and (2.1). Thus $x = \langle 1, e \rangle - s_*\langle 1, e \rangle \in \ker s_*$ and $\text{sgn}_\beta x = 2 - \text{sgn}_{\alpha s_*}\langle 1, e \rangle$ while $\text{sgn}_\gamma x = -\text{sgn}_{\alpha s_*}\langle 1, e \rangle$. Hence $x \notin P(\beta) \cap P(\gamma)$. We may assume $x \notin P(\beta)$.

We claim $\text{ann}_{WF}(x) \subset P(\alpha)$. Suppose $r \in WF$ and $rx = 0$. Then $rx \in P(\beta)$ and so $r \in P(\beta) \cap WF = P(\alpha)$. This proves the claim, and since $P(\alpha)$ is a minimal prime, shows $P(\alpha) \in \text{Asf}(\ker s_*) = \text{Att}(\ker s_*)$.

Next, suppose $P(\alpha) \in \text{Att}(\ker s_*)$. Assume, if possible, that $|X(\alpha)| = 1$. Denote by α also its unique extension to K . Suppose $\text{ann}_{WF}(x) \subset P(\alpha)$ for some $x \in \ker s_*$. We may assume $s = \text{tr}_{K/F}$ by (2.1). Thus $0 = \text{sgn}_{\alpha s_*}(x) = \text{sgn}_\alpha x$ by [15, III 4.5]. Hence $x \in P(\alpha)$.

Let $A = \{\delta \in X_K \mid x \in P(\delta)\}$; A is clopen. The complement A' is clopen and so is $B = \varepsilon_{K/F}(A')$, where $\varepsilon_{K/F}(Q) = Q \cap F$, by the Open Mapping Theorem [6, 4.9]. By the Normality Theorem [4, 3.2], there exists an $r \in WF$ such that $\text{sgn}_\delta r = 0$ if $\delta \in B$ and $\text{sgn}_\delta(r) = 2^n$ if $\delta \notin B$ (some fixed n). We note that $\alpha \notin B$ since $\alpha \in A$, $\alpha \notin A'$ and $\varepsilon_{K/F}^{-1}(\alpha) = \{\alpha\}$ is disjoint from A' .

Let $\delta \in X_K$. If $\delta \in A'$ then $\beta \equiv \varepsilon_{K/F}(\delta) \in B$ and so $\text{sgn}_\delta(rx) = 0$, as $\text{sgn}_\delta(r) = \text{sgn}_\beta(r) = 0$. If $\delta \in A$ then $\text{sgn}_\delta(rx) = 0$ as $\text{sgn}_\delta(x) = 0$. Hence $rx \in W_t K$ and $2^k rx = 0$ for some k . That is, we have $2^k r \in \text{ann}_{WF}(x) \subset P(\alpha)$. But $\text{sgn}_\alpha(2^k r) = 2^{k+n}$, as $\alpha \notin B$, a contradiction. \square

COROLLARY 2.6. *Suppose $\ker s_* \neq 0$. The following are equivalent:*

- (1) $\ker s_* \subset W_t K$.
- (2) $M(K/F) \subset W_t K$.
- (3) Every ordering on F extends uniquely to K .
- (4) $\text{tr}_*(1)$ is a unit.
- (5) $\text{Att}(\ker s_*) = \{IF\}$.

Proof. (1) \leftrightarrow (2) follows as $\ker s_*$ generates $M(K/F)$ by (1.1). (3) \leftrightarrow (4) is [15, III 4.5] and [11, VIII 6.4].

(1) \rightarrow (3). Let $\alpha \in X_F$ and let $\beta_1, \beta_2 \in X(\alpha)$. Choose any $e \in K^\cdot$. We assume $s_*\langle 1 \rangle = \langle 1 \rangle$. Then $\langle e \rangle - s_*\langle e \rangle \in \ker s_* \subset W_t K$ and so $0 = \text{sgn}_{\beta_i}\langle e \rangle - \text{sgn}_{\alpha s_*}\langle e \rangle$ for $i = 1, 2$. Thus $\text{sgn}_{\beta_1}e = \text{sgn}_{\beta_2}e$ for all $e \in K^\cdot$. Hence $\beta_1 = \beta_2$.

(3) \rightarrow (1). Let $\alpha \in X_K$ and set $\beta = \alpha \cap F$. Then $\text{sgn}_\beta \text{tr}_*(m) = \text{sgn}_\alpha(m)$ for any $m \in WK$ (tr is the trace $\text{tr}_{K/F}$). Thus if $m \in \ker s_*$ then $\text{sgn}_\alpha m = 0$ and so $m \in W_t K$. Thus $\ker \text{tr}_* \subset W_t K$ and hence $\ker s_* \subset W_t K$.

(3) \rightarrow (5). We have $\text{Att}(\ker s_*) \neq \emptyset$ by (2.1). But (2.2) and (2.5) show only IF could be attached to $\ker s_*$. Lastly, (5) \rightarrow (3) is (2.5). \square

For a field E and form $\phi \in WE$ we write $D(\phi)$, or $D_E(\phi)$ if we need more precision, for the elements of E represented by ϕ . For a positive integer m we will write $D(m)$ for $D(m\langle 1 \rangle)$. Lastly, $D(\infty) = \bigcup_{n \geq 1} D(m)$.

COROLLARY 2.7. *Let $s \in \text{Hom}(K, F)^\cdot$ and suppose $s_*\langle 1 \rangle = \langle 1 \rangle$. Suppose also that $\dim(s_*\langle x \rangle)_{an} = 1$ for all $x \in K^\cdot$. Then:*

- (1) s_* is a ring homomorphism.
- (2) $m(K/F) = \ker s_* = M(K/F) = (\{\langle 1, -y \rangle \mid y \in U\})$.
- (3) $U \subset D_K(\infty)$.
- (4) Every ordering on F extends uniquely to K .
- (5) $\text{Att}(\ker s_*) = \{IF\}$.
- (6) For $a \in G(F)$, $D_K\langle 1, -a \rangle = D_F\langle 1, -a \rangle(D_K\langle 1, -a \rangle \cap U)$.

Proof. We have $s_*\langle x \rangle = \langle N_{K/F}(x) \rangle$ by (1.3) and so s_* is a ring homomorphism. Then $\ker s_*$ is an ideal which gives (2) by (1.1) and (1.6), noting that $\langle 1 \rangle - s_*\langle y \rangle \in \ker s_* \cap WF = 0$. By (1.5) $m(K/F) \subset W_t K$ and so if $y \in U$ then $\langle 1, -y \rangle \in W_t K$. Hence $U \subset D_K(\infty)$. Parts (4), (5) follow from (2.6) as $\ker s_* \subset W_t K$.

Lastly, let $bx \in D_K\langle 1, -a \rangle$ where $b \in G(F)$ and $x \in U$. Then $\langle\langle -a, -b \rangle\rangle = \langle\langle -a, -x \rangle\rangle$. Apply s_* to get

$$\langle\langle -a, -b \rangle\rangle = s_*\langle\langle -a, -b \rangle\rangle = \langle\langle -a \rangle\rangle s_*\langle\langle -x \rangle\rangle = 0.$$

Hence $b \in D_K\langle 1, -a \rangle \cap G(F) = D_F\langle 1, -a \rangle$. Then $x \in D\langle 1, -a \rangle \cap U$. \square

REMARK. (2.7) applies in the following cases:

- (1) $I^2F = 0$ (e.g. $\text{tr. d.}_{\mathbb{C}} F = 1$). Here we may write any $s_*\langle x \rangle = \langle N_{K/F}(x) \rangle + \phi$ where $\phi \in I^2F = 0$.
- (2) $G(K) = \{1, a\}G(F)$. This follows from (1.4).

COROLLARY 2.8. *If every ordering on F extends uniquely to K then $G(K)/G(F) \approx D_K(\infty)/D_F(\infty)$.*

Proof. We may assume $WK \neq WF$. $\text{Att}(WK/WF) = \{IF\}$ by (2.6) and so WF is an IF -primary submodule of WK . In particular, multiplication by $2(1)$ is locally nilpotent on WK/WF . That is, if $x \in G(K)$ then $2^m\langle x \rangle \in WF$ for some m . Hence $ax \in D_K(2^m)$ for some $a \in G(F)$. So $G(K) = G(F)D_K(\infty)$ and $G(K)/G(F) \approx D_K(\infty)/D_K(\infty) \cap G(F) = D_K(\infty)/D_F(\infty)$. \square

The condition (2.3) telling when IF is attached to $\ker s_*$ is not easy to check. We give some examples. Clearly $IF \in \text{Att}(\ker s_*)$ if F is non-real and $WK \neq WF$. For an example with F real, take $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt[3]{2})$. \mathbb{Q} has a unique ordering α which extends uniquely, so $P(\alpha) \notin \text{Att}(\ker s_*)$ by (2.6). Also $\ker s_* \neq 0$ as $\sqrt[3]{2} \notin \mathbb{Q} \cdot K^2$. Thus $\text{Att}(\ker s_*) = \{IF\}$.

For an example with $IF \notin \text{Att}(\ker s_*)$, consider the Pythagorean SAP field K with automorphism σ of odd order n constructed by Ware [16]. If $F = K^\sigma$ then K/F is Galois of degree n . As $|X(P)| > 1$ for $P \in X_F$ we have $WK \neq WF$, while the fact that $W_iK = 0$ implies $IF \notin \text{Att}(\ker s_*)$.

In general, the property $IF \notin \text{Att}(\ker s_*)$ is restrictive. We close this section by examining some of its consequences.

LEMMA 2.9. *Let $[K : F] = 2k + 1$ and choose s such that $s_*(1) = \langle 1 \rangle$. Suppose $IF \notin \text{Att}(\ker s_*)$. Then:*

- (1) $D_K(\infty) = D_F(\infty)K^2$.
- (2) If $N_{K/F}(w) \in (-1)^k F \cdot K^2$ then $D_F(\infty) \subset D_K\langle 1, -w \rangle$.
- (3) $W_iK = W_iF$.

Proof. (1) Let $w \in D_K(\infty)$ so that $\langle 1, -w \rangle \in W_iK$. Now $s_*(1, -w) \in W_iF$. Thus $\langle 1, -w \rangle - s_*\langle 1, -w \rangle \in W_iK \cap \ker s_* = 0$ by (2.3). Then $s_*\langle 1, -w \rangle = \langle 1, -w \rangle$, $w \in F \cdot K^2$ and $w \in D_F(\infty)K^2$.

(2) We have $\det(s_*\langle w \rangle) = N_{K/F}(w) = (-1)^k$. Then

$$\det(\langle w \rangle - s_*\langle w \rangle) = (-1)^{k+1}w \quad \text{and} \quad d(\langle w \rangle - s_*\langle w \rangle) = w.$$

Hence $\langle w \rangle - s_*\langle w \rangle = \langle 1, -w \rangle + \phi$ for some $\phi \in I^2K$. If $x \in D_F(\infty)$

then $\langle 1, -x \rangle (\langle w \rangle - s_*(w)) \in \ker s_* \cap W_t K = 0$. By the Arason-Pfister theorem, $\langle 1, -x \rangle \langle 1, -w \rangle = 0$ and $x \in D_K(\langle 1, -w \rangle)$.

(3) $W_t K$ is generated by $\langle 1, -w \rangle$, $w \in D_K(\infty)$. Apply (1). \square

COROLLARY 2.10. *If $IF \notin \text{Att}(\ker s_*)$ then $m(K/F) = 0$. In particular, WK embeds into a fiber product of copies of WF . If $|X_F| < \infty$ then we need only finitely many copies.*

Proof. If $\phi \in m(K/F)$, $\phi \neq 0$ then $\phi \in W_t K$ by (1.5) and $\phi \in \ker s_*$. This contradicts (2.3). Thus $m(K/F) = 0$. Write $G(K) = \text{gr}\{xi|i \in I\} \cdot G(F)$, where $\text{gr}(S)$ is the group generated by S . Set $s_i(y) = \text{tr}_{K/F}(x_i y)$ for all $y \in K$. Then $WK \rightarrow \prod_I WF$ by $\phi \mapsto (\dots, (s_i)(\phi), \dots)$ is injective.

Suppose $|X_F| < \infty$. Then $|X_K| < \infty$ also. Write $X_K = \{Q_1, \dots, Q_n\}$. Now $\bigcap Q_i = D_K(\infty) = D_F(\infty) \dot{K}^2$ by (2.9). Hence

$$[G(K) : G(F)] \leq [\dot{K} : \bigcap Q_i] \leq 2^n.$$

Thus WK embeds into n copies of WF . \square

COROLLARY 2.11. *Suppose $IF \notin \text{Att}(\ker s_*)$.*

(1) *If $|X_F| < \infty$ then WK is a finitely generated WF -module.*

(2) *If WF is noetherian then so is WK .* \square

COROLLARY 2.12. *$WF \approx WK$ iff every ordering on F extends uniquely and $\ker s_* \cap W_t K = 0$.*

Proof. By (2.2), (2.3) and (2.5) we have $\text{Att}(\ker s_*) = 0$. Then $\ker s_* = 0$ by (2.1). \square

REMARK. There is a partial converse to (2.8). If $W_t K = W_t F$ then $IF \notin \text{Att}(\ker s_*)$. Namely, if $\phi \in W_t K \cap \ker s_*$ then $\phi \in WF$ and so $\phi = s_*(\phi) = 0$. Thus $W_t K \cap \ker s_* = 0$ and $IF \notin \text{Att}(\ker s_*)$.

3. $\ker s_*$ as a projective module.

LEMMA 3.1. (1) *$\ker s_*$ is projective iff WK is projective.*

(2) *If $\ker s_*$ is free then WK is free.*

Proof. We may assume $s_*(\langle 1 \rangle) = \langle 1 \rangle$ by (1.1). Then both parts follow from $WK \approx WF \oplus \ker s_*$. \square

The trace of an R -module M is:

$$\text{tr } M = \left\{ \sum f_i(m_i) \mid f_i \in \text{Hom}_R(M, R), m_i \in M \right\}.$$

We refer to [7] for basic facts about $\text{tr } M$.

PROPOSITION 3.2. *Suppose $\ker s_*$ is projective and $\ker s_* \neq 0$. Then:*

- (1) $\text{tr}(\ker s_*) = WF$,
- (2) $\text{ann}_{WF}(\ker s_*) = 0$.

Proof. (1) $\text{tr}(\ker s_*)$ is an ideal so if $\text{tr}(\ker s_*) \neq WF$ then $\text{tr}(\ker s_*)$ is contained in a maximal ideal of WF . We check the two cases.

Suppose $\text{tr}(\ker s_*) \subset IF$. Choose $x \in K$ such that $s_*\langle 1 \rangle = s_*\langle x \rangle = \langle 1 \rangle$. (This is possible by (1.4) since otherwise $L(s) = \{\langle 1 \rangle\}$ and $U = \{\langle 1 \rangle\}$. But then for any $x \in K^*$, $x \in N_{K/F}(x)K^2 \subset F \cdot K^2$, as $N_{K/F}(xN_{K/F}(x)) \in K^2$. This implies $WK = WF$ and $\ker s_* = 0$, contrary to the assumption). Then $\langle 1, -x \rangle \in \ker s_*$. We have $IF \cdot \ker s_* = \ker s_*$ by [7, 3.30(a)] while $\langle 1, -x \rangle \in \ker s_* \setminus I^2K$ and $IF \cdot \ker s_* \subset I^2K$. Thus $\text{tr}(\ker s_*) \not\subset IF$.

Next suppose $\text{tr}(\ker s_*) \subset P(\alpha, p)$ for some $\alpha \in X_F$ and odd prime p . Let $m \geq 1$ be the largest integer with $\text{tr}(\ker s_*) \subset P(\alpha, p^m)$; a maximum exists since $\bigcap_m P(\alpha, p^m) \subset P(\alpha) \subset IF$. Now $\text{tr}(\ker s_*) = (\text{tr } \ker s_*)^2$ by [7, 3.30(a)]. Hence $\text{tr}(\ker s_*) \subset P(\alpha, p^m)^2 \subset P(\alpha, p^{2m})$, a contradiction. Thus $\text{tr}(\ker s_*) \not\subset P(\alpha, p)$ and so $\text{tr}(\ker s_*) = WF$.

(2) Clearly $\text{tr}(\ker s_*) = WF$ is a finitely generated ideal, so $\text{ann}_{WF}(\ker s_*)$ is generated by an idempotent [7, 3.30(b)]. Only 0 and 1 are idempotent in WF [11, VIII 6.8] and clearly $\text{ann}_{WF}(\ker s_*) \neq R$ as $\ker s_* \neq 0$. Thus $\text{ann}_{WF}(\ker s_*) = 0$. \square

THEOREM 3.3. *Suppose F is real and $\ker s_* \neq 0$. If some ordering on F extends uniquely to K then $\ker s_*$ is not projective.*

Proof. Suppose $\ker s_*$ is projective. Then $\text{ann}_{WF}(\ker s_*) = 0$ by (3.2). Let P be a prime ideal attached to $WF \approx WF/\text{ann}_{WF}(\ker s_*)$. Now $(\ker s_*)_P$ is $(WF)_P$ -free and so:

$$\text{ann}_{(WF)_P}(\ker s_*)_P = 0 = (\text{ann}_{WF}(\ker s_*))(WF)_P.$$

Then P is attached to $\ker s_*$ [13, Lemma 2]. That is, $\text{Att}(WF) \subset \text{Att}(\ker s_*)$.

To complete the proof we need only check that every $P(\alpha)$, $\alpha \in X_F$, is attached to WF , viewed as a WF -module. This would yield a contradiction to (2.5). Let $\alpha \in X_F$ and choose $a >_\alpha 0$ with $a \notin F^2$. Then $0 \neq \langle 1, -a \rangle \in \text{ann}\langle 1, a \rangle$ and $\text{ann}\langle 1, a \rangle \subset P(\alpha)$. Since $P(\alpha)$ is a minimal prime ideal we have $P(\alpha) \in \text{Att}(WF)$. In the case that

$a >_\alpha 0$ implies $a \in F^{\cdot 2}$ we have $X_F = \{\alpha\}$ and $G(F) = \{\pm 1\}$. Thus $WF = \mathbf{Z}$, $P(\alpha) = \{0\} = \text{ann } 2$, so that again $P(\alpha) \in \text{Att}(WF)$. \square

COROLLARY 3.4. *Suppose $\ker s_*$ is a non-zero projective WF -module. If $W_i F \neq 0$ then $\ker s_* \cap W_i K \neq 0$.*

Proof. The proof of (3.3) shows $\text{Att}(WF) \subset \text{Att}(\ker s_*)$. If $W_i F \neq 0$ then $IF \in \text{Att}(WF)$ by (2.3) and so $IF \in \text{Att}(\ker s_*)$. This implies $\ker s_* \cap W_i K \neq 0$ by (2.3). \square

If no ordering on F extends uniquely to K (for example if K/F is Galois) then it is possible for $\ker s_*$ to be WF -projective—even for WK to be WF -free.

PROPOSITION 3.5. *There is a real field F and a Galois extension K of F of degree 3 such that:*

- (1) WF and WK are noetherian,
- and
- (2) WK is WF -free.

Proof. Let $\alpha = \alpha_1, \alpha_2, \alpha_3$ be the roots of $x^3 - 3x + 1 \in \mathbf{Q}[x]$. Note that $\mathbf{Q}(\alpha)/\mathbf{Q}$ is Galois. Let F be a maximal field in $\overline{\mathbf{Q}} \cap \mathbf{R}$ not containing α ($\overline{\mathbf{Q}}$ is the algebraic closure of \mathbf{Q}). F is real with the ordering induced by \mathbf{R} . Moreover $G(F) = \{\pm 1\}$. Namely, if $a \in F$, $a > 0$ then $F(\sqrt{a}) \subset \overline{\mathbf{Q}} \cap \mathbf{R}$ and $\alpha \notin F(\sqrt{a})$ as $\deg \alpha = 3$. Hence, by maximality, $F(\sqrt{a}) = F$ and $a \in F^2$.

Let $K = F(\alpha)$. Since $x^3 - 3x + 1$ is irreducible over F , by construction, K/F is Galois of degree 3. We claim that K is Pythagorean. Suppose not. Let $\beta \in \sum K^2$, $\beta \notin K^2$. Note that $\beta \notin F$, as $\beta \in \sum K^2$ implies $\beta > 0$ and so $\beta \in F$ would yield $\beta \in F^2$. Thus $F(\alpha) = F(\beta) = K$. Let σ generate $\text{Gal}(K/F)$ and set $\beta_i = \sigma^i(\beta)$, $i = 0, 1, 2$ ($\beta_0 = \beta$). We note that each β_i is in $\sum K^2$. If $g(x) = \text{irr}(\beta, F)$ then $g(x^2) = \text{irr}(\sqrt{\beta}, F)$. Thus $L = F(\sqrt{\beta_0}, \sqrt{\beta_1}, \sqrt{\beta_2})$ is Galois over F , contains $K = F(\beta)$ and is contained in $\overline{\mathbf{Q}} \cap \mathbf{R}$.

Now $[L : F] = 3 \cdot 2^r$ for some $r = 1, 2$ or 3 . Let P be a Sylow 3-subgroup and let $F(Q)$ be the fixed field. Then $F(Q) \subset \overline{\mathbf{Q}} \cap \mathbf{R}$ and $\alpha \notin F(Q)$ as $\deg \alpha = 3$ while $\deg Q = 2^r$. This contradicts the maximality of F .

Hence K is Pythagorean, and SAP since $K \subset \overline{\mathbf{Q}}$ [4, Example 1, p. 1177]. F has a unique ordering so K has 3 orderings. Hence $|G(K)| = 8$ and $WK \approx \mathbf{Z} \sqcap \mathbf{Z} \sqcap \mathbf{Z}$ which is free over $\mathbf{Z} \approx WF$. \square

The example of (3.5) yields another case where $IF \notin \text{Att}(\ker s_*)$. Indeed $\text{Att}(\ker s_*) = \{P(\alpha)\}$. Also (3.5) is another example of a Pythagorean field with an automorphism of odd order (cf. [16]).

We will show that the situation of (3.5), namely, K/F Galois, WK noetherian and WK WF -free, is impossible if F is non-real and non-rigid. Weaker results hold with fewer restrictions on K and F so we begin with no assumptions on K or F .

LEMMA 3.6. *Suppose WK is a free WF -module. Then for some index set I there exists $\phi_i \in WK$ for $i \in I$ such that:*

- (1) $WK = \bigoplus_I WF \cdot \phi_i$,
- (2) $\phi_i = \langle \alpha_i \rangle + \psi_i$ where $\alpha_i \in K$ and $\psi_i \in I^2K$, and
- (3) $G(K) = A \times G(F)$, where A is the group generated by the α_i , $i \in I$.

Proof. We have $WK = \bigoplus_I WF \cdot \phi_i$ for some collection $\{\phi_i \in WK \mid i \in I\}$. Clearly at least one ϕ_i , say ϕ_1 , is odd dimensional. For any even dimensional ϕ_j replace ϕ_j by $\phi_j - \phi_1$. We may thus assume (1) and (2) hold.

Now $G(K) = A \cdot G(F)$ since if $x \in G(K)$ then $\langle x \rangle = \sum r_i \phi_i$ and so $x = \det \langle x \rangle = \pm \prod \det(r_i) \alpha_i \in A \cdot G(F)$. We claim that by replacing some ϕ_i by $a \phi_i$, $a \in G(F)$, we may assume $A \cap G(F) = 1$.

This is clearer if we write the \mathbf{Z}_2 -vector space $G(K)$ additively. We wish to show that there exist a_i ($i \in I$) in the subspace $G(F)$ such that $\text{span}\{\alpha_i + a_i \mid i \in I\} \cap G(F) = \{0\}$. Choose any complementary subspace $G(F)'$. Then every α_i has a unique expression $\alpha_i = a_i + a'_i$ for some $a_i \in G(F)$ and $a'_i \in G(F)'$. Use these a_i . \square

PROPOSITION 3.7. *Suppose F is non-real and WK is a free WF -module. Then for all $f \in G(F)$, $f \neq 1$, we have $D_K \langle 1, -f \rangle = D_F \langle 1, -f \rangle$.*

Proof. Write $WK = \bigoplus_I WF \cdot \phi_i$ as in (3.6). Each odd dimensional form is a unit as F is non-real. Multiplication by ϕ_1^{-1} is an WF -module isomorphism and $\{\phi_1^{-1} \phi_i \mid i \in I\}$ satisfies (1), (2), (3) of (3.6). We may thus assume $\phi_1 = \langle 1 \rangle$. The result is clear if $WK = WF$ so we may assume $|I| \geq 2$. Write $G(K) = A \times G(F)$ as in (3.6) and let $\alpha \in A$.

Claim. $WK = WF \cdot \langle 1 \rangle \oplus WF \cdot \langle \alpha \rangle \oplus M$, for some WF -module M .

We have $\langle \alpha \rangle = r_1 \langle 1 \rangle + \sum_{i \geq 2} r_i \phi_i$. If all r_i ($i \geq 2$) are even dimensional then by determinants $\alpha \in G(F)$, contradicting (3.6). We may thus assume r_2 is odd dimensional. Since F is non-real, r_2 is a unit in WF . We have:

$$r_2^{-1} \langle \alpha \rangle = r_2^{-1} r_1 \langle 1 \rangle + \phi_2 + \sum_{i \geq 3} r_2^{-1} r_i \phi_i.$$

Set $M = \bigoplus_{i \geq 3} WF \cdot \phi_i$. Then $\phi_2 \in WF \cdot \langle 1 \rangle + WF \cdot \langle \alpha \rangle + M$, hence $WK = WF \cdot \langle 1 \rangle + WF \cdot \langle \alpha \rangle + M$. Moreover, if:

$$s_1 \langle 1 \rangle + s_2 \langle \alpha \rangle + m = 0 \quad (m \in M)$$

then

$$\begin{aligned} s_1 \langle 1 \rangle + s_2 (r_1 \langle 1 \rangle + r_2 \phi_2 + m') + m &= 0 \\ (s_1 + s_2 r_1) \langle 1 \rangle + s_2 r_2 \phi_2 + s_2 m' + m &= 0. \end{aligned}$$

But $\langle 1 \rangle = \phi_1$ and ϕ_i ($i \geq 2$) are independent. Thus $s_2 r_2 = 0$. Again r_2 is a unit so $s_2 = 0$. Thus $s_1 = 0$ and $m = 0$. This proves the Claim.

Now say $f \in G(F)$, $f \neq 1$. Let $x \in D_K \langle 1, -f \rangle$, $x \notin G(F)$. Then $x = g\alpha$ for some $g \in G(F)$ and $\alpha \in A$, $\alpha \neq 1$. But then $\langle 1, -f \rangle \langle 1 \rangle = \langle g \rangle \langle 1, -f \rangle \langle \alpha \rangle$ contradicting the Claim. Thus $D_K \langle 1, -f \rangle \subset G(F)$ and so $D_K \langle 1, -f \rangle = D_F \langle 1, -f \rangle$. \square

In the following, $B(F)$ denotes the basic part, namely those $a \in F$ with either $a = \pm 1$, a or $-a$ not rigid (cf. [12]).

THEOREM 3.8. *Suppose F is non-real and $G(F)$ is finite. If $\ker s_*$ is a finitely generated projective WF -module then either:*

- (1) $WK \approx WF[A]$ where $A = G(K)/G(F)$ or
- (2) $B(F) = \{\pm 1\}$ and $WF \approx \mathbf{Z}_n[C]$ with $n = 2$ or 4 and C a group or exponent two.

Proof. WF is a local ring so $\ker s_*$, hence WK , is finitely generated free. Suppose $B(F) \neq \{\pm 1\}$. Choose $f \in B(F) \setminus \{\pm 1\}$. Set $X_1(K) = D_K \langle 1, -f \rangle$. Then $X_1(K) = X_1(F) = D_F \langle 1, -f \rangle$ by (3.7). For $i \geq 2$ and a field E let $X_i(E) = \bigcup D_E \langle 1, -a \rangle$, over $a \in X_{i-1}(E) \setminus \{1\}$. Then by [2, 2.4]

$$B(K) = \pm(X_1(K)X_2(K)^2 \cup -X_1(K)X_3(K)) = B(F) \subset G(F).$$

The result is then standard, see [12, 5.19]. And if $B(F) = \{\pm 1\}$ then WF is classified as given [12, 5.21]. \square

REMARK. If $WK = WF[A]$, as in (3.8)(1), then WK is clearly a free WF -module. Suppose $B(F) = \{\pm 1\}$ as in (3.8)(2) and $B(K) \cap G(F) = \{\pm 1\}$. We may write $G(K) = B \times C$ where $B(K) \subset B$ and $G(F) = \pm C$. Then any form in WK may be written uniquely as $\sum \langle b_i c_i \rangle = \sum \langle c_{i1} \rangle \cdot \langle b_1 \rangle + \sum \langle c_{i2} \rangle \cdot \langle b_2 \rangle + \dots$. Thus again WK is a free WF -module. However, we know of no example of an odd degree extension K/F with $WK \neq WF$ and either (3.8)(1) or (2) occurring.

We obtain a slightly weaker result if WF is not noetherian.

LEMMA 3.9. *Let $WK = \bigoplus_I WF \cdot \phi_i$ as in (3.6). Let $\alpha, \beta \in A \setminus \{1\}$ be distinct and let $a, b, c, d \in G(F)$. If $b\alpha \in D\langle 1, -a\beta \rangle$ and $d\alpha \in D\langle 1, -c\beta \rangle$ then $b = d$ and $a = c$.*

Proof. We have

$$\begin{aligned} 0 &= \langle \langle -c\beta, -d\alpha \rangle \rangle \equiv \langle \langle -ac, -d\alpha \rangle \rangle - \langle \langle -a\beta, -d\alpha \rangle \rangle \\ &\equiv \langle \langle -ac, -d\alpha \rangle \rangle - \langle \langle -a\beta, -bd \rangle \rangle \pmod{I^3 K}. \end{aligned}$$

Thus $\langle \langle -ac, -d\alpha \rangle \rangle = \langle \langle -bd, -a\beta \rangle \rangle$. Apply linkage [12, 1.14]:

$$\langle \langle -ac, -d\alpha \rangle \rangle = \langle \langle -ac, -x \rangle \rangle = \langle \langle -bd, -x \rangle \rangle = \langle \langle -bd, -a\beta \rangle \rangle$$

for some $x \in K^*$. Now $x \in D\langle 1, -abcd \rangle$. If $ac \neq bd$ then $x \in G(F)$ by (3.7). But $xd\alpha \in D\langle 1, -ac \rangle$ which forces $a = c$, by (3.7) again. Similarly $xa\beta \in D\langle 1, -bd \rangle$ yields $b = d$. Suppose then that $ac = bd$. Now $xd\alpha \in D\langle 1, -ac \rangle$ gives $x \in \alpha G(F)$ (unless $a = c$ and so $b = d$). And $xa\beta \in D\langle 1, -bd \rangle$ gives $x \in \beta G(F)$ (unless $b = d$ and so $a = c$). But $\alpha G(F) \cap \beta G(F) = \emptyset$. Hence $a = c$ and $b = d$. \square

THEOREM 3.10. *Suppose F is non-real and $G(F)$ is infinite. If $\ker s_*$ is a finitely generated projective WF -module then either:*

- (1) $WK \approx WF[A]$, with $A = G(K)/G(F)$ or
- (2) $|B(F)| < \infty$ and $R = R_0[C]$ for some Witt ring R_0 and infinite group C of exponent 2.

Proof. If $|B(F)| < \infty$ then R is as described [12, 5.19]. Suppose $B(F)$ is infinite. Let $\alpha \in A$, $\alpha \neq 1$. We will show α is bi-rigid.

Suppose α is not rigid (the argument for $-\alpha$ is similar). Then $\alpha \in B(K)$ and for all $f \in B(F)$, $f\alpha$ is not bi-rigid. Hence there exist infinitely many f with $f\alpha$ not rigid (that is, if $f\alpha$ is rigid then $-f\alpha$ is not rigid). But A is finite, as WK is finitely generated over WF , so there exist distinct f, g in F and $\beta \in A \setminus \{1, \alpha\}$ such

that $b\beta \in D\langle 1, -f\alpha \rangle$, $d\beta \in D\langle 1, -g\alpha \rangle$ for some $b, d \in F$. This contradicts (4.9). \square

LEMMA 3.11. *If t_1, \dots, t_n , and all $t_i t_j$ ($i \neq j$) are rigid then $D\langle t_1, \dots, t_n \rangle = \{t_1, \dots, t_n\}$.*

Proof. By induction on n . Suppose $n = 2$.

$$D\langle t_1, t_2 \rangle = t_1 D\langle 1, t_1 t_2 \rangle = t_1 \{1, t_1 t_2\} = \{t_1, t_2\}.$$

For $n > 2$ we have by induction:

$$D\langle t_1, \dots, t_n \rangle = \bigcup_{i=1}^{n-1} D\langle t_i, t_n \rangle = \{t_1, \dots, t_n\}. \quad \square$$

LEMMA 3.12. *Let K/F be finite Galois (not necessarily of odd degree). Let $t \in K \setminus FK^2$. Then at least one of t , tt^g ($g \in \text{Gal}(K/F)$) is not rigid.*

Proof. Suppose t and all tt^g are rigid. Note t^g is rigid as $D\langle 1, t \rangle^g = D\langle 1, t^g \rangle$. Also if $g, h \in \text{Gal}(K/F)$ are distinct then $t^g t^h = g(tt^{hg^{-1}})$ is rigid. Hence by (3.11) $D(\sum_G \langle t^g \rangle) = \{t^g | g \in \text{Gal}(K/F)\}$. But $\sum \langle t^g \rangle = \text{tr}_* \langle t \rangle \in WF$. Hence some $t^g \in G(F)$. But then $t \in G(F)$, a contradiction. \square

THEOREM 3.13. *Let F be non-real and suppose that either (i) $G(F)$ is finite and $B(F) \neq \{\pm 1\}$ or (ii) $G(F)$ is infinite and $B(F)$ is infinite. Let K/F be Galois of odd degree. Then neither WK nor $\ker s_*$ are finitely generated projective WF -modules.*

Proof. If WK is a finitely generated projective WF -module then (3.8), (3.10) imply $B(K) \subset FK^2$ and hence if $t \in K \setminus FK^2$ with $K = F(t)$ then t and all tt^g ($g \in \text{Gal}(K/F)$) are bi-rigid. Namely if $tt^g \in FK^2$, say $tt^g = at$, then $g^2(t) = a(at) = t$. Thus t is fixed by g^2 . As g has odd order, t is fixed by g . But then $K = F(t)$. This contradicts (3.12). \square

Ware [16, 1.6] shows a rigid field cannot be the Galois odd degree extension. (3.13) improves this slightly: even the case $WK \approx WF[A]$, $A = G(K)/G(F)$ cannot arise.

In a different direction we have:

PROPOSITION 3.14. *Suppose WK is a noetherian, injective WF -module. Then F is non-real and WF is Gorenstein (that is, $|\text{ann } IF| = 2$).*

Proof. WK injective implies its direct summand WF is injective. Thus WF has injective dimension 0 and so Krull dimension 0. Thus F is non-real. Further, WF is Gorenstein (cf. [1], [9]). \square

4. Noetherian extensions. We have given several examples of odd degree extensions K/F where WK is a finitely generated WF -module. This is necessarily the case when X_F is finite and $IF \notin \text{Att}(\ker s_*)$ by (2.11). We collect here several results on the possible values of $[G(K) : G(F)]$.

PROPOSITION 4.1. *Let $[K : F] = p$ be an odd prime and suppose K/F is Galois. If $[G(K) : G(F)] = 2^k$ then $p | 2^k - 1$.*

Proof. Let $G = \text{Gal}(K/F)$ and let σ generate G . G acts on $G(K)/G(F)$. Suppose $xG(F)$ is a fixed point. Then $N_{K/F}(x) \in x^p G(F) = xG(F)$ and so $x \in G(F)$. If $x \notin G(F)$ then the orbit $\{\sigma^i(xG(F)) | i \in \mathbf{Z}\}$ has order p (there is no stabilizer as G is simple). Thus p divides $2^k - 1$. \square

EXAMPLE. Let p be an odd prime and set $n = 2^p - 1$. Let K be \mathbf{Q}_2 with the n th roots of unity adjoined. Then K/\mathbf{Q}_2 is Galois of degree p [14, Prop. 16, p. 77]. By [11, p. 161] we have $[G(K) : G(\mathbf{Q}_2)] = 2^{p-1}$. This gives the minimal value of $[G(K) : G(F)]$ for p such that the order of 2 mod p is $p - 1$ (thus for $p = 3, 5, 11, 13, 19, 29, 37, 53, 59$ etc.).

COROLLARY 4.2. *Let $[K : F] = p_1 p_2 \cdots p_t$ with the p_i 's prime (not necessarily distinct). Let k_i be the least positive integer such that $p_i | 2^{k_i} - 1$. If K/F is Galois and $G(K) \neq G(F)$ then $[G(K) : G(F)] \geq 2^w$, where $w = k_1 + \cdots + k_t$.*

Proof. We use induction on t . The case $t = 1$ is (4.1) and if $t > 1$ then choose an intermediate normal extension L and apply the result to K/L and L/F . \square

When p is a Mersenne prime (i.e., $p = 2^k - 1$) then the minimal (non-trivial) square class extension for a Galois extension of degree p is $p + 1$. In this case we may improve (1.5).

PROPOSITION 4.3. *Suppose K/F is Galois and that $[K : F] = p$ where $p = 2^k - 1$ is a prime. If $[G(K) : G(F)] = p + 1$ then $m(K/F) \subset \text{ann}(2^k\langle 1 \rangle)$.*

Proof. Choose $s \in \text{Hom}(K, F)$ with $s_*\langle 1 \rangle = \langle 1 \rangle$. There is an $x \in G(K)$ with $\text{tr}_*\langle x \rangle = s_*\langle 1 \rangle = \langle 1 \rangle$. Now $(-1)^{p-1/2} = \det \text{tr}_*\langle x \rangle = N_{K/F}(x)$. Since $p = 2^k - 1$ ($k \geq 2$) we have $N_{K/F}(x) = -1$. Write $G(K) = U \times G(F)$ as in §1. There is only one (non-trivial) orbit in $G(K)/G(F)$. Thus $U = \{1, x_1, \dots, x_p\}$ where $\sigma(x_i) = x_{i+1}$ (here σ generates $\text{Gal}(K/F)$ and $x_{p+1} \equiv x_1$). We may assume $x_1 = -x$ and so $\text{tr}_*\langle x_1 \rangle = \langle -1 \rangle$.

Let $\psi = \phi_0 + \sum_{i=1}^p \langle x_i \rangle \phi_i \in m(K/F)$, where $\phi_0, \dots, \phi_p \in WF$. Then:

$$0 = \text{tr}_* \psi = p\phi_0 - \sum_{i=1}^p \phi_i,$$

$$0 = \text{tr}_*\langle x_1 \rangle \psi = p\phi_1 - \phi_0 - \sum_{i=2}^p \phi_i.$$

Subtraction yields $p(\phi_0 - \phi_1) - (\phi_1 - \phi_0) = 0$ and so $2^k(\phi_0 - \phi_1) = 0$. Similarly, $2^k(\phi_i - \phi_j) = 0$ for all i, j .

Now $\langle -1 \rangle = \text{tr}_*\langle x_1 \rangle = \langle x_1, x_2, \dots, x_p \rangle$. Thus $\langle x_p \rangle = -\langle 1, x_1, \dots, x_{p-1} \rangle$. Then $\psi = \phi_0 + \langle x_1 \rangle \phi_1 + \dots + \langle x_{p-1} \rangle \phi_{p-1} - \langle 1, x_1, \dots, x_{p-1} \rangle \phi_p = (\phi_0 - \phi_p) + \langle x_1 \rangle (\phi_1 - \phi_p) + \dots + \langle x_{p-1} \rangle (\phi_{p-1} - \phi_p)$. Thus $2^k \psi = 0$. □

(4.3) applies when $[K : F] = 3$ and $[G(K) : G(F)] = 4$. See after (4.1) for an example of such an extension. We can improve (4.3) in this case (see the second example after (1.1)).

COROLLARY 4.4. *Suppose K/F is Galois with $[K : F] = 3$ and $[G(K) : G(F)] = 4$. Write $U = \{1, x, y, xy\}$. Then:*

- (1) $m(K/F) = \{\phi_0\langle x \rangle + \phi_2\langle y \rangle \mid \phi_i \in WF, 4\phi_i = 0 \text{ and } \phi_0 + \phi_1 + \phi_2 = 0\}$.
- (2) $m(K/F) = 0$ iff $D_F(4) \subset D_K\langle 1, -x \rangle \cap D_K\langle 1, -y \rangle$.
- (3) If F is non-real and $m(K/F) = 0$ then $x, y \in D_K(2)$.

Proof. (1) Follows from the proof of (4.3). Suppose $m(K/F) = 0$. If $w \in D_F(4)$ then for $\phi = \langle 1, -w \rangle$ we have $4\phi = 0$ and $\langle 1, -x \rangle \phi \in m(K/F) = 0$. Thus $w \in D_K\langle 1, -x \rangle$, and similarly $w \in D_K\langle 1, -y \rangle$.

If $D_F(4) \subset D_K\langle 1, -x \rangle \cap D_K\langle 1, -y \rangle$ and $\psi = \phi_0 + \phi_1\langle x \rangle + \phi_2\langle y \rangle \in m(K/F)$ then

$$\begin{aligned}\psi &= \phi_0 + \phi_1\langle x \rangle - (\phi_0 + \phi_1)\langle y \rangle \\ &= \phi_0\langle 1, -y \rangle + \langle x \rangle\phi_1\langle 1, -xy \rangle = 0 + 0 = 0,\end{aligned}$$

as $\phi_i \in \text{ann}(4)$ which is generated by $\langle 1, -w \rangle$, $w \in D_F(4)$. Thus $m(K/F) = 0$, proving (2).

To prove (3) note that (2) implies $D_F(2^{2+k}) \subset D_K(2^k\langle\langle -x \rangle\rangle) \cap D_K(2^k\langle\langle -y \rangle\rangle)$. If $D_F(4) = G(F)$ then $-1 \in D_K\langle 1, -x \rangle \cap D_K\langle 1, -y \rangle$ and $x, y \in D(2)$. Otherwise, say $D_F(2^{k+1}) \neq G(F)$ and $D_F(2^{k+2}) = G(F)$ for some $k \geq 1$. Then $-1 \in D(2^k\langle\langle -x \rangle\rangle)$ and $2^{k+1}\langle\langle -x \rangle\rangle = 0$. Thus $x \in D(2^{k+1}) \subset D(2^{k-1}\langle\langle -x \rangle\rangle)$. So $-1 \in D(2^{k-1}\langle\langle -x \rangle\rangle)$ and $2^k\langle\langle -x \rangle\rangle = 0$. Continue until $2\langle 1, -x \rangle = 0$. Similarly $2\langle 1, -y \rangle = 0$. \square

We have only a few results for non-Galois extensions.

PROPOSITION 4.5. *Let L be the normal closure of K/F . If L is real then $[K : F] \leq [G(K) : G(F)]$.*

Proof. Let $X_E(P)$ denote the set of extensions of an ordering P to a field E . Let $Q \in X_L$ and set $P = Q \cap F$, $V = Q \cap K$. Then $|X_L(p)| = [L : F]$ as L/F is Galois, and $|X_L(V)| = [L : K]$. Then $|X_K(P)| = [L : F]/[L : K] = [K : F]$.

Let $h(S)$ denote the number of subgroups of $G(K)$ of index 2 containing a set S . Let $P \in X_F$. Then $h(P) = |G(K)/P| - 1 = 2[G(K) : G(F)] - 1$. Also $h(P \cup \{-1\}) = [G(K) : G(F)] - 1$. Thus there are $[G(K) : G(F)]$ many subgroups of index 2 in $G(K)$, containing P but missing -1 . These are the only possible choices for extensions of P to K . Hence $[K : F] = |X_K(p)| \leq [G(K) : G(F)]$. \square

We close with a detailed study of the smallest possible case: $[K : F] = 3$ and $[G(K) : G(F)] = 2$. We know of no such extensions.

LEMMA 4.6. *Suppose $[K : F] = 3$ and K/F is separable but not Galois. Let L be the normal closure of K . Then:*

(1) *There exists a field E such that $F \subset E \subset L$, $[L : E] = 3$ and L/E is Galois.*

(2) $[G(K) : G(F)] = \frac{[G(L) : G(E)]}{[D_K\langle 1, -g \rangle : D_F\langle 1, -g \rangle]}$, for some $g \in G(F)$.

(3) $[G(K) : G(F)] \leq [G(L) : G(E)]$.

Proof. We have $[L : F] = 6$. Thus there exists a normal subgroup H of $\text{Gal}(L/F)$ of order 3. Let E be the fixed field of H . Then $[L : E] = 3$ and $E = F(\sqrt{g})$ for some $g \in G(F) \setminus \{1\}$. Suppose $K = F(e)$. Then $e \notin E$ and so $L = F(\sqrt{g})$. By [11, VII, 3.4]:

$$\begin{aligned} [G(E) : G(F)] &= \frac{1}{2} |D_F\langle 1, -g \rangle| \\ [G(L) : G(K)] &= \frac{1}{2} |D_K\langle 1, -g \rangle|. \end{aligned}$$

Hence the formula in (2) holds. (3) follows from (2). □

LEMMA 4.7. *Suppose $G(K) = \{1, a\}G(F)$. Set $H = D\langle 1, -a \rangle \cap G(F)$. Then for $f \in G(F)$:*

$$\begin{aligned} D_K\langle 1, -f \rangle &= \begin{cases} D_F\langle 1, -f \rangle & \text{if } f \notin H, \\ \{1, a\}D_F\langle 1, -f \rangle & \text{if } f \in H, \end{cases} \\ D_K\langle 1, -af \rangle &= \{1, -af\}(D_F\langle 1, -f \rangle \cap H). \end{aligned}$$

Proof. By (1.4) there is an $s \in \text{Hom}(K/F)$ with $s_*\langle 1 \rangle = s_*(a) = \langle 1 \rangle$. (2.7)(6) then gives the computation of $D_K\langle 1, -f \rangle$. Clearly $D_K\langle 1, -af \rangle = \{1, -af\}(D_K\langle 1, -af \rangle \cap G(F))$. Then $g \in D_K\langle 1, -af \rangle \cap G(F)$ iff $af \in D_K\langle 1, -g \rangle$ iff $g \in D_F\langle 1, -f \rangle$ and $g \in H$. Thus $D_K\langle 1, -af \rangle = \{1, -af\}(D_F\langle 1, -f \rangle \cap H)$. □

PROPOSITION 4.8. *Suppose $[K : F] = 3$ and $G(K) = \{1, a\}G(F)$. Then:*

- (1) $|D\langle 1, -a \rangle \cap G(F)| \neq 1$;
- (2) *If $|D\langle 1, -a \rangle \cap G(F)| = 2$ then either:*
 - (i) $\text{rad}(F) \neq 1$, or
 - (ii) WF and WK are group ring extensions, or
 - (iii) *There is a non-real Witt ring R_0 such that $WF = \mathbf{Z} \sqcap R_0$ and $WK = \mathbf{Z} \sqcap R_0[\{1, a\}]$. In particular, $|X_F| = |X_K| = 1$.*

Proof. (1) Suppose $|D\langle 1, -a \rangle \cap G(F)| = 1$. Then (4.7) implies a is bi-rigid. Thus $WK = WF[\{1, a\}]$ is a group ring extension. Let L be the normal closure of K . Then $L = K(\sqrt{g})$ for some $g \in G(F)$. Set $E = F(\sqrt{g})$. Now $D_K\langle 1, -g \rangle = D_F\langle 1, -g \rangle$ so that $[G(K) : G(F)] = [G(L) : G(E)]$ by (4.6). But (4.1) implies $[G(L) : G(E)] \geq 4$, a contradiction.

(2) Write $D\langle 1, -a \rangle \cap G(F) = \{1, f\}$ and suppose $\text{rad}(F) = 1$; in particular, $D_F\langle 1, -f \rangle \neq G(F)$. If $x \in G(F) - D_F\langle 1, -f \rangle$ then

$D\langle 1, -ax \rangle = \{1, -ax\}$ by (4.7). Thus if there exists $g, -g \in G(F) - D_F\langle 1, -f \rangle$ then ag is bi-rigid. Now $f \in D\langle 1, -a \rangle$ so a is not bi-rigid and hence $g = a \cdot ag$ is bi-rigid. From $D_K\langle 1, -g \rangle = D_F\langle 1, -g \rangle$ we see that both WF and WK are group rings (with $\{1, g\}$ the group). This gives (ii).

So we may assume for all $g \in G(F)$ that either g or $-g$ is in $D\langle 1, -f \rangle$. Thus $[G(F) : D_F\langle 1, -f \rangle] = 2$ and $-1 \notin D_F\langle 1, -f \rangle$. In particular, $D_F\langle 1, -f \rangle$ is an ordering on F . From $G(F) = \{1, f\} \times D_F\langle 1, -f \rangle$ we get $WF = \mathbf{Z} \cap R_0$ for some Witt ring R_0 .

We also have that $D_K\langle 1, -f \rangle = \{1, a\}D_F\langle 1, -f \rangle$ has index 2, in $G(K)$, and misses -1 . Thus $D_K\langle 1, -f \rangle$ is an ordering. Again, $G(K) = \{1, f\} \times D_K\langle 1, -f \rangle$. Now in $D_K\langle 1, -f \rangle$, $D\langle 1, a \rangle = \{1, a\}$ and $D\langle 1, -af \rangle = \{1, -af\}$. Hence $WK = \mathbf{Z} \cap R_0[\{1, a\}]$.

Lastly, (2.7) implies $\text{Att}(\ker s_*) = \{IF\}$. Then (2.7) and (2.8) yield $|D_K(\infty)/D_F(\infty)| = 2$. Now $D_F(\infty) = 1 \times D_L(\infty)$, where $R_0 = WL$, and $D_K(\infty) = 1 \times D_L(\infty)$ unless $a \in D_L(\infty)$. But this only occurs if $-1 \in D_L(\infty)$. Hence R_0 is non-real and $|X_K| = |X_F| = 1$. \square

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vsv@math.ucla.edu

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Los Angeles, CA 90024-1555
christ@math.ucla.edu

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University of Utah
Salt Lake City, UT 84112
clemens@math.utah.edu

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093
tenright@ucsd.edu

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

R. FINN
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

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