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## BILINEAR OPERATORS ON $L^\infty(G)$ OF LOCALLY COMPACT GROUPS

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Let  $G$  and  $H$  be compact groups. We study in this paper the space  $\text{Bil}^\sigma = \text{Bil}^\sigma(L^\infty(G), L^\infty(H))$ . That space consists of all bounded bilinear functionals on  $L^\infty(G) \times L^\infty(H)$  that are weak\* continuous in each variable separately. We prove, among other things, that  $\text{Bil}^\sigma$  is isometrically isomorphic to a closed two-sided ideal in  $\text{BM}(G, H)$ . In the case of abelian  $G, H$ , we show that  $\text{Bil}^\sigma$  does not have an approximate identity and that  $\widehat{G} \times \widehat{H}$  is dense in the maximal ideal space of  $\text{Bil}^\sigma$ . Related results are given.

**0. Introduction.** Let  $V$  and  $W$  be Banach spaces over the complex numbers, and let  $\text{Bil}(V, W)$  denote the space of bounded bilinear functions  $F: V \times W \rightarrow C$ . Then this is a Banach space under the usual vector space operators and the norm

$$\|F\| = \sup\{|F(x, y)| : x \in V, y \in W, \|x\| = \|y\| = 1\}.$$

Furthermore  $\text{Bil}(V, W)$  may be identified with the dual space of  $V \otimes W$ , the projective tensor product of  $V$  and  $W$ . When  $X$  and  $Y$  are locally compact Hausdorff spaces, then elements in  $\text{Bil}(C_0(X), C_0(Y))$ , also denoted by  $\text{BM}(X, Y)$ , are called *bimeasures* (see Graham and Schreiber [7] and Gilbert, Ito and Schreiber [4]).

If  $V$  and  $W$  are dual Banach spaces, we let  $\text{Bil}^\sigma(V, W)$  denote all  $F \in \text{Bil}(V, W)$  such that  $x \mapsto F(x, y)$  and  $y \mapsto F(x, y)$  are continuous when  $V$  and  $W$  have the weak\*-topology. Then, as readily checked,  $\text{Bil}^\sigma(V, W)$  is a norm-closed subspace of  $\text{Bil}(V, W)$ . It is the purpose of this paper to study  $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$  when  $G$  and  $H$  are compact groups.

In §1, we shall give some general results on

$$\text{Bil}^\sigma(L^\infty(X, \mu), L^\infty(Y, \nu))$$

when  $X, Y$  are locally compact Hausdorff spaces and  $\mu, \nu$  are positive regular Borel measures on  $X$  and  $Y$ , respectively. In §2, we show that if  $G$  and  $H$  are compact groups, then  $\text{Bil}^\sigma = \text{Bil}^\sigma(L^\infty(G), L^\infty(H))$

is isometrically isomorphic to a closed ideal in  $\text{BM}(G, H)$  with multiplication as defined in [2]. Furthermore,  $\text{Bil}^\sigma$  has a dense subset consisting of bilinear functionals  $F$  such that their Grothendieck measures  $\mu_g, \nu_g$  are such that  $d\mu_g/dm_G$  and  $d\nu_g/dm_H$  are bounded away from 0 and from infinity (here  $m_G$  and  $m_H$  denote Haar measure on the respective groups). In §3, we shall concentrate on the case when  $G$  and  $H$  are both compact and abelian. We shall show that in this case  $\widehat{G} \times \widehat{H}$  is dense in the maximal ideal space of  $\text{Bil}^\sigma$  and that  $\text{Bil}^\sigma$  is a symmetric Banach algebra. Furthermore  $\text{Bil}^\sigma$  does not have an (even unbounded) approximate identity when  $G$  and  $H$  are infinite, compact. In §4, we shall list some open problems related to  $\text{Bil}^\sigma$ .

The space  $\text{Bil}^\sigma(U, V)$  has been studied in a different context by Effros [3]. A consequence of Theorem 3.7 (below) is that  $\text{Bil}^\sigma$  has no virtual diagonals; see the Remark following Theorem 3.7.

**1. The space  $\text{Bil}^\sigma$ .** If  $X$  is a locally compact Hausdorff space, we let  $L^\infty(X)$ ,  $C(X)$ ,  $C_0(X)$ , and  $C_{00}(X)$  be the spaces of bounded functions on  $X$  which are, respectively, Borel measurable, continuous, continuous with limit zero at infinity and continuous with compact support. The supremum norm on each of those spaces will be denoted by  $\|\cdot\|_\infty$ . If  $X$  and  $Y$  are locally compact Hausdorff spaces, we write  $V_0(X, Y) = C_0(X) \widehat{\otimes} C_0(Y)$ , the projective tensor product of  $C_0(X)$  and  $C_0(Y)$ . Then the space  $\text{BM}(X, Y)$  may be identified with the dual Banach space of  $V_0(X, Y)$ .

Throughout this section  $X$  and  $Y$  will denote locally compact Hausdorff spaces and  $\mu, \nu$  will denote positive regular Borel measures on  $X$  and  $Y$  constructed from a fixed positive functional on  $C_{00}(X)$  and  $C_{00}(Y)$ , respectively (see [9, §11]). We will write  $L^\infty(\mu)$  and  $L^\infty(\nu)$  for  $L^\infty(X, \mu)$  and  $L^\infty(Y, \nu)$  respectively. In this case,  $L^\infty(\mu) = L^1(\mu)^*$ , and  $L^\infty(\nu) = L^1(\nu)^*$ . We will write  $\text{Bil}^\sigma$  for  $\text{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$ . As usual, the norms for spaces  $L^p$ ,  $1 \leq p < \infty$ , will be denoted by  $\|\cdot\|_p$ . When  $G$  is a locally compact group,  $L^p(G)$  will denote the  $L^p$ -space defined with respect to a fixed left Haar measure  $m_G$  on  $G$ .

**PROPOSITION 1.1.**  *$\text{Bil}^\sigma$  consists exactly of the bilinear functionals  $F$  such that, for all  $x \in L^\infty(\mu)$  and all  $y \in L^\infty(\nu)$ ,  $f \mapsto F(f, y)$ , for  $f \in L^\infty(\mu)$ , is given by integration against an element of  $L^1(\mu)$  and  $g \mapsto F(x, g)$ , for  $g \in L^\infty(\nu)$ , is given by integration against an element of  $L^1(\nu)$ .*

*Proof.* Let  $F \in \text{Bil}^\sigma$ . Fix  $y \in L^\infty(\nu)$ . Since  $f \mapsto F(f, y)$  is weak\* continuous in  $f$ ,  $f \mapsto F(f, y)$  must belong to the dual space of  $L^\infty(\mu)$ , when  $L^\infty(\mu)$  is given the weak\* topology, that is,  $f \mapsto F(f, y)$  belongs to  $L^1(\mu)$ . The same argument applies to  $g \mapsto F(x, g)$ , for  $g \in L^\infty(\nu)$ .

On the other hand, suppose that, the bilinear functional  $F$  is such that for all  $x \in L^\infty(\mu)$ ,  $y \in L^\infty(\nu)$ ,  $f \mapsto F(f, y)$ , for  $f \in L^\infty(\mu)$ , is given by integration against an element of  $L^1(\mu)$  and  $g \mapsto F(x, g)$ , for  $g \in L^\infty(\nu)$ , is given by integration against an element of  $L^1(\nu)$ . Then for each fixed  $y \in L^\infty(\nu)$ ,  $f \mapsto F(f, y)$  is weak\* continuous in  $f$ , and for each fixed  $x \in L^\infty(\mu)$ ,  $g \mapsto F(x, g)$  is weak\* continuous in  $g$ . Hence,  $F \in \text{Bil}^\sigma$ . □

**PROPOSITION 1.2.** *Let  $\omega$  be a non-negative, finite regular Borel measure on  $X \times Y$ . Then  $\omega \in \text{Bil}^\sigma$  if and only if the projection of  $\omega$  onto  $X$  is absolutely continuous with respect to  $\mu$  and the projection of  $\omega$  onto  $Y$  is absolutely continuous with respect to  $\nu$ .*

*Proof.* If  $\omega$  has the projection property, then it obviously has the weak\* continuity property that is required for membership in  $\text{Bil}^\sigma$ .

On the other hand, suppose that  $\omega \in \text{Bil}^\sigma$ . Then  $f \mapsto \int (f \otimes 1) d\omega$  is a non-negative, locally finite, regular Borel measure on  $X$  that is the projection of  $\omega$  on  $X$ . Also,  $f \mapsto \int (f \otimes 1) d\omega$  is weak\* continuous from  $L^\infty(\mu)$  to  $\mathbb{C}$ . If the projection of  $\omega$  (let us call it  $\omega'$ ) were not absolutely continuous with respect to  $\mu$ , then we could find a sequence of functions  $f_n$  in  $C(X)$  such that  $0 \leq f_n \leq 1$ ,  $f_n \rightarrow 0$  a.e.  $d\mu$  and  $\int f_n d\omega' \not\rightarrow 0$ . Of course, that sequence  $f_n \rightarrow 0$  weak\* in  $L^\infty(\mu)$ , so

$$\int (f \otimes 1) d\omega \rightarrow 0,$$

a contradiction. [More abstractly, we could just point out that any linear functional on  $L^\infty(\mu)$  that is weak\* continuous is necessarily given by integration against an element of  $L^1(\mu)$ , by general Banach space duality.]

A similar argument shows that the projection of  $\omega$  on  $Y$  is absolutely continuous with respect to  $\nu$ . □

**LEMMA 1.3.** *Let  $R, S$  be von Neumann algebras, and let  $A, B$  be weak\* dense  $C^*$ -subalgebras of  $R, S$ , respectively. Then the mapping given by restricting  $\text{Bil}^\sigma(R, S)$  to  $(A \hat{\otimes} B)$  is an isometry; that is,  $\text{Bil}^\sigma(R, S)$  may be identified with a closed subspace of  $(A \hat{\otimes} B)^*$ .*

*Proof.* Let  $F \in \text{Bil}^\sigma(R, S)$ ,  $\varepsilon > 0$ , and let  $x \in R$ ,  $y \in S$  be of norm one such that  $|F(x, y) - \|F\|| < \varepsilon/3$ . By the Kaplansky density theorem [14, Theorem 4.8], there exist nets  $x_\alpha \rightarrow x$  and  $y_\beta \rightarrow y$  with  $x_\alpha$  all belonging to the unit ball of  $A$  and  $y_\beta$  all in the unit ball of  $B$ . By the weak\*-weak\* continuity of  $F$ ,  $F(x, y) = \lim_\alpha F(x_\alpha, y)$ . Hence, for some  $\alpha_0$  we have  $|F(x_{\alpha_0}, y) - F(x, y)| < \varepsilon/3$ . Similarly, there exists a  $\beta_0$  such that  $|F(x_{\alpha_0}, y_{\beta_0}) - F(x_{\alpha_0}, y)| < \varepsilon/3$ . Hence  $|F(x_{\alpha_0}, y_{\beta_0}) - \|F\|| < \varepsilon$ , and the result follows.  $\square$

**COROLLARY 1.4.** *The restriction of elements of  $\text{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$  to the space  $C_0(X) \hat{\otimes} C_0(Y)$  is an isometry. In particular,  $\text{Bil}^\sigma$  may be identified with a closed subspace of  $\text{BM}(X, Y)$ .*

We define  $\mathcal{L}^\infty(X)$  to be the space of all bounded Borel functions on  $X$ .

If  $\varphi_X \in \mathcal{L}^\infty(X)$ , and  $f_1 = f_2$  locally  $\mu$ -a.e., then  $\varphi_X f_1 = \varphi_X f_2$  locally  $\mu$ -a.e. In particular, for any  $f \in L^\infty(\mu)$ ,  $\varphi_X f$  defines an element in  $L^\infty(\mu)$ , and the map  $f \mapsto \varphi_X f$  is weak\*-weak\* continuous.

Given  $\varphi_X \in \mathcal{L}^\infty(X)$ ,  $\varphi_Y \in \mathcal{L}^\infty(Y)$ , and  $F \in \text{Bil}^\sigma$  we define a bounded bilinear functional  $\varphi \cdot F$  on  $L^\infty(\mu) \times L^\infty(\nu)$  by

$$\langle \varphi \cdot F, (f, g) \rangle = \langle F, (\varphi_X f, \varphi_Y g) \rangle$$

for  $f \in L^\infty(X)$  and  $g \in L^\infty(Y)$ . Then  $\varphi \cdot F \in \text{Bil}^\sigma$  and

$$\|\varphi \cdot F\| \leq \|F\| \|\varphi_X\|_\infty \|\varphi_Y\|_\infty.$$

We recall that the support of a bimeasure is the smallest closed subset  $Q$  in  $X \times Y$  such that  $\langle h, F \rangle = 0$  for all  $h \in V_0(X, Y)$  for which  $h \equiv 0$  in a neighborhood of  $Q$ .

The following three results are variants (as indicated) of known facts. The proofs are essentially identical to those cited.

**PROPOSITION 1.5** [7, Lemma 1.4]. *The set of elements of  $\text{Bil}^\sigma$  that have compact support is norm dense in  $\text{Bil}^\sigma$ .*

**PROPOSITION 1.6** [7, Lemma 1.5]. *Let  $X'$  (resp.  $Y'$ ) be a closed subspace of  $Y$  (resp.  $Y$ ) and  $\mu'$ ,  $\nu'$  denote the restrictions of  $\mu$ ,  $\nu$  to those closed subspaces. Then there is a projection of norm one from  $\text{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$  onto the space  $\text{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu'))$ .*

The image in  $\text{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu'))$  of a bimeasure is called the *restriction of the bimeasure to  $X' \times Y'$*  and is written  $F|_{X' \times Y'}$ .

**COROLLARY 1.7.** *Let  $G$  (resp.  $H$ ) be a locally compact group and  $G'$  (resp.  $H'$ ) an open subgroup. Then there is a norm one projection from  $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$  onto  $\text{Bil}^\sigma(L^\infty(G'), L^\infty(H'))$*

A bimeasure  $F$  is *discrete* if there exist sequences of finite subsets  $A_n$  of  $X$  and  $B_n$  of  $Y$  such that  $F = \lim_n F|_{A_n \times B_n}$  (norm limit). A bimeasure is *continuous* if its restriction to every product of finite sets is zero. Obviously,  $\text{BM}_c$  and  $\text{BM}_d$  are norm closed vector spaces. The set of discrete bimeasures is denoted  $\text{BM}_d(X, Y)$  and the set of continuous bimeasures is denoted  $\text{BM}_c(X, Y)$ . Graham and Schreiber showed that topologically  $\text{BM}(X, Y) = \text{BM}_d(X, Y) \oplus \text{BM}_c(X, Y)$  [7, Theorem 1.8].

**PROPOSITION 1.8.** *If either  $\mu$  or  $\nu$  is a continuous measure, then  $\text{Bil}^\sigma$  is contained in  $\text{BM}_c(X, Y)$ . In particular,  $\text{Bil}^\sigma$  is a proper subset of  $\text{BM}(X, Y)$ .*

*Proof.* Let  $F \in \text{Bil}^\sigma$ . By Proposition 1.5, we may assume that  $F$  is supported on a compact set  $X' \times Y'$ , so we will not distinguish between  $F$  and  $F|_{X' \times Y'}$ . We write  $F = F_1 + F_2$ , where  $F_1$  is continuous and  $F_2$  is discrete. Let  $A_n \subset X'$  (resp.  $B_n \subset Y'$ ) be increasing sequences of finite subsets such that  $F_2 = \lim_n F|_{A_n \times B_n}$ . Let  $A = \bigcup A_n$ . Suppose that  $\mu$  is a continuous measure. Then  $\mu(A) = 0$ . By Lusin's Theorem [12, p. 54], (and enlarging  $A$  if necessary) there exists a sequence of continuous functions  $\{f_j\}$  such that  $0 \leq f_j \leq 1$  for all  $j$ ,  $f_j \rightarrow 0$  on  $A$ ,  $f_j \rightarrow 1$  on  $X' \setminus A$  (pointwise in both cases), and the  $f_n$  are supported in a common compact superset of  $X' \times Y'$ . It follows that for each integer  $n$ , every  $f \in C_0(X)$  and every  $g \in C_0(Y)$ ,

$$F(f, g) = \lim_j F(f_j f, g) = \lim_j (F_1 + F_2)(f_j f, g),$$

and

$$F_2|_{A_n \times B_n}(f, g) = \lim_j F_2(f_j f, g) = 0.$$

The first equality above follows from the weak\* continuity of  $F$  and the second from the fact that  $f_j f \rightarrow 0$  on  $A_n$  combined with the dominated convergence theorem. Thus,  $F_2(f, g) = 0$  for all  $f, g$ , so  $F_2 = 0$ . □

**LEMMA 1.9.** *Let  $\mu$  and  $\nu$  be non-negative, locally finite, regular Borel measures on the locally compact spaces  $X, Y$ , respectively. Then*

for any  $F \in \text{Bil}^\sigma$  there exist  $p \in L^1(\mu)$  and  $q \in L^1(\nu)$  such that  $p \geq 0$ ,  $q \geq 0$ ,  $\|p\|_1 = \|q\|_1 = 1$  and

$$(1.1) \quad |F(f, g)| \leq K \|F\| \left( \int |f|^2 p \, d\mu \right)^{1/2} \left( \int |g|^2 q \, d\nu \right)^{1/2}$$

for all  $f \in L^\infty(\mu)$  and  $g \in L^\infty(\nu)$ , where  $K$  is a universal constant.

*Proof.* Suppose that  $F \in \text{Bil}^\sigma$ . By Proposition 1.5, we know that  $F$  has  $\sigma$ -compact support. We thus may assume that  $\mu$  and  $\nu$  are  $\sigma$ -finite (since they are locally finite). [Indeed, let the support of  $F$  be  $\bigcup_{j=1}^\infty X_j \times Y_j$ , where the  $X_j, Y_j$  are compact. Let  $\mu_j$  (resp.  $\nu_j$ ) be the restriction of  $\mu$  to  $X_j$  (resp.  $Y_j$ ). The assumption of local finiteness implies that  $\mu_j, \nu_j$  are  $\sigma$ -finite measures.] Of course,  $L^\infty(\mu)$  does not change if we replace  $\mu$  by an equivalent probability measure. Also, weak\* topologies on the  $L^\infty$  space induced by the two measures (the probability measure and the original measure) are identical, by the uniqueness of the predual of  $L^\infty(\mu)$  (see [14, p. 135]). Let the support of  $F$  be  $\bigcup_{j=1}^\infty X_j \times Y_j$ , where the  $X_j, Y_j$  are compact. Let  $\mu_j$  (resp.  $\nu_j$ ) be the restriction of  $\mu$  to  $X_j$  (resp.  $Y_j$ ). The assumption of local finiteness implies that  $\mu_j, \nu_j$  are finite measures. We may assume that  $\mu_1$  and  $\nu_1$  have norm  $\frac{1}{2}$  and that  $\|\mu_{j+1} - \mu_j\| = 2^{-j}$  and similarly for the  $\nu_j$  for all  $j$ . Hence,  $F \in \text{Bil}^\sigma(L^\infty(\sum \mu_j), L^\infty(\sum \nu_j))$ . Thus, we may assume that  $\mu$  and  $\nu$  are probability measures.

Let a Grothendieck measure pair  $\mu', \nu'$  for  $F$  be given. Then the pair  $\mu', \nu'$  has the property that

$$(1.2) \quad |F(f, g)| \leq K \|F\| \|f\|_{L^2(\mu')} \|g\|_{L^2(\nu')} \\ \text{for all } f \in C(X), g \in C(Y),$$

where  $K$  is the usual complex Grothendieck constant. Furthermore,  $\mu'$  is a probability measure on  $X'$  and  $\nu'$  is a probability measure on  $Y'$ .

Let  $\mu' = \mu_a + \mu_s$ , where  $\mu_a$  is absolutely continuous with respect to  $\mu$  and  $\mu_s$  is singular with respect to  $\mu$ . Let  $A, B$  be a partition of  $X$  into two disjoint Borel sets such that  $\mu(B) = 0$ , and

$$\mu_a(E) = \mu'(A \cap E) \quad \text{and} \quad \mu_s(E) = \mu'(B \cap E) \quad \text{for all Borel } E \subset X.$$

Let  $f \in L^\infty(\mu)$  have norm one. By Lusin's Theorem [12, p. 54], there exists a sequence  $\{f_n\}$  in  $C(X)$  such that  $\|f_n\| \leq 1$  for all  $n$  and  $f(x)\chi_A(x) = \lim_{n \rightarrow \infty} f_n$  pointwise a.e.  $d(\mu + \mu_s)$ . We note that  $f\chi_A = f$   $\mu$ -a.e. and  $f\chi_A = 0$   $d\mu_s$ -a.e. Hence, for each  $h \in L^1(\mu)$ ,

$f_n \cdot h \rightarrow f \cdot h$  pointwise  $d\mu$ -a.e. and  $|f_n \cdot h| \leq |f \cdot h| d\mu$ -a.e. for all  $n$ . By the dominated convergence theorem (and here we need the actual finiteness of  $\mu$ ),  $\int f_n \cdot h d\mu \rightarrow \int f \cdot h d\mu$ . That is,

$$(1.3) \quad f_n \rightarrow f \text{ in the weak* topology of } L^\infty(\mu).$$

Since  $f_n \rightarrow 0$  pointwise a.e.  $d\mu_s$ ,  $|f_n| \rightarrow 0$  pointwise a.e.  $d\mu_s$ . Since  $|f_n|^2 \leq 1$ , the dominated convergence theorem again implies that

$$(1.4) \quad \begin{aligned} \int |f_n|^2 d\mu_s &\rightarrow \int |f|^2 d\mu_s = 0 \quad \text{and} \\ \int |f_n|^2 d\mu_a &\rightarrow \int |f|^2 d\mu_a. \end{aligned}$$

Hence  $\int |f_n|^2 d\mu' \rightarrow \int |f|^2 d\mu_a$ . Also, by (1.2),

$$(1.5) \quad |F(f_n, g)| \leq K \|F\| \|f_n\|_{L^2(\mu')} \|g\|_{L^2(\nu')} \quad \text{for all } g \in C(Y).$$

Now,  $F(f_n, g) \rightarrow F(f, g)$  by (1.3) and

$$\begin{aligned} \|f_n\|_{L^2(\mu)}^2 &= \int |f_n \chi_A|^2 d\mu' \\ &\rightarrow \int |f|^2 d\mu' \\ &= \int |f|^2 d\mu_a + \int |f|^2 d\mu_s \\ &= \int |f|^2 d\mu_a, \end{aligned}$$

by (1.4). Therefore,

$$|F(f, g)| \leq K \|F\| \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu')},$$

by (1.5).

A similar argument applied to  $g \in L^\infty(\nu)$  gives

$$|F(f, g)| \leq K \|F\| \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu_a)}. \quad \square$$

Let  $f$  be a Borel function on the locally compact space  $X$ , and  $\omega$  be a non-negative, locally finite, regular Borel measure on  $X$ . We say that  $f$  is *bounded away from 0 and  $\infty$*  if there exist constants  $0 < c < C < \infty$  such that  $c \leq f(x) \leq C$  a.e.  $d\omega$ .

**LEMMA 1.10.** *Let  $\mu$  and  $\nu$  denote regular Borel locally measures on the locally compact spaces  $X$  and  $Y$ . Then  $\text{Bil}^\sigma$  has a dense subset consisting of the bilinear functionals  $F$  such that their Grothendieck*

measures  $\mu_g, \nu_g$  are such that  $d\mu_g/d\mu$  and  $d\nu_g/d\nu$  are bounded away from zero and away from  $\infty$ .

*Proof.* Let  $F \in \text{Bil}^\sigma$ . We may assume that  $\mu$  and  $\nu$  are probability measures and that we have a Grothendieck measure pair  $\mu_g, \nu_g$  for  $F$  with  $\mu_g \ll \mu$  and  $\nu_g \ll \nu$ . The validity of this second assumption follows from Lemma 1.9.

Now, by (1.1, and using the notation of Lemma 1.9.), if  $A$  is a Borel subset of  $X$  and  $B$  is a Borel subset of  $Y$ , then

$$|\langle f\chi_A \otimes g\chi_B, F \rangle| \rightarrow 0 \quad \text{as } \mu(A) \rightarrow 0, \text{ and/or } \nu(B) \rightarrow 0,$$

by the Lebesgue dominated convergence theorem.

Thus, given  $n > 0$ , define the Borel sets  $A_n, B_n$  by

$$A_n = \{x \in X : p(x) \notin [1/n, n]\}$$

and

$$B_n = \{y \in Y : q(y) \notin [1/n, n]\},$$

where  $p, q$  are as in Lemma 1.9.

Then  $\mu(A_n) \rightarrow 0$  and  $\nu(B_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $\delta > 0$  be given. Then there exists  $n > 0$  such that

$$|\langle f\chi_{A_n} \otimes g\chi_{B_n}, u \rangle| \leq \frac{\delta}{4} \|f\|_\infty \|g\|_\infty \quad \text{for all } f \in L^\infty(\mu), g \in L^\infty(\nu).$$

We let  $F_1 = (\chi_{A_n} \otimes \chi_{B_n})\mu \times \nu + ((1 - \chi_{A_n}) \otimes (1 - \chi_{B_n}))F$ . It is then clear that  $\|F - F_1\| \leq \delta$ .  $\square$

**2. Locally compact groups.** In this section,  $G$  and  $H$  will be locally compact groups, not both discrete. We now write  $\text{Bil}^\sigma$  in place of  $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$ . We study the properties of the particular space  $\text{Bil}^\sigma$ , where we are already using the group structure to define  $\text{Bil}^\sigma$ . We remind the reader that we continue the identification of  $\text{Bil}^\sigma$  with a closed subspace of  $\text{BM}_c(G, H)$  (see Corollary 1.4 and Proposition 1.8).

Furthermore, by Proposition 1.1,  $\text{Bil}^\sigma$  consists of the bilinear functionals  $F$  such that, for all  $x \in L^\infty(m_G), y \in L^\infty(m_H), f \mapsto F(f, y)$  ( $f \in L^\infty(m_G)$ ) is given by integration against an element of  $L^1(\mu)$  and  $g \mapsto F(x, g)$  ( $g \in L^\infty(m_H)$ ) is given by integration against an element of  $L^1(\nu)$ .

We note that  $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$  is a  $(L^\infty(m_G), L^\infty(m_H))$  module in the sense that the (obviously bounded) operations  $(g \cdot F)$

and  $F \cdot f$  are defined by

$$(g \cdot F)(h, k) = F(h, gk) \quad \text{and} \quad (F \cdot f)(h, k) = F(fh, k)$$

for all  $F \in \text{Bil}(L^\infty(m_G), L^\infty(m_H))$ ,  $f, h \in L^\infty(m_G)$  and  $g, k \in L^\infty(m_H)$ .

Also,  $\text{Bil}^\sigma$  is a closed submodule of the  $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$ .

We define  $L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu) =_{\text{def}} (\text{Bil}^\sigma)^*$ . Then  $L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu)$  is a dual  $(L^\infty(m_G), L^\infty(m_H))$ -module when the operations are defined by

$$\langle g \cdot M, F \rangle = \langle M, g \cdot F \rangle \quad \text{and} \quad \langle M \cdot f, F \rangle = \langle M, F \cdot f \rangle,$$

where  $M \in L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu)$ ,  $F \in \text{Bil}^\sigma$ ,  $f \in L^\infty(\mu)$  and  $g \in L^\infty(\nu)$ .

A dual module is *normal* if the mappings

$$\begin{aligned} f &\mapsto f \cdot M \text{ from } L^\infty(\mu) \rightarrow L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu) \quad \text{and} \\ g &\mapsto M \cdot g \text{ from } L^\infty(\nu) \rightarrow L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu) \end{aligned}$$

are both weak\*-weak\* continuous.

**THEOREM 2.1.** *Let  $G$  and  $H$  be locally compact groups. Then  $\text{Bil}^\sigma$  is an ideal in  $\text{BM}(G, H)$ . Also,  $\text{Bil}^\sigma$  is a normal  $(L^\infty(G), L^\infty(H))$  module.*

*Proof.* Immediate from Lemma 1.9 and the facts that (i)  $\text{BM}(G, H)$  is an algebra under convolution (see [7, 2.5] or [4, 2.4]) and (ii) that the Grothendieck measures for a convolution product may be taken to be the convolutions of the Grothendieck measures of the factors [4, loc. cit].

The last assertion is a consequence of [3, Lemma 2.2] and Lemma 1.9 above. □

**REMARKS 2.2.** (a) Note that the mapping

$$\theta: L^\infty(G) \otimes L^\infty(H) \rightarrow L^\infty(G) \otimes^\sigma L^\infty(H)$$

defined by  $\theta(f \otimes g)(F) = F(f, g)$  is one-to-one. Hence, we may identify the space  $L^\infty(G) \otimes L^\infty(H)$  with its image in  $L^\infty(G) \hat{\otimes}^\sigma L^\infty(H)$ . That image is weak\* dense.

Furthermore, if  $M \in L^\infty(G) \hat{\otimes}^\sigma L^\infty(H)$  of norm one, then there is a net  $M_\alpha = \sum \lambda_i^\alpha (f_i^\alpha \otimes g_i^\alpha)$ , with the  $f_i$ 's and  $g_i$ 's in their respective unit balls, the  $\lambda_i$ 's nonnegative with sum one, such that  $M_\alpha \rightarrow M$  in the weak\* topology. (See [3, p. 139 and p. 141].)

(b) There is a unique weak\*-continuous extension to  $L^\infty(G) \otimes^\sigma L^\infty(H)$  of the multiplication map

$$\pi: L^\infty(G) \otimes L^\infty(G) \rightarrow L^\infty(G)$$

given by  $f \otimes g \mapsto f \cdot g$  (see [3, p. 142]).

**THEOREM 2.3.** *Let  $G$  and  $H$  be compact groups. Then  $\text{Bil}^\sigma$  has a dense subset consisting of the bilinear functionals  $F$  such that their Grothendieck measures  $\mu, \nu$  are such that  $d\mu/dm_G$  and  $d\nu/dm_H$  are bounded away from zero and away from  $\infty$ .*

*Proof.* Immediate from Lemma 1.10. □

**LEMMA 2.4.** *Let  $\mu$  and  $\nu$  be continuous probability measures on the locally compact spaces  $X$  and  $Y$  respectively. Then there is a projection of norm one from  $\text{BM}(X, Y)$  onto  $\text{Bil}^\sigma$ .*

*Proof.* It is well-known (and easy to see) that  $\text{BM}(X, Y)$  may be imbedded isometrically in  $\text{Bil}(M(X)^*, M(Y)^*)$ . Let  $f_0 \in M(X)^*$ , be such that  $f_0$  is one a.e. with respect to (the image of)  $\mu$  and zero with respect to (the image of) all measures on  $X$  that are singular with respect to  $\mu$ . Define  $g_0 \in M(X)^*$  analogously. Then the composition of  $F \mapsto (f_0 \times g_0)F$  with the restriction of the resulting element to  $C(Y) \times C(Y)$  is a linear norm-reducing mapping  $P$  of  $\text{BM}(X, Y)$ . Furthermore,  $PF = F$  for all  $F \in \text{Bil}^\sigma$ . Finally, (straightforward computations show that)  $f \mapsto PF(f, g)$  is absolutely continuous with respect to  $\mu$  for all  $g \in C(Y)$  and that  $g \mapsto PF(f, g)$  is absolutely continuous with respect to  $\nu$  for all  $f \in C(X)$ . That is,  $PF \in \text{Bil}^\sigma$ . It follows that  $P$  is the required projection. □

**THEOREM 2.5.** *Let  $G$  and  $H$  be locally compact groups. Then there is no projection from  $\text{Bil}^\sigma$  onto the closed subspace of  $\text{Bil}^\sigma$  generated by  $L^1(G \times H)$ .*

*Proof.* This is immediate from [7, Theorem 1] and Lemma 2.4 above. □

**LEMMA 2.6.** *Let  $G$  be a compact group and  $U$  an open subset of  $G$ . Then there exists an integer  $n \geq 1$  such that  $U^n$  is an open subgroup of  $G$ .*

*Proof.* Let  $y \in U$ . The closed semigroup  $H$  generated by  $y$  is a compact semigroup. Therefore  $H$  contains an idempotent [2, 1.8]; that idempotent is necessarily the identity of  $G$ . (Alternatively, we can apply the fact [2, 1.10] that a compact subsemigroup of a group is a subgroup, so  $e \in H$ , which is, in fact, a group.) In any case,  $e$  is in the closure of  $\{y^l\}$ .

Let  $V$  be any symmetric neighborhood of  $e$ . We may assume that  $V$  is so small that  $yV \subseteq U$ . Then

$$y^l V \subseteq (yV)^l \subseteq U^l.$$

Since  $\{y^l\}$  accumulates at  $e$ , there are large  $l$ 's such that  $y^l \in V^{-1} = V$ . Therefore  $y^{-l} \in V$ , so

$$e = y^l y^{-l} \in y^l V \subseteq U^l.$$

That is,  $e \in U^l$ . Thus, we may assume  $e \in U^{lm}$  for all  $m > 0$ . In particular, the sets  $U^{lm}$  are increasing. Again, consider the closed subgroup  $H$  generated by  $y \in U^{lm_0}$  for some  $m_0 > 0$ . ( $H$  is a subgroup by [2, 1.10].) If that closed subgroup is finite, then eventually it is contained in  $U^{lm}$  for some  $m \geq m_0$ . Otherwise, every element of it is an accumulation point of the set  $\{y^n : n > 0\}$ . (That also follows from the fact that a compact semigroup in a compact group is necessarily a group.) Hence, every element of  $H$  belongs to some  $U^{lm}$ . This argument applies to every element of  $\bigcup_{m \geq 1} U^{lm}$ . That is, the group  $K = \bigcup_{m \geq 1} U^{lm}$ .

Since  $\bigcup_{m \geq 1} U^{lm}$  is a group, and open, it is also a closed subgroup, and therefore it is compact. Therefore  $\bigcup_{m \geq 1} U^{lm} = U^{lm(0)}$  for some  $m(0)$ .

By the monotonicity of the  $U^{lm}$ ,  $K = U^{lm(0)}$  □

We can now give a variant of Lemma 1.10.

**THEOREM 2.7.** (1) *Let  $G, H$  be compact and connected groups. Then the set of those  $u \in \text{Bil}^\sigma$  for which there is an  $n \geq 1$  for which the Grothendieck measures for  $u^n$  are Haar measure is a dense subset of  $\text{Bil}^\sigma$ .*

(2) *Let  $G, H$  be compact groups. Then the set of those  $F \in \text{Bil}^\sigma$  for which there is an  $n \geq 1$  for which the Grothendieck measures for  $F^n$  are Haar measure on an open subgroup of  $G$  is a dense subset of  $\text{Bil}^\sigma$ .*

*Proof.* Let  $\mu, \nu$  be Grothendieck measures for  $F$ . Then  $\mu = (f + g)m_G$ , where  $f$  is continuous and  $\|g\|$  is small. Similarly for  $\nu$ . Then the Grothendieck measures for  $F^n$  are  $\mu^n$  and  $\nu^n$ . By Lemma 2.6,  $f^n > 0$  on an open subgroup of  $G$ . We may throw away the terms involving  $g$  in  $(f + g)^n$ , thus obtaining the required conclusion for both (i) and (ii). □

**3. Compact abelian groups.** Suppose that  $G$  and  $H$  are compact abelian groups with character groups  $\widehat{G}$  and  $\widehat{H}$ , respectively.

Let  $u \in \text{BM}(G, H)$ . The *Fourier transform*  $\hat{u}$  of  $u$  is defined by

$$\hat{u}(\gamma, \rho) = \langle \bar{\gamma} \otimes \bar{\rho}, u \rangle, \quad \text{for all } \gamma \in \widehat{G}, \rho \in \widehat{H}.$$

Then  $\hat{u}$  is well-defined and  $\|\hat{u}\|_\infty \leq \|u\|$  (see [7, p. 97]).

**REMARK.** The multipliers of  $\text{Bil}^\sigma$  are exactly the elements of  $\text{BM}(G, H)$ .

This is immediate upon taking weak\* limits, since the unit ball of  $\text{Bil}^\sigma$  is dense in the unit ball of  $\text{BM}$ , even though (see below)  $\text{Bil}^\sigma$  does not have an approximate identity. Here are some details.

We first note that the measures in the unit ball of  $\text{Bil}^\sigma$  are weak\* dense in that ball (one proof of that is known as Riemann sums for double integrals; another is known as “bounded spectral synthesis” for sets whose union is a Kronecker set [13, Corollary 4]). The argument in the “bounded spectral synthesis” form easily adapts to the case of approximation by measures belonging to a fixed  $L$ -space that is weak\* dense in  $M(G \times H)$ . Hence, the measures in the unit ball of  $\text{Bil}^\sigma$  are dense in the unit ball of  $\text{BM}(G, H)$ .

Suppose that  $\varphi$  is a function defined on  $\widehat{G} \times \widehat{H}$  such that  $\varphi \hat{u}$  is the Fourier transform (see below) of an element of  $\text{Bil}^\sigma$  for all  $u \in \text{Bil}^\sigma$ . Then  $\|\varphi \hat{u}\| \leq C \|u\|$  for all  $u$  and some constant  $C$ . We note that the set Fourier-Stieltjes transform of  $\text{BM}(G, H)$  is closed under bounded pointwise convergence (that follows from a diagonalization argument and the fact that the unit ball of  $\text{BM}(G, H)$  is compact in the weak\* topology). By taking weak\* limits (within the unit ball), we conclude that  $\varphi$  is a multiplier of  $\text{BM}(G, H)$ . Since  $\text{BM}(G, H)$  has an identity, the remark follows.  $\square$

Suppose that we have a  $u$  whose Grothendieck measures  $\mu, \nu$  are such that  $d\mu/dm_G$  and  $d\nu/dm_H$  are bounded away from zero and away from  $\infty$ . Then, by using that and the Plancherel Theorem, we can identify  $L^2(\mu)$  with  $L^2(\widehat{G})$  and  $L^2(\nu)$  with  $L^2(\widehat{H})$ . Using those identifications, we can explicitly compute the linear mapping  $T: L^2(\widehat{G}) \rightarrow L^2(\widehat{H})$ . Here,  $T$  is the mapping associated with the Grothendieck measures. Of course, we have lost information about the constant in the Grothendieck inequality. The new mapping  $T$  is given by:

$$(T\hat{f})(\rho) = \sum_{\gamma} \hat{u}(\gamma, \rho) \hat{f}(\gamma),$$

where  $\hat{f} \in L^2(\hat{G})$ . That follows at once from the fact that

$$\langle u, f \otimes g \rangle = \sum_{\gamma, \rho} \hat{u}(\gamma, \rho) \hat{f}(\gamma) \hat{g}(\rho), \quad \text{for all } f \in C(G), \quad g \in C(H),$$

which, in turn, is an easy calculation from

$$\langle u, f \otimes g \rangle = \left\langle u, \left( \sum \hat{f}(\gamma) \bar{\gamma} \right) \otimes \left( \sum \hat{g}(\rho) \bar{\rho} \right) \right\rangle.$$

The norm of the new  $T$  is now bounded by the product of three numbers: the norm of the old  $T$ , the supremum of  $d\mu/dm_G$ , and the reciprocal of the infimum of  $d\nu/dm_H$ .

**PROPOSITION 3.1.** *Let  $G$  and  $H$  be compact abelian groups. Let  $u \in \text{Bil}^\sigma$ . Suppose that  $u$  has  $\mu, \nu$  for its Grothendieck measures with  $d\mu/dm_G$  and  $d\nu/dm_H$  both bounded away from zero and away from  $\infty$ . Then there exists a constant  $C > 0$  such that  $\sum_\gamma |\hat{u}(\gamma, \rho)|^2 < C$  for every fixed  $\rho \in \hat{H}$  and  $\sum_\rho |\hat{u}(\gamma, \rho)|^2 < C$  for every fixed  $\gamma \in \hat{G}$ .*

*Proof.* By the discussion preceding the statement of Proposition 3.1, we see that there is a linear transformation  $T: L^2(G) \rightarrow L^2(H)$  such that

$$\langle u, f \otimes g \rangle = \langle Tf, g \rangle \quad \text{for all } f \in C(G) \text{ and all } g \in C(H).$$

(This transformation is the composition of the transformation discussed above with two Plancherel transformations.) Then

$$\sum_\gamma |\hat{u}(\gamma, \rho)|^2 \leq \|T^*(\rho)\|_2^2 \leq \|T\|. \quad \square$$

**COROLLARY 3.2.** *Let  $u \in \text{Bil}^\sigma$ , where  $G, H$  are compact abelian groups. Then for every  $\varepsilon > 0$  there exists  $N > 0$  such that for each  $\rho \in \hat{H}$ ,*

$$\text{Card}\{\gamma : |\hat{u}(\gamma, \rho)| > \varepsilon\} \leq N.$$

*Proof.* We fix  $\varepsilon > 0$ . Let  $v$  be such that  $\|u - v\| < \varepsilon/3$  and such that  $v$  satisfies the hypotheses of Proposition 3.1. We let  $N$  be any integer greater than  $9C/\varepsilon^2$  (the  $C$  is from Proposition 3.1 applied to  $v$ ). Then  $|\hat{u}(\gamma, \rho)| > \varepsilon$  implies  $|\hat{v}(\gamma, \rho)| > \varepsilon/3$ , and that can occur at most  $9C/\varepsilon^2$  times.  $\square$

**THEOREM 3.3.** *Let  $u \in \text{Bil}^\sigma$ , where  $G, H$  are compact abelian groups. Then the spectral radius of  $u$  is*

$$\sup_{\gamma \in \hat{G}, \rho \in \hat{H}} |\hat{u}(\gamma, \rho)|.$$

*Proof.* By Theorem 2.3, we may assume that there is a Grothendieck measure pair  $\mu, \nu$  for  $u$  such that  $d\mu/dm_G$  and  $d\nu/dm_H$  are both bounded away from zero and infinity. Thus, we may assume that there is bounded linear transformation  $T: L^2(G) \rightarrow L^2(H)$  such that  $\langle u, f \otimes g \rangle = \langle Tf, g \rangle$  for all  $f \in C(G)$  and  $g \in C(H)$ . Furthermore, for all continuous  $f$  on  $G$ ,  $g$  on  $H$ ,

$$(3.1) \quad \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)| |\hat{f}(\gamma)| |\hat{g}(\rho)| = |\langle u * \tilde{u}, f \otimes g \rangle| \\ \leq C' \|u\|^2 \|f\|_2 \|g\|_2,$$

where  $C'$  is the product of four numbers:  $K$  (the Grothendieck constant), the norm of  $u$ , the supremum of  $d\mu/dm_G$ , and the reciprocal of the infimum of  $d\nu/dm_H$ .

Let  $f_1$  denote the Radon-Nikodym derivative  $d\mu/dm_G$ . Then  $f_1$  has  $L^1$ -norm 1 and is bounded away from zero and infinity. Therefore, the  $n$ th convolution powers of  $f_1$  converge to 1 uniformly, by Lemma 3.4 below. The same applies to  $g_1 = d\nu/dm_H$ .

That means that the Grothendieck measures (call them  $\mu_n, \nu_n$ ) for  $u^n$  become closer and closer to Haar measures, so the norm of the isomorphisms (and of their inverses) between  $L^2(\mu_n)$  and  $L^2(\hat{G})$  on the one hand, and  $L^2(\nu_n)$  and  $L^2(\hat{H})$  on the other hand, approach one. Thus, for sufficiently large  $n$ , we may assume that

$$\|u^n\|_{\text{Bil}^\sigma} \leq C \sup\{|\langle u^n, f \otimes g \rangle| : \|f\|_2 \|g\|_2 \leq 1\},$$

where  $C$  does not depend on  $n$ , and the supremum is taken over all  $f, g$  of uniform norm one.

But

$$\langle u^n, f \otimes g \rangle = \sum_{\gamma, \rho} \hat{u}^n(\gamma, \rho) \hat{f}(\gamma) \hat{g}(\rho).$$

Therefore

$$|\langle u^n, f \otimes g \rangle| \leq \|\hat{u}^{n-2}\|_\infty \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)| |\hat{f}(\gamma)| |\hat{g}(\rho)|.$$

It follows that

$$\|u^n\|_{\text{Bil}^\sigma} \leq C \|\hat{u}^{n-2}\|_\infty \sup \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)| |\hat{f}(\gamma)| |\hat{g}(\rho)|,$$

where  $C$  does not depend on  $n$ , and the supremum is taken over all  $f, g$  of uniform norm one. By (3.1),

$$\sup_{\|f\|_\infty \leq 1, \|g\|_\infty \leq 1} \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)| |\hat{f}(\gamma)| |\hat{g}(\rho)| = \sup |\langle u * \tilde{u}, f \otimes g \rangle| \\ \leq C' \|u\|^2 \|f\|_2 \|g\|_2,$$

so  $\|u^n\|_{\text{Bil}^\sigma} \leq C'' \|\hat{u}^{n-2}\|_\infty$  for all  $n$ .

The conclusion about the spectral radius now follows easily.  $\square$

LEMMA 3.4. *Let  $f$  be a bounded non-negative Borel function on the compact group  $G$  that is bounded away from zero and has  $L^1$ -norm one. Then the sequence of convolution powers of  $f$  converges uniformly to 1.*

*Proof.* Since  $f$  is bounded,  $f \in L^2(G)$  and  $\hat{f} \in L^2(\widehat{G})$ . Therefore  $f^2 = f * f$  has an absolutely convergent Fourier series, so, in particular,  $\hat{f} \in c_0(\widehat{G})$ . Since  $f > 0$  and  $\|f\|_1 = 1$ ,  $\hat{f}(0) = 1$ . We apply the Lebesgue Dominated Convergence Theorem to  $\hat{f}^n$  (with  $|\hat{f}|^2$  being the dominating function and  $n > 2$ ) to conclude that  $\hat{f}^n$  converges in  $l^1$ -norm to a function  $f'$  that is equal to the characteristic function of a finite subset of  $\widehat{G}$  (finite because  $\hat{f} \in c_0(\widehat{G})$ ). Of course, that means that  $f^n$  converges uniformly to a function  $f_1$  that is non-zero everywhere (the infimum of  $f^n$  is increasing with  $n$ ). Thus,  $f_1 m_G$  is an idempotent probability measure. By [11, 3.2.4],  $f_1 m_G$  is Haar measure on a compact subgroup of  $G$ . Since  $\hat{f}_1 = f'$  has finite support, that subgroup has finite index. If the index were greater than 1,  $f_1$  would be zero somewhere, a contradiction. Therefore  $f_1 = 1$  everywhere.  $\square$

COROLLARY 3.5. *Let  $G$  and  $H$  be compact abelian groups. Then  $\widehat{G} \times \widehat{H}$  is dense in the maximal ideal space of  $\text{Bil}^\sigma$  and  $\text{Bil}^\sigma$  is a symmetric Banach algebra*

*Proof.* This is a standard argument: the result is more or less immediate from Theorem 3.3. Here are the details.

We first note that  $\text{Bil}^\sigma$  is self-adjoint. For if  $S \in \text{Bil}^\sigma$  is such that its Gelfand transform  $\widehat{S}$  is real on  $\widehat{G} \times \widehat{H}$ , but not real on all of  $\Delta \text{Bil}^\sigma$  (the maximal ideal space), then for an appropriate  $k > 1$ ,  $\exp(ikS)$  has Gelfand transform larger than one at that non-real value, but has Fourier-Stieltjes transform at most one, thus contradicting Theorem 3.3.

Since the space of Gelfand transforms  $\widehat{\text{Bil}^\sigma}$  is self-adjoint and separating, it is uniformly dense in  $C_0(\Delta \text{Bil}^\sigma)$ . If  $\widehat{G} \times \widehat{H}$  were not dense in  $\Delta \text{Bil}^\sigma$ , then there would be a continuous function  $f$  on  $\Delta \text{Bil}^\sigma$  such that  $\|f\|_\infty = 1$  and  $|f| < 1/2$  on  $\widehat{G} \times \widehat{H}$ . By estimating  $f$  uniformly by an element of  $\text{Bil}^\sigma$ , we again contradict Theorem 3.3.  $\square$

We now give an example of an element of  $\text{Bil}^\sigma$ . The example is simple; we use it to show that  $\text{Bil}^\sigma$  does not have approximate identities, even unbounded ones.

Let  $\mu$  and  $\nu$  denote regular Borel probability measures on the locally compact spaces  $X$  and  $Y$ . Suppose that  $\{\gamma_\alpha\}$  is an orthonormal basis for  $L^2(\mu)$ , and that  $\{\rho_\beta\}$  is an orthonormal basis for  $L^2(\nu)$ . Let subsequences of those bases be chosen. Let  $F(\gamma_\alpha, \rho_\beta)$  be defined by

$$F(\gamma_{\alpha_j}, \rho_{\beta_k}) = \begin{cases} 2^{-k/2}, & 2^k \leq j \leq 2^{k+1} - 1 \text{ and } j \geq 1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F(\gamma_\alpha, \rho_\beta) = 0 \text{ if there is no pair } j, k \text{ with } \alpha = \alpha_j \text{ and } \beta = \beta_k.$$

**PROPOSITION 3.6.** *With the above hypotheses,*

- (1)  *$F$  is a bilinear functional on  $L^2(\mu) \times L^2(\nu)$  that is bounded by 1;*
- (2)  *$F$  represents an element of  $\text{Bil}^\sigma$ ; and*
- (3) *Grothendieck measures for  $F$  are given by  $\mu, \nu$ .*

*Proof.* For the first part, let  $x, y \in L^2(\mu) \times L^2(\nu)$ , and let  $x_j = \langle x, \gamma_{\alpha_j} \rangle$  for all  $j$  and  $y_k = \langle y, \rho_{\beta_k} \rangle$  for all  $k$ . Let also  $F_{j,k} = F(\gamma_{\alpha_j}, \rho_{\beta_k})$ . Then

$$F(x, y) = \sum_k \sum_{j=2^k}^{2^{k+1}-1} F_{j,k} x_j y_k.$$

We may assume that the  $x_j$  and  $y_k$  are non-negative. For each  $k$ ,

$$\sum_{j=2^k}^{2^{k+1}-1} F_{j,k} x_j \leq \left( \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2},$$

by the Cauchy-Schwarz inequality. Therefore,

$$\begin{aligned} |F(x, y)| &\leq \sum_k \left( \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} y_k \\ &\leq \left( \sum_k \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} \left( \sum_k y_k^2 \right)^{1/2}. \end{aligned}$$

That is,

$$(3.2) \quad F(x, y) \leq \|x\|_{L^2(\mu)} \|y\|_{L^2(\nu)}.$$

For the second assertion, by the first part and the fact that  $\mu, \nu$  are probability measures,  $|F(x, y)| \leq \|x\|_\infty \|y\|_\infty$  for all  $x \in L^\infty(\mu), y \in L^\infty(\nu)$ . Hence  $F$  represents an element of  $\text{Bil}(L^\infty(\mu), L^\infty(\nu))$ . We must show that  $F$  is weak\* continuous in each variable separately. Suppose that  $x_\lambda \rightarrow x$  weak\* in  $L^\infty(\mu)$  and that  $y \in L^\infty(\nu)$ . Note that  $L^\infty(\mu) \subseteq L^2(\mu) \subseteq L^1(\mu)$ . By the latter containment,  $x_\lambda$  converges weak\* in  $L^2(\mu)$ . Since  $L^\infty(\mu)$  is dense in  $L^2(\mu)$ ,

$$x_\lambda \rightarrow x \text{ weakly in } L^2(\mu).$$

Let

$$z = \sum_k \langle y, \rho_{\beta_k} \rangle \left( \sum_{j=2^k}^{2^{k+1}-1} \langle x, \gamma_{\alpha_j} \rangle \gamma_{\alpha_j} \right).$$

Then  $z \in L^2(\mu)$  and  $\langle w, z \rangle = F(w, y)$  for all  $w \in L^\infty(\mu)$ . Since  $z \in L^2(\mu)$ ,

$$\lim_\lambda F(x_\lambda, y) = \lim_\lambda \langle x_\lambda, z \rangle = \langle x, z \rangle = F(x, y).$$

The weak\* continuity in  $y$  is proved identically.

For the last assertion, we just apply (3.2) that  $\mu$  and  $\nu$  have the required property. □

**THEOREM 3.7.** *Let  $G$  and  $H$  be infinite compact abelian groups. Then  $\text{Bil}^\sigma$  does not have an (even unbounded) approximate identity.*

**REMARK.** A virtual diagonal for a Banach algebra  $A$  is a bounded net  $\{m_\alpha\}$  in  $A \hat{\otimes} A$  such that  $\lim_\alpha (m_\alpha a - a m_\alpha) = 0$  and  $\lim \pi(m_\alpha) a = a$  for each  $a \in A$ , where  $\pi(a \otimes b) = ab$ . The Banach algebra  $A$  is amenable if and only if  $A$  has a virtual diagonal. If  $A$  is amenable, then  $A$  has a bounded approximate identity. Hence,  $\text{Bil}^\sigma$  is never amenable when  $G, H$  are compact abelian groups. See [1, p. 243] and [10, p. 50, Ex. 36].

*Proof.* Let the elements of  $\hat{G}$  be denoted by  $\gamma_\alpha$  and the elements of  $\hat{H}$  be denoted by  $\rho_\beta$ . We apply the example of Proposition 3.6, only replacing  $\mu$  with  $m_G$  and  $\nu$  with  $m_H$ . Suppose that  $L \in \text{Bil}^\sigma$  were such that  $\|L * F - F\| \leq \frac{1}{2K}$ , where  $K$  is the usual complex Grothendieck constant.

By [4, 2.4], Grothendieck measures for a convolution of bimeasures are the convolution of Grothendieck measures of the factors. Combining that with the third item of Proposition 3.6, we see that Grothendieck measures for  $L * F - F = (L - \delta_0) * F$  are exactly Haar measure. That is, for all  $x \in L^2(G)$  and  $y \in L^2(H)$ ,

$$(3.3) \quad |\langle L * F - F, x \otimes y \rangle| \leq K \|L * F - F\| \|x\|_2 \|y\|_2.$$

For simplicity, denote  $F(\gamma_{\alpha_j}, \rho_{\beta_k})$  by  $F_{j,k}$  and  $L(\gamma_{\alpha_j}, \rho_{\beta_k})$  by  $L_{j,k}$ . For each  $k$ , let us compare the values of  $L * F$  and  $F$  at  $\gamma_{\alpha_j}, \rho_{\beta_k}$ , for  $2^k \leq j \leq 2^{k+1} - 1$ .

We will apply when  $x$  is the element of  $L^2(G)$  such that the Fourier transform of  $x$  is  $2^{-k/2} e^{-\theta(j,k)}$ , where  $\theta(j,k)$  is the argument of  $L_{j,k} - 1$  if that difference is non-zero, and zero otherwise and  $y = \rho_{\beta_k}$ . Then

$$(3.4) \quad \begin{aligned} \langle L * F - F, x \otimes y \rangle &= \sum_{j=2^k}^{2^{k+1}-1} |L_{j,k} - 1| 2^{-k} \\ &\leq K \|L * F - F\| \|x\|_2 \|y\|_2 \leq \frac{1}{2}. \end{aligned}$$

Therefore, for at least half the terms in (3.4),  $|L_{j,k} - 1| \leq \frac{1}{2}$ . That means that

$$(3.5) \quad |L_{j,k}| \geq \frac{1}{2}$$

for at least  $2^{k-1}$  terms. For  $k$  sufficiently large, that contradicts Corollary 3.2.  $\square$

When  $G$  is a compact abelian group,  $L^1(G)$  has a dense subset consisting of elements whose Fourier transforms have finite support. That is not possible for  $\text{Bil}^\sigma$ , since the characteristic function of any graph of a one-to-one function from  $\widehat{G}$  to  $\widehat{H}$  is the Fourier transform of an element of  $\text{Bil}^\sigma$ . In view of Corollary 3.2, one might hope that “finitely supported” could be replaced by “summable on sets of the form  $\gamma \times \widehat{H}$ , with uniform bound on the sums.” That is not possible, as the next result asserts.

**THEOREM 3.8.** *Let  $G$  and  $H$  be compact abelian groups. Then the set of elements  $L$  in  $\text{Bil}^\sigma$  for which  $\sup_\rho \sum_\gamma |L(\gamma, \rho)| < \infty$  is not dense.*

*Proof.* We adapt the proof of Theorem 3.7, using the same  $F$  as there.

Suppose that  $L \in \text{Bil}^\sigma$  is close to  $F$ . Then the Grothendieck measures for  $L$  must be close (in an  $L^2$  sense) to those of  $F$ , that is they must be near to the respective Haar measures. That means that if  $\|F - L\|$  is sufficiently small, then

$$|\langle F - L, x \otimes y \rangle| \leq 2K\|F - L\| \|x\|_2 \|y\|_2$$

for all  $x \in L^2(G), y \in l^2(G)$ .

Suppose that  $\|L - F\| < \frac{\epsilon}{2K}$ . Suppose also that  $\sup_\rho \sum_\gamma |\widehat{L}(\gamma)| < \infty$ . Then for sufficiently large  $k$ ,  $|L_{j,k}| < 2^{-1-k/2}$  for at least half the  $j$  in the range  $2^k \leq j \leq 2^{k+1} - 1$ .

Then

$$\sum_{2^k \leq j \leq 2^{k+1} - 1} |L_{j,k} - F_{j,k}| 2^{-k/2} \geq 2^{-k/2} 2^{-1-k/2} 2^{k-1} = 2^{-2}$$

(evaluate at the same  $x, y$  as in the proof of Theorem 3.7). That implies that  $|L_{j,k} - 2^{-k/2}| < \epsilon$ , which is impossible for small  $\epsilon$ .  $\square$

**4. Problems.** We list in this section some open questions.

- (1) What happens if  $L^\infty$  is replaced with  $LUC(G)$ ?  $C(G)$ ? [And one looks at the corresponding spaces defined via weak\* limits?]
- (2) What happens when we replace  $L^\infty$  with  $VN(G)$ ?
- (3) Does either  $\text{Bil}^\sigma$  or  $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$  characterize the underlying groups? (Wendel's Theorem.)
- (4) Same question for  $\text{BM}(G, H)$ .
- (5) What is the dual of  $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$ ?

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