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BILINEAR OPERATORS ON $L^\infty(G)$ OF LOCALLY COMPACT GROUPS

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Let G and H be compact groups. We study in this paper the space $\mathrm{Bil}^\sigma = \mathrm{Bil}^\sigma(L^\infty(G), L^\infty(H))$. That space consists of all bounded bilinear functionals on $L^\infty(G) \times L^\infty(H)$ that are weak* continuous in each variable separately. We prove, among other things, that Bil^σ is isometrically isomorphic to a closed two-sided ideal in $\mathrm{BM}(G, H)$. In the case of abelian G, H, we show that Bil^σ does not have an approximate identity and that $\widehat{G} \times \widehat{H}$ is dense in the maximal ideal space of Bil^σ. Related results are given.

0. Introduction. Let V and W be Banach spaces over the complex numbers, and let $\mathrm{Bil}(V, W)$ denote the space of bounded bilinear functions $F: V \times W \to C$. Then this is a Banach space under the usual vector space operators and the norm

$$\|F\| = \sup\{|F(x, y)| : x \in V,\ y \in W,\ \|x\| = \|y\| = 1\}.$$

Furthermore $\mathrm{Bil}(V, W)$ may be identified with the dual space of $V \hat{\otimes} W$, the projective tensor product of V and W. When X and Y are locally compact Hausdorff spaces, then elements in $\mathrm{Bil}(C_0(X), C_0(Y))$, also denoted by $\mathrm{BM}(X, Y)$, are called *bimeasures* (see Graham and Schreiber [7] and Gilbert, Ito and Schreiber [4]).

If V and W are dual Banach spaces, we let $\mathrm{Bil}^\sigma(V, W)$ denote all $F \in \mathrm{Bil}(V, W)$ such that $x \mapsto F(x, y)$ and $y \mapsto F(x, y)$ are continuous when V and W have the weak*-topology. Then, as readily checked, $\mathrm{Bil}^\sigma(V, W)$ is a norm-closed subspace of $\mathrm{Bil}(V, W)$. It is the purpose of this paper to study $\mathrm{Bil}^\sigma(L^\infty(G), L^\infty(H))$ when G and H are compact groups.

In §1, we shall give some general results on

$$\mathrm{Bil}^\sigma(L^\infty(X, \mu), L^\infty(Y, \nu))$$

when X, Y are locally compact Hausdorff spaces and μ, ν are positive regular Borel measures on X and Y, respectively. In §2, we show that if G and H are compact groups, then $\mathrm{Bil}^\sigma = \mathrm{Bil}^\sigma(L^\infty(G), L^\infty(H))$

is isometrically isomorphic to a closed ideal in $\mathrm{BM}(G, H)$ with multiplication as defined in [**2**]. Furthermore, Bil^σ has a dense subset consisting of bilinear functionals F such that their Grothendieck measures μ_g, ν_g are such that $d\mu_g/dm_G$ and $d\nu_g/dm_H$ are bounded away from 0 and from infinity (here m_G and m_H denote Haar measure on the respective groups). In §3, we shall concentrate on the case when G and H are both compact and abelian. We shall show that in this case $\widehat{G} \times \widehat{H}$ is dense in the maximal ideal space of Bil^σ and that Bil^σ is a symmetric Banach algebra. Furthermore Bil^σ does not have an (even unbounded) approximate identity when G and H are infinite, compact. In §4, we shall list some open problems related to Bil^σ.

The space $\mathrm{Bil}^\sigma(U, V)$ has been studied in a different context by Effros [**3**]. A consequence of Theorem 3.7 (below) is that Bil^σ has no virtual diagonals; see the Remark following Theorem 3.7.

1. The space Bil^σ. If X is a locally compact Hausdorff space, we let $L^\infty(X)$, $C(X)$, $C_0(X)$, and $C_{00}(X)$ be the spaces of bounded functions on X which are, respectively, Borel measurable, continuous, continuous with limit zero at infinity and continuous with compact support. The supremum norm on each of those spaces will be denoted by $\|\cdot\|_\infty$. If X and Y are locally compact Hausdorff spaces, we write $V_0(X, Y) = C_0(X)\hat{\otimes}C_0(Y)$, the projective tensor product of $C_0(X)$ and $C_0(Y)$. Then the space $\mathrm{BM}(X, Y)$ may be identified with the dual Banach space of $V_0(X, Y)$.

Throughout this section X and Y will denote locally compact Hausdorff spaces and μ, ν will denote positive regular Borel measures on X and Y constructed from a fixed positive functional on $C_{00}(X)$ and $C_{00}(Y)$, respectively (see [**9**, §11]). We will write $L^\infty(\mu)$ and $L^\infty(\nu)$ for $L^\infty(X, \mu)$ and $L^\infty(Y, \nu)$ respectively. In this case, $L^\infty(\mu) = L^1(\mu)^*$, and $L^\infty(\nu) = L^1(\nu)^*$. We will write Bil^σ for $\mathrm{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$. As usual, the norms for spaces L^p, $1 \le p < \infty$, will be denoted by $\|\cdot\|_p$. When G is a locally compact group, $L^p(G)$ will denote the L^p-space defined with respect to a fixed left Haar measure m_G on G.

PROPOSITION 1.1. Bil^σ *consists exactly of the bilinear functionals* F *such that, for all* $x \in L^\infty(\mu)$ *and all* $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$, *for* $f \in L^\infty(\mu)$, *is given by integration against an element of* $L^1(\mu)$ *and* $g \mapsto F(x, g)$, *for* $g \in L^\infty(\nu)$, *is given by integration against an element of* $L^1(\nu)$.

Proof. Let $F \in \mathrm{Bil}^\sigma$. Fix $y \in L^\infty(\nu)$. Since $f \mapsto F(f, y)$ is weak* continuous in f, $f \mapsto F(f, y)$ must belong to the dual space of $L^\infty(\mu)$, when $L^\infty(\mu)$ is given the weak* topology, that is, $f \mapsto F(f, y)$ belongs to $L^1(\mu)$. The same argument applies to $g \mapsto F(x, g)$, for $g \in L^\infty(\nu)$.

On the other hand, suppose that, the bilinear functional F is such that for all $x \in L^\infty(\mu)$, $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$, for $f \in L^\infty(\mu)$, is given by integration against an element of $L^1(\mu)$ and $g \mapsto F(x, g)$, for $g \in L^\infty(\nu)$, is given by integration against an element of $L^1(\nu)$. Then for each fixed $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$ is weak* continuous in f, and for each fixed $x \in L^\infty(\mu)$, $g \mapsto F(x, g)$ is weak* continuous in g. Hence, $F \in \mathrm{Bil}^\sigma$. □

PROPOSITION 1.2. *Let ω be a non-negative, finite regular Borel measure on $X \times Y$. Then $\omega \in \mathrm{Bil}^\sigma$ if and only if the projection of ω onto X is absolutely continuous with respect to μ and the projection of ω onto Y is absolutely continuous with respect to ν.*

Proof. If ω has the projection property, then it obviously has the weak* continuity property that is required for membership in Bil^σ.

On the other hand, suppose that $\omega \in \mathrm{Bil}^\sigma$. Then $f \mapsto \int (f \otimes 1)\, d\omega$ is a non-negative, locally finite, regular Borel measure on X that is the projection of ω on X. Also, $f \mapsto \int (f \otimes 1)\; d\omega$ is weak* continuous from $L^\infty(\mu)$ to $\mathbf{C}$. If the projection of ω (let us call it ω') were not absolutely continuous with respect to μ, then we could find a sequence of functions f_n in $C(X)$ such that $0 \leq f_n \leq 1$, $f_n \to 0$ a.e. $d\mu$ and $\int f_n\, d\omega' \not\to 0$. Of course, that sequence $f_n \to 0$ weak* in $L^\infty(\mu)$, so

$$\int (f \otimes 1)\, d\omega \to 0,$$

a contradiction. [More abstractly, we could just point out that any linear functional on $L^\infty(\mu)$ that is weak* continuous is necessarily given by integration against an element of $L^1(\mu)$, by general Banach space duality.]

A similar argument shows that the projection of ω on Y is absolutely continuous with respect to μ. □

LEMMA 1.3. *Let R, S be von Neumann algebras, and let A, B be weak* dense C^*-sublagebras of R, S, respectively. Then the mapping given by restricting $\mathrm{Bil}^\sigma(R, S)$ to $(A \hat{\otimes} B)$ is an isometry; that is, $\mathrm{Bil}^\sigma(R, S)$ may be identified with a closed subspace of $(A \hat{\otimes} B)^*$.*

Proof. Let $F \in \operatorname{Bil}^\sigma(R, S)$, $\varepsilon > 0$, and let $x \in R$, $y \in S$ be of norm one such that $|F(x, y) - \|F\|| < \varepsilon/3$. By the Kaplansky density theorem [**14**, Theorem 4.8], there exist nets $x_\alpha \to x$ and $y_\beta \to y$ with x_α all belonging to the unit ball of A and y_β all in the unit ball of B. By the weak*-weak* continuity of F, $F(x, y) = \lim_\alpha F(x_\alpha, y)$. Hence, for some α_0 we have $|F(x_{\alpha_0}, y) - F(x, y)| < \varepsilon/3$. Similarly, there exists a β_0 such that $|F(x_{\alpha_0}, y_{\beta_0}) - F(x_{\alpha_0}, y)| < \varepsilon/3$. Hence $|F(x_{\alpha_0}, y_{\beta_0}) - \|F\|| < \varepsilon$, and the result follows. □

COROLLARY 1.4. *The restriction of elements of* $\operatorname{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$ *to the space* $C_0(X)\hat{\otimes}C_0(Y)$ *is an isometry. In particular,* $\operatorname{Bil}^\sigma$ *may be identified with a closed subspace of* $\operatorname{BM}(X, Y)$.

We define $\mathscr{L}^\infty(X)$ to be the space of all bounded Borel functions on X.

If $\varphi_X \in \mathscr{L}^\infty(X)$, and $f_1 = f_2$ locally μ-a.e., then $\varphi_X f_1 = \varphi_X f_2$ locally μ-a.e. In particular, for any $f \in L^\infty(\mu)$, $\varphi_X f$ defines an element in $L^\infty(\mu)$, and the map $f \mapsto \varphi_X f$ is weak*-weak* continuous.

Given $\varphi_X \in \mathscr{L}^\infty(X)$, $\varphi_Y \in \mathscr{L}^\infty(Y)$, and $F \in \operatorname{Bil}^\sigma$ we define a bounded bilinear functional $\varphi \cdot F$ on $L^\infty(\mu) \times L^\infty(\nu)$ by

$$\langle \varphi \cdot F, (f, g)\rangle = \langle F, (\varphi_X f, \varphi_Y g)\rangle$$

for $f \in L^\infty(X)$ and $g \in L^\infty(\nu)$. Then $\varphi \cdot F \in \operatorname{Bil}^\sigma$ and

$$\|\varphi \cdot F\| \le \|F\| \, \|\varphi_X\|_\infty \|\varphi_Y\|_\infty.$$

We recall that the support of a bimeasure is the smallest closed subset Q in $X \times Y$ such that $\langle h, F\rangle = 0$ for all $h \in V_0(X, Y)$ for which $h \equiv 0$ in a neighborhood of Q.

The following three results are variants (as indicated) of known facts. The proofs are essentially identical to those cited.

PROPOSITION 1.5 [7, Lemma 1.4]. *The set of elements of* $\operatorname{Bil}^\sigma$ *that have compact support is norm dense in* $\operatorname{Bil}^\sigma$.

PROPOSITION 1.6 [7, Lemma 1.5]. *Let* X' (*resp.* Y') *be a closed subspace of* Y (*resp.* Y) *and* μ', ν' *denote the restrictions of* μ, ν *to those closed subspaces. Then there is a projection of norm one from* $\operatorname{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu))$ *onto the space* $\operatorname{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu'))$.

The image in $\operatorname{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu'))$ of a bimeasure is called the *restriction* of the bimeasure to $X' \times Y'$ and is written $F|_{X' \times Y'}$.

COROLLARY 1.7. *Let G (resp. H) be a locally compact group and G' (resp. H') an open subgroup. Then there is a norm one projection from* $\mathrm{Bil}^\sigma(L^\infty(G), L^\infty(H))$ *onto* $\mathrm{Bil}^\sigma(L^\infty(G'), L^\infty(H'))$

A bimeasure F is *discrete* if there exist sequences of finite subsets A_n of X and B_n of Y such that $F = \lim_n F|_{A_n\times B_n}$ (norm limit). A bimeasure is *continuous* if its restriction to every product of finite sets is zero. Obviously, BM_c and BM_d are norm closed vector spaces. The set of discrete bimeasures is denoted $\mathrm{BM}_d(X, Y)$ and the set of continuous bimeasures is denoted $\mathrm{BM}_c(X, Y)$. Graham and Schreiber showed that topologically $\mathrm{BM}(X, Y) = \mathrm{BM}_d(X, Y) \oplus \mathrm{BM}_c(X, Y)$ [7, Theorem 1.8].

PROPOSITION 1.8. *If either μ or ν is a continuous measure, then* Bil^σ *is contained in* $\mathrm{BM}_c(X, Y)$. *In particular,* Bil^σ *is a proper subset of* $\mathrm{BM}(X, Y)$.

Proof. Let $F \in \mathrm{Bil}^\sigma$. By Proposition 1.5, we may assume that F is supported on a compact set $X'\times Y'$, so we will not distinguish between F and $F|_{X'\times Y'}$. We write $F = F_1 + F_2$, where F_1 is continuous and F_2 is discrete. Let $A_n \subset X'$ (resp. $B_n \subset Y'$) be increasing sequences of finite subsets such that $F_2 = \lim_n F|_{A_n\times B_n}$. Let $A = \bigcup A_n$. Suppose that μ is a continuous measure. Then $\mu(A) = 0$. By Lusin's Theorem [12, p. 54], (and enlarging A if necessary) there exists a sequence of continuous functions $\{f_j\}$ such that $0 \le f_j \le 1$ for all j, $f_j \to 0$ on A, $f_j \to 1$ on $X\backslash A$ (pointwise in both cases), and the f_n are supported in a common compact superset of $X' \times Y'$. It follows that for each integer n, every $f \in C_0(X)$ and every $g \in C_0(Y)$,

$$F(f, g) = \lim_j F(f_j f, g) = \lim_j (F_1 + F_2)(f_j f, g),$$

and

$$F_2|_{A_n\times B_n}(f, g) = \lim_j F_2(f_j f, g) = 0.$$

The first equality above follows from the weak* continuity of F and the second from the fact that $f_j f \to 0$ on A_n combined with the dominated convergence theorem. Thus, $F_2(f, g) = 0$ for all f, g, so $F_2 = 0$. □

LEMMA 1.9. *Let μ and ν be non-negative, locally finite, regular Borel measures on the locally compact spaces X, Y, respectively. Then* *for any* $F \in \mathrm{Bil}^\sigma$ *there exist* $p \in L^1(\mu)$ *and* $q \in L^1(\nu)$ *such that* $p \geq 0$, $q \geq 0$, $\|p\|_1 = \|q\|_1 = 1$ *and*

$$(1.1) \qquad |F(f, g)| \leq K\|F\| \left(\int |f|^2 p\, d\mu\right)^{1/2} \left(\int |g|^2 q\, d\nu\right)^{1/2}$$

for all $f \in L^\infty(\mu)$ *and* $g \in L^\infty(\nu)$, *where* K *is a universal constant.*

Proof. Suppose that $F \in \mathrm{Bil}^\sigma$. By Proposition 1.5, we know that F has σ-compact support. We thus may assume that μ and ν are σ-finite (since they are locally finite). [Indeed, let the support of F be $\bigcup_{j=1}^\infty X_j \times Y_j$, where the X_j, Y_j are compact. Let μ_j (resp. ν_j) be the restriction of μ to X_j (resp. Y_j). The assumption of local finiteness implies that μ_j, ν_j are σ-finite measures.] Of course, $L^\infty(\mu)$ does not change if we replace μ by an equivalent probability measure. Also, weak* topologies on the L^∞ space induced by the two measures (the probability measure and the original measure) are identical, by the uniqueness of the predual of $L^\infty(\mu)$ (see [**14**, p. 135]). Let the support of F be $\bigcup_{j=1}^\infty X_j \times Y_j$, where the X_j, Y_j are compact. Let μ_j (resp. ν_j) be the restriction of μ to X_j (resp. Y_j). The assumption of local finiteness implies that μ_j, ν_j are finite measures. We may assume that μ_1 and ν_1 have norm $\frac{1}{2}$ and that $\|\mu_{j+1} - \mu_j\| = 2^{-j}$ and similarly for the ν_j for all j. Hence, $F \in \mathrm{Bil}^\sigma(L^\infty(\sum \mu_j), L^\infty(\sum \nu_j))$. Thus, we may assume that μ and ν are probability measures.

Let a Grothendieck measure pair μ', ν' for F be given. Then the pair μ', ν' has the property that

$$(1.2) \qquad |F(f, g)| \leq K\|F\| \, \|f\|_{L^2(\mu')} \|g\|_{L^2(\nu')} \quad \text{for all } f \in C(X), \ g \in C(Y),$$

where K is the usual complex Grothendieck constant. Furthermore, μ' is a probability measure on X' and ν' is a probability measure on Y'.

Let $\mu' = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to μ and μ_s is singular with respect to μ. Let A, B be a partition of X into two disjoint Borel sets such that $\mu(B) = 0$, and

$$\mu_a(E) = \mu'(A \cap E) \quad \text{and} \quad \mu_s(E) = \mu'(B \cap E) \quad \text{for all Borel } E \subset X.$$

Let $f \in L^\infty(\mu)$ have norm one. By Lusin's Theorem [**12**, p. 54], there exists a sequence $\{f_n\}$ in $C(X)$ such that $\|f_n\| \leq 1$ for all n and $f(x)\chi_A(x) = \lim_{n\to\infty} f_n$ pointwise a.e. $d(\mu + \mu_s)$. We note that $f\chi_A = f$ μ-a.e. and $f\chi_A = 0$ $d\mu_s$-a.e. Hence, for each $h \in L^1(\mu)$,

$f_n \cdot h \to f \cdot h$ pointwise $d\mu$-a.e. and $|f_n \cdot h| \le |f \cdot h| d\mu$-a.e. for all n. By the dominated convergence theorem (and here we need the actual finiteness of μ), $\int f_n \cdot h \, d\mu \to \int f \cdot h \, d\mu$. That is,

$$f_n \to f \text{ in the weak* topology of } L^\infty(\mu). \tag{1.3}$$

Since $f_n \to 0$ pointwise a.e. $d\mu_s$, $|f_n| \to 0$ pointwise a.e. $d\mu_s$. Since $|f_n|^2 \le 1$, the dominated convergence theorem again implies that

$$\begin{aligned} &\int |f_n|^2 \, d\mu_s \to \int |f|^2 \, d\mu_s = 0 \quad \text{and} \\ &\int |f_n|^2 \, d\mu_a \to \int |f|^2 \, d\mu_a. \end{aligned} \tag{1.4}$$

Hence $\int |f_n|^2 \, d\mu' \to \int |f_n|^2 \, d\mu_a$. Also, by (1.2),

$$|F(f_n, g)| \le K \|F\| \, \|f_n\|_{L^2(\mu')} \|g\|_{L^2(\nu')} \quad \text{for all } g \in C(Y). \tag{1.5}$$

Now, $F(f_n, g) \to F(f, g)$ by (1.3) and

$$\begin{aligned} \|f_n\|^2_{L^2(\mu)} &= \int |f_n \chi_A|^2 \, d\mu' \\ &\to \int |f|^2 \, d\mu' \\ &= \int |f|^2 \, d\mu_a + \int |f|^2 \, d\mu_s \\ &= \int |f|^2 \, d\mu_a, \end{aligned}$$

by (1.4). Therefore,

$$|F(f, g)| \le K \|F\| \, \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu')},$$

by (1.5).

A similar argument applied to $g \in L^\infty(\nu)$ gives

$$|F(f, g)| \le K \|F\| \, \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu_a)}. \qquad \square$$

Let f be a Borel function on the locally compact space X, and ω be a non-negative, locally finite, regular Borel measure on X. We say that f is *bounded away from* 0 *and* ∞ if there exist constants $0 < c < C < \infty$ such that $c \le f(x) \le C$ a.e. $d\omega$.

LEMMA 1.10. *Let μ and ν denote regular Borel locally measures on the locally compact spaces X and Y. Then* Bil^σ *has a dense subset consisting of the bilinear functionals F such that their Grothendieck* *measures* μ_g, ν_g *are such that* $d\mu_g/d\mu$ *and* $d\nu_g/d\nu$ *are bounded away from zero and away from* ∞.

Proof. Let $F \in \mathrm{Bil}^\sigma$. We may assume that μ and ν are probability measures and that we have a Grothendieck measure pair μ_g, ν_g for F with $\mu_g \ll \mu$ and $\nu_g \ll \nu$ The validity of this second assumption follows from Lemma 1.9.

Now, by (1.1, and using the notation of Lemma 1.9,), if A is a Borel subset of X and B is a Borel subset of Y, then

$$|\langle f\chi_A \otimes g\chi_B, F\rangle| \to 0 \quad \text{as } \mu(A) \to 0, \text{ and/or } \nu(B) \to 0,$$

by the Lebesgue dominated convergence theorem.

Thus, given $n > 0$, define the Borel sets A_n, B_n by

$$A_n = \{x \in X : p(x) \notin [1/n, n]\}$$

and

$$B_n = \{y \in Y : q(x) \notin [1/n, n]\},$$

where p, q are as in Lemma 1.9.

Then $\mu(A_n) \to 0$ and $\nu(B_n) \to 0$ as $n \to \infty$.

Let $\delta > 0$ be given. Then there exists $n > 0$ such that

$$|\langle f\chi_{A_n} \otimes g\chi_{B_n}, u\rangle| \le \frac{\delta}{4}\|f\|_\infty \|g\|_\infty \quad \text{for all } f \in L^\infty(\mu),\ \ g \in L^\infty(\nu).$$

We let $F_1 = (\chi_{A_n} \otimes \chi_{B_n})\mu \times \nu + ((1 - \chi_{A_n}) \otimes (1 - \chi_{B_n}))F$. It is then clear that $\|F - F_1\| \le \delta$. □

2. Locally compact groups. In this section, G and H will be locally compact groups, not both discrete. We now write Bil^σ in place of $\mathrm{Bil}^\sigma(L^\infty(G), L^\infty(H))$. We study the properties of the particular space Bil^σ, where we are already using the group structure to define Bil^σ. We remind the reader that we continue the identification of Bil^σ with a closed subspace of $\mathrm{BM}_c(G, H)$ (see Corollary 1.4 and Proposition 1.8).

Furthermore, by Proposition 1.1, Bil^σ consists of the bilinear functionals F such that, for all $x \in L^\infty(m_G)$, $y \in L^\infty(m_H)$, $f \mapsto F(f, y)$ ($f \in L^\infty(m_G)$) is given by integration against an element of $L^1(\mu)$ and $g \mapsto F(x, g)$ ($g \in L^\infty(m_H)$) is given by integration against an element of $L^1(\nu)$.

We note that $\mathrm{Bil}(L^\infty(m_G), L^\infty(m_H))$ is a $(L^\infty(m_G), L^\infty(m_H))$ module in the sense that the (obviously bounded) operations $(g \cdot F)$

and $F\cdot f$ are defined by

$$(g\cdot F)(h,k)=F(h,gk) \quad\text{and}\quad (F\cdot f)(h,k)=F(fh,k)$$

for all $F\in \mathrm{Bil}(L^\infty(m_G), L^\infty(m_H))$, $f, h\in L^\infty(m_G)$ and $g, k\in L^\infty(m_H)$.

Also, Bil^σ is a closed submodule of the $\mathrm{Bil}(L^\infty(m_G), L^\infty(m_H))$.

We define $L^\infty(\mu)\hat{\otimes}^\sigma L^\infty(\nu) =_{\mathrm{def}} (\mathrm{Bil}^\sigma)^*$. Then $L^\infty(\mu)\hat{\otimes}^\sigma L^\infty(\nu)$ is a dual $(L^\infty(m_G), L^\infty(m_H))$-module when the operations are defined by

$$\langle g\cdot M, F\rangle=\langle M, g\cdot F\rangle \quad\text{and}\quad \langle M\cdot f, F\rangle=\langle M, F\cdot f\rangle,$$

where $M\in L^\infty(\mu)\hat{\otimes}^\sigma L^\infty(\nu)$, $F\in\mathrm{Bil}^\sigma$, $f\in L^\infty(\mu)$ and $g\in L^\infty(\nu)$.

A dual module is *normal* if the mappings

$$f\mapsto f\cdot M \text{ from } L^\infty(\mu)\to L^\infty(\mu)\hat{\otimes}^\sigma L^\infty(\nu) \quad\text{and}$$
$$g\mapsto M\cdot g \text{ from } L^\infty(\nu)\to L^\infty(\mu)\hat{\otimes}^\sigma L^\infty(\nu)$$

are both weak*-weak* continuous.

THEOREM 2.1. *Let G and H be locally compact groups. Then* Bil^σ *is an ideal in* $\mathrm{BM}(G, H)$. *Also,* Bil^σ *is a normal* $(L^\infty(G), L^\infty(H))$ *module.*

Proof. Immediate from Lemma 1.9 and the facts that (i) $\mathrm{BM}(G, H)$ is an algebra under convolution (see [7, 2.5] or [**4**, 2.4]) and (ii) that the Grothendieck measures for a convolution product may be taken to be the convolutions of the Grothendieck measures of the factors [**4**, *loc. cit*].

The last assertion is a consequence of [**3**, Lemma 2.2] and Lemma 1.9 above. □

REMARKS 2.2. (a) Note that the mapping

$$\theta\colon L^\infty(G)\otimes L^\infty(H)\to L^\infty(G)\otimes^\sigma L^\infty(H)$$

defined by $\theta(f\otimes g)(F)=F(f, g)$ is one-to-one. Hence, we may identify the space $L^\infty(G)\otimes L^\infty(H)$ with its image in $L^\infty(G)\hat{\otimes}^\sigma L^\infty(H)$. That image is weak* dense.

Furthermore, if $M\in L^\infty(G)\hat{\otimes}^\sigma L^\infty(H)$ of norm one, then there is a net $M_\alpha=\sum\lambda_i^\alpha(f_i^\alpha\otimes g_i^\alpha)$, with the f_i's and g_i's in their respective unit balls, the λ_i's nonnegative with sum one, such that $M_\alpha\to M$ in the weak* topology. (See [**3**, p. 139 and p. 141].)

(b) There is a unique weak*-continuous extension to $L^\infty(G)\otimes^\sigma L^\infty(H)$ of the multiplication map

$$\pi\colon L^\infty(G)\otimes L^\infty(G)\to L^\infty(G)$$

given by $f\otimes g\mapsto f\cdot g$ (see [**3**, p. 142]).

THEOREM 2.3. *Let G and H be compact groups. Then* Bil^σ *has a dense subset consisting of the bilinear functionals F such that their Grothendieck measures μ, ν are such that $d\mu/dm_G$ and $d\nu/dm_H$ are bounded away from zero and away from ∞.*

Proof. Immediate from Lemma 1.10. □

LEMMA 2.4. *Let μ and ν be continuous probability measures on the locally compact spaces X and Y respectively. Then there is a projection of norm one from* $\mathrm{BM}(X, Y)$ *onto* Bil^σ.

Proof. It is well-known (and easy to see) that $\mathrm{BM}(X, Y)$ may be imbedded isometrically in $\mathrm{Bil}(M(X)^*, M(Y)^*)$. Let $f_0 \in M(X)^*$, be such that f_0 is one a.e. with respect to (the image of) μ and zero with respect to (the image of) all measures on X that are singular with respect to μ. Define $g_0 \in M(X)^*$ analogously. Then the composition of $F \mapsto (f_0 \times g_0)F$ with the restriction of the resulting element to $C(Y) \times C(Y)$ is a linear norm-reducing mappping P of $\mathrm{BM}(X, Y)$. Furthermore, $PF = F$ for all $F \in \mathrm{Bil}^\sigma$. Finally, (straightforward computations show that) $f \mapsto PF(f, g)$ is absolutely continuous with respect to μ for all $g \in C(Y)$ and that $g \mapsto PF(f, g)$ is absolutely continuous with respect to μ for all $f \in C(X)$. That is, $PF \in \mathrm{Bil}^\sigma$. It follows that P is the required projection. □

THEOREM 2.5. *Let G and H be locally compact groups. Then there is no projection from* Bil^σ *onto the closed subspace of* Bil^σ *generated by $L^1(G \times H)$.*

Proof. This is immediate from [7, Theorem 1] and Lemma 2.4 above. □

LEMMA 2.6. *Let G be a compact group and U an open subset of G. Then there exists an integer $n \geq 1$ such that U^n is an open subgroup of G.*

Proof. Let $y \in U$. The closed semigroup H generated by y is a compact semigroup. Therefore H contains an idempotent [2, 1.8]; that idempotent is necessarily the identity of G. (Alternatively, we can apply the fact [2, 1.10] that a compact subsemigroup of a group is a subgroup, so $e \in H$, which is, in fact, a group.) In any case, e is in the closure of $\{y^l\}$.

Let V be any symmetric neighborhood of e. We may assume that V is so small that $yV \subseteq U$. Then

$$y^l V \subseteq (yV)^l \subseteq U^l.$$

Since $\{y^l\}$ accumulates at e, there are large l's such that $y^l \in V^{-1} = V$. Therefore $y^{-l} \in V$, so

$$e = y^l y^{-l} \in y^l V \subseteq U^l.$$

That is, $e \in U^l$. Thus, we may assume $e \in U^{lm}$ for all $m > 0$. In particular, the sets U^{lm} are increasing. Again, consider the closed subgroup H generated by $y \in U^{lm_0}$ for some $m_0 > 0$. (H is a subgroup by [2, 1.10].) If that closed subgroup is finite, then eventually it is contained in U^{lm} for some $m \geq m_0$. Otherwise, every element of it is an accumulation point of the set $\{y^n : n > 0\}$. (That also follows from the fact that a compact semigroup in a compact group is necessarily a group.) Hence, every element of H belongs to some U^{lm}. This argument applies to every element of $\bigcup_{m\geq 1} U^{lm}$. That is, the group $K = \bigcup_{m\geq 1} U^{lm}$.

Since $\bigcup_{m\geq 1} U^{lm}$ is a group, and open, it is also a closed subgroup, and therefore it is compact. Therefore $\bigcup_{m\geq 1} U^{lm} = U^{lm(0)}$ for some $m(0)$.

By the monotonicity of the U^{lm}, $K = U^{lm(0)}$ □

We can now give a variant of Lemma 1.10.

THEOREM 2.7. (1) *Let G, H be compact and connected groups. Then the set of those $u \in \mathrm{Bil}^\sigma$ for which there is an $n \geq 1$ for which the Grothendieck measures for u^n are Haar measure is a dense subset of* Bil^σ.

(2) *Let G, H be compact groups. Then the set of those $F \in \mathrm{Bil}^\sigma$ for which there is an $n \geq 1$ for which the Grothendieck measures for F^n are Haar measure on an open subgroup of G is a dense subset of* Bil^σ.

Proof. Let μ, ν be Grothendieck measures for F. Then $\mu = (f+g)m_G$, where f is continuous and $\|g\|$ is small. Similarly for ν. Then the Grothendieck measures for F^n are μ^n and ν^n. By Lemma 2.6, $f^n > 0$ on an open subgroup of G. We may throw away the terms involving g in $(f+g)^n$, thus obtaining the required conclusion for both (i) and (ii). □

3. Compact abelian groups. Suppose that G and H are compact abelian groups with character groups $\widehat{G}$ and $\widehat{H}$, respectively.

Let $u \in \mathrm{BM}(G, H)$. The *Fourier transform* $\hat{u}$ of u is defined by

$$\hat{u}(\gamma, \rho) = \langle \bar{\gamma} \otimes \bar{\rho}, u \rangle, \quad \text{for all } \gamma \in \widehat{G}, \ \rho \in \widehat{H}.$$

Then $\hat{u}$ is well-defined and $\|\hat{u}\|_\infty \le \|u\|$ (see [7, p. 97]).

REMARK. The multipliers of Bil^σ are exactly the elements of $\mathrm{BM}(G, H)$.

This is immediate upon taking weak* limits, since the unit ball of Bil^σ is dense in the unit ball of BM, even though (see below) Bil^σ does not have an approximate identity. Here are some details.

We first note that the measures in the unit ball of Bil^σ are weak* dense in that ball (one proof of that is known as Riemann sums for double integrals; another is known as "bounded spectral synthesis" for sets whose union is a Kronecker set [**13**, Corollary 4]). The argument in the "bounded spectral synthesis" form easily adapts to the case of approximation by measures belonging to a fixed L-space that is weak* dense in $M(G \times H)$. Hence, the measures in the unit ball of Bil^σ are dense in the unit ball of $\mathrm{BM}(G, H)$.

Suppose that φ is a function defined on $\widehat{G} \times \widehat{H}$ such that $\varphi\hat{u}$ is the Fourier transform (see below) of an element of Bil^σ for all $u \in \mathrm{Bil}^\sigma$. Then $\|\varphi\hat{u}\| \le C\|u\|$ for all u and some constant C. We note that the set Fourier-Stieltjes transform of $\mathrm{BM}(G, H)$ is closed under bounded pointwise convergence (that follows from a diagonalization argument and the fact that the unit ball of $\mathrm{BM}(G, H)$ is compact in the weak* topology). By taking weak* limits (within the unit ball), we conclude that φ is a multiplier of $\mathrm{BM}(G, H)$. Since $\mathrm{BM}(G, H)$ has an identity, the remark follows. □

Suppose that we have a u whose Grothendieck measures μ, ν are such that $d\mu/dm_G$ and $d\nu/dm_H$ are bounded away from zero and away from ∞. Then, by using that and the Plancherel Theorem, we can identify $L^2(\mu)$ with $L^2(\widehat{G})$ and $L^2(\nu)$ with $L^2(\widehat{H})$. Using those identifications, we can explicitly compute the linear mapping $T\colon L^2(\widehat{G}) \to L^2(\widehat{H})$. Here, T is the mapping associated with the Grothendieck measures. Of course, we have lost information about the constant in the Grothendieck inequality. The new mapping T is given by:

$$(T\hat{f})(\rho) = \sum_{\gamma} \hat{u}(\gamma, \rho)\hat{f}(\gamma),$$

where $\hat{f} \in L^2(\widehat{G})$. That follows at once from the fact that

$$\langle u, f \otimes g\rangle = \sum_{\gamma, \rho} \hat{u}(\gamma, \rho)\hat{f}(\gamma)\hat{g}(\rho), \quad \text{for all } f \in C(G),\ g \in C(H),$$

which, in turn, is an easy calculation from

$$\langle u, f \otimes g\rangle = \left\langle u, \left(\sum \hat{f}(\gamma)\bar{\gamma}\right) \otimes \left(\sum \hat{g}(\rho)\bar{\rho}\right)\right\rangle.$$

The norm of the new T is now bounded by the product of three numbers: the norm of the old T, the supremum of $d\mu/dm_G$, and the reciprocal of the infimum of $d\nu/dm_H$.

PROPOSITION 3.1. *Let G and H be compact abelian groups. Let $u \in \mathrm{Bil}^\sigma$. Suppose that u has μ, ν for its Grothendieck measures with $d\mu/dm_G$ and $d\nu/dm_H$ both bounded away from zero and away from ∞. Then there exists a constant $C > 0$ such that $\sum_\gamma |\hat{u}(\gamma, \rho)|^2 < C$ for every fixed $\rho \in \widehat{H}$ and $\sum_\rho |\hat{u}(\gamma, \rho)|^2 < C$ for every fixed $\gamma \in \widehat{G}$.*

Proof. By the discussion preceding the statement of Proposition 3.1, we see that there is a linear transformation $T\colon L^2(G) \to L^2(H)$ such that

$$\langle u, f \otimes g\rangle = \langle Tf, g\rangle \quad \text{for all } f \in C(G) \text{ and all } g \in C(H).$$

(This transformation is the composition of the transformation discussed above with two Plancherel transformations.) Then

$$\sum_\gamma |\hat{u}(\gamma, \rho)|^2 \le \|T^*(\rho)\|_2^2 \le \|T\|. \qquad \square$$

COROLLARY 3.2. *Let $u \in \mathrm{Bil}^\sigma$, where G, H are compact abelian groups. Then for every $\varepsilon > 0$ there exists $N > 0$ such that for each $\rho \in \widehat{H}$,*

$$\mathrm{Card}\{\gamma : |\hat{u}(\gamma, \rho)| > \varepsilon\} \le N.$$

Proof. We fix $\varepsilon > 0$. Let v be such that $\|u - v\| < \varepsilon/3$ and such that v satisfies the hypotheses of Proposition 3.1. We let N be any integer greater than $9C/\varepsilon^2$ (the C is from Proposition 3.1 applied to v). Then $|\hat{u}(\gamma, \rho)| > \varepsilon$ implies $|\hat{v}(\gamma, \rho)| > \varepsilon/3$, and that can occur at most $9C/\varepsilon^2$ times. $\square$

THEOREM 3.3. *Let $u \in \mathrm{Bil}^\sigma$, where G, H are compact abelian groups. Then the spectral radius of u is*

$$\sup_{\gamma \in \widehat{G}, \rho \in \widehat{H}} |\hat{u}(\gamma, \rho)|.$$

Proof. By Theorem 2.3, we may assume that there is a Grothendieck measure pair μ, ν for u such that $d\mu/dm_G$ and $d\nu/dm_H$ are both bounded away from zero and infinity. Thus, we may assume that there is bounded linear transformation $T\colon L^2(G) \to L^2(H)$ such that $\langle u, f\otimes g\rangle = \langle Tf, g\rangle$ for all $f \in C(G)$ and $g \in C(H)$. Furthermore, for all continuous f on G, g on H,

$$\sum_{\gamma,\rho} |\hat{u}^2(\gamma, \rho)||\hat{f}(\gamma)|\,|\hat{g}(\rho)| = |\langle u * \tilde{u}, f \otimes g\rangle| \le C'\|u\|^2\|f\|_2\|g\|_2, \tag{3.1}$$

where C' is the product of four numbers: K (the Grothendieck constant), the norm of u, the supremum of $d\mu/dm_G$, and the reciprocal of the infimum of $d\nu/dm_H$.

Let f_1 denote the Radon-Nikodym derivative $d\mu/dm_G$. Then f_1 has L^1-norm 1 and is bounded away from zero and infinity. Therefore, the nth convolution powers of f_1 converge to 1 uniformly, by Lemma 3.4 below. The same applies to $g_1 = d\nu/dm_H$

That means that the Grothendieck measures (call them μ_n, ν_n) for u^n become closer and closer to Haar measures, so the norm of the isomorphisms (and of their inverses) between $L^2(\mu_n)$ and $L^2(\widehat{G})$ on the one hand, and $L^2(\nu_n)$ and $L^2(\widehat{H})$ on the other hand, approach one. Thus, for sufficiently large n, we may assume that

$$\|u^n\|_{\mathrm{Bil}^\sigma} \le C \sup\{|\langle u^n, f \otimes g\rangle| : \|f\|_2\|g\|_2 \le 1\},$$

where C does not depend on n, and the supremum is taken over all f, g of uniform norm one.

But

$$\langle u^n, f \otimes g\rangle = \sum_{\gamma,\rho} \hat{u}^n(\gamma, \rho)\hat{f}(\gamma)\hat{g}(\rho).$$

Therefore

$$|\langle u^n, f \otimes g\rangle| \le \|\hat{u}^{n-2}\|_\infty \sum_{\gamma,\rho} |\hat{u}^2(\gamma, \rho)|\,|\hat{f}(\gamma)|\,|\hat{g}(\rho)|.$$

It follows that

$$\|u^n\|_{\mathrm{Bil}^\sigma} \le C\,\|\hat{u}^{n-2}\|_\infty \sup \sum_{\gamma,\rho} |\hat{u}^2(\gamma, \rho)||\hat{f}(\gamma)|\,|\hat{g}(\rho)|,$$

where C does not depend on n, and the supremum is taken over all f, g of uniform norm one. By (3.1),

$$\sup_{\|f\|_\infty \le 1, \|g\|_\infty \le 1} \sum_{\gamma,\rho} |\hat{u}^2(\gamma, \rho)|\,|\hat{f}(\gamma)|\,|\hat{g}(\rho)| = \sup|\langle u * \tilde{u}, f \otimes g\rangle| \le C'\|u\|^2\|f\|_2\|g\|_2,$$

so $\|u^n\|_{\mathrm{Bil}^\sigma} \le C''\,\|\hat{u}^{n-2}\|_\infty$ for all n.

The conclusion about the spectral radius now follows easily. □

LEMMA 3.4. *Let f be a bounded non-negative Borel function on the compact group G that is bounded away from zero and has L^1-norm one. Then the sequence of convolution powers of f converges uniformly to* 1.

Proof. Since f is bounded, $f \in L^2(G)$ and $\hat{f} \in L^2(\widehat{G})$. Therefore $f^2 = f*f$ has an absolutely convergent Fourier series, so, in particular, $\hat{f} \in c_0(\widehat{G})$. Since $f > 0$ and $\|f\|_1 = 1$, $\hat{f}(0) = 1$. We apply the Lebesgue Dominated Convergence Theorem to $\hat{f}^n$ (with $|\hat{f}|^2$ being the dominating function and $n > 2$) to conclude that $\hat{f}^n$ converges in l^1-norm to a function f' that is equal to the characteristic function of a finite subset of $\widehat{G}$ (finite because $\hat{f} \in c_0(\widehat{G})$). Of course, that means that f^n converges uniformly to a function f_1 that is non-zero everywhere (the infimum of f^n is increasing with n). Thus, $f_1 m_G$ is an idempotent probability measure. By **[11**, 3.2.4], $f_1 m_G$ is Haar measure on a compact subgroup of G. Since $\hat{f_1} = f'$ has finite support, that subgroup has finite index. If the index were greater than 1, f_1 would be zero somewhere, a contradiction. Therefore $f_1 = 1$ everywhere. □

COROLLARY 3.5. *Let G and H be compact abelian groups. Then $\widehat{G} \times \widehat{H}$ is dense in the maximal ideal space of* Bil^σ *and* Bil^σ *is a symmetric Banach algebra*

Proof. This is a standard argument: the result is more or less immediate from Theorem 3.3. Here are the details.

We first note that Bil^σ is self-adjoint. For if $S \in \mathrm{Bil}^\sigma$ is such that its Gelfand transform $\widehat{S}$ is real on $\widehat{G}\times\widehat{H}$, but not real on all of $\Delta\,\mathrm{Bil}^\sigma$ (the maximal ideal space), then for an appropriate $k > 1$, $\exp(ikS)$ has Gelfand transform larger than one at that non-real value, but has Fourier-Stieltjes transform at most one, thus contradicting Theorem 3.3.

Since the space of Gelfand transforms $\mathrm{Bil}^{\sigma\wedge}$ is self-adjoint and separating, it is uniformly dense in $C_0(\Delta\,\mathrm{Bil}^\sigma)$. If $\widehat{G}\times\widehat{H}$ were not dense in $\Delta\,\mathrm{Bil}^\sigma$, then there would be a continuous function f on $\Delta\,\mathrm{Bil}^\sigma$ such that $\|f\|_\infty = 1$ and $|f| < 1/2$ on $\widehat{G} \times \widehat{H}$. By estimating f uniformly by an element of Bil^σ, we again contradict Theorem 3.3. □

We now give an example of an element of Bil^σ. The example is simple; we use it to show that Bil^σ does not have approximate identities, even unbounded ones.

Let μ and ν denote regular Borel probability measures on the locally compact spaces X and Y. Suppose that $\{\gamma_\alpha\}$ is an orthonormal basis for $L^2(\mu)$, and that $\{\rho_\beta\}$ is an orthonormal basis for $L^2(\nu)$. Let subsequences of those bases be chosen. Let $F(\gamma_\alpha, \rho_\beta)$ be defined by

$$F(\gamma_{\alpha_j}, \rho_{\beta_k}) = \begin{cases} 2^{-k/2}, & 2^k \le j \le 2^{k+1} - 1 \text{ and } j \ge 1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F(\gamma_\alpha, \rho_\beta) = 0 \text{ if there is no pair } j,\ k \text{ with } \alpha = \alpha_j \text{ and } \beta = \beta_k.$$

PROPOSITION 3.6. *With the above hypotheses,*

(1) *F is a bilinear functional on $L^2(\mu) \times L^2(\nu)$ that is bounded by* 1;
(2) *F represents an element of* Bil^σ; *and*
(3) *Grothendieck measures for F are given by μ, ν.*

Proof. For the first part, let $x, y \in L^2(\mu) \times L^2(\nu)$, and let $x_j = \langle x, \gamma_{\alpha_j}\rangle$ for all j and $y_k = \langle y, \rho_{\beta_k}\rangle$ for all k. Let also $F_{j,k} = F(\gamma_{\alpha_j}, \rho_{\beta_k})$. Then

$$F(x, y) = \sum_k \sum_{j=2^k}^{2^{k+1}-1} F_{j,k} x_j y_k.$$

We may assume that the x_j and y_k are non-negative. For each k,

$$\sum_{j=2^k}^{2_{k+1}-1} F_{j,k} x_j \le \left(\sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2},$$

by the Cauchy-Schwarz inequality. Therefore,

$$\begin{aligned} |F(x, y)| &\le \sum_k \left(\sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} y_k \\ &\le \left(\sum_k \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} \left(\sum_k y_k^2 \right)^{1/2}. \end{aligned}$$

That is,

$$F(x, y) \le \|x\|_{L^2(\mu)}\|y\|_{L^2(\nu)}. \tag{3.2}$$

For the second assertion, by the first part and the fact that μ, ν are probability measures, $|F(x, y)| \le \|x\|_\infty \|y\|_\infty$ for all $x \in L^\infty(\mu)$, $y \in L^\infty(\nu)$. Hence F represents an element of $\mathrm{Bil}(L^\infty(\mu), L^\infty(\nu))$. We must show that F is weak* continuous in each variable separately. Suppose that $x_\lambda \to x$ weak* in $L^\infty(\mu)$ and that $y \in L^\infty(\nu)$. Note that $L^\infty(\mu) \subseteq L^2(\mu) \subseteq L^1(\mu)$. By the latter containment, x_λ converges weak* in $L^2(\mu)$ Since $L^\infty(\mu)$ is dense in $L^2(\mu)$,

$$x_\lambda \to x \text{ weakly in } L^2(\mu).$$

Let

$$z = \sum_k \langle y, \rho_{\beta_k}\rangle \left(\sum_{j=2^k}^{2^{k+1}-1} \langle x, \gamma_{\alpha_j}\rangle \gamma_{\alpha_j} \right).$$

Then $z \in L^2(\mu)$ and $\langle w, z\rangle = F(w, y)$ for all $w \in L^\infty(\mu)$. Since $z \in L^2(\mu)$,

$$\lim_\lambda F(x_\lambda, y) = \lim_\lambda \langle x_\lambda, z\rangle = \langle x, z\rangle = F(x, y).$$

The weak* continuity in y is proved identically.

For the last assertion, we just apply (3.2) that μ and ν have the required property. □

THEOREM 3.7. *Let G and H be infinite compact abelian groups. Then* Bil^σ *does not have an (even unbounded) approximate identity.*

REMARK. A virtual diagonal for a Banach algebra A is a bounded net $\{m_\alpha\}$ in $A\hat{\otimes}A$ such that $\lim_\alpha(m_\alpha a - a m_\alpha) = 0$ and $\lim \pi(m_\alpha)a = a$ for each $a \in A$, where $\pi(a \otimes b) = ab$. The Banach algebra A is amenable if and only if A has a virtual diagonal. If A is amenable, then A has a bounded approximate identity. Hence, Bil^σ is never amenable when G, H are compact abelian groups. See [**1**, p. 243] and [**10**, p. 50, Ex. 36].

Proof. Let the elements of $\widehat{G}$ be denoted by γ_α and the elements of $\widehat{H}$ be denoted by ρ_β. We apply the example of Proposition 3.6, only replacing μ with m_G and ν with m_H. Suppose that $L \in \mathrm{Bil}^\sigma$ were such that $\|L * F - F\| \le \frac{1}{2K}$, where K is the usual complex Grothendieck constant.

By [4, 2.4], Grothendieck measures for a convolution of bimeasures are the convolution of Grothendieck measures of the factors. Combining that with the third item of Proposition 3.6, we see that Grothendieck measures for $L * F - F = (L - \delta_0) * F$ are exactly Haar measure. That is, for all $x \in L^2(G)$ and $y \in L^2(H)$,

$$(3.3) \qquad |\langle L * F - F, x \otimes y\rangle| \leq K\|L * F - F\| \, \|x\|_2 \, \|y\|_2.$$

For simplicity, denote $F(\gamma_{\alpha_j}, \rho_{\beta_k})$ by $F_{j,k}$ and $L(\gamma_{\alpha_j}, \rho_{\beta_k})$ by $L_{j,k}$. For each k, let us compare the values of $L * F$ and F at γ_{α_j}, ρ_{β_k}, for $2^k \leq j \leq 2^{k+1} - 1$.

We will apply when x is the element of $L^2(G)$ such that the Fourier transform of x is $2^{-k/2}e^{-\theta(j,k)}$, where $\theta(j, k)$ is the argument of $L_{j,k} - 1$ if that difference is non-zero, and zero otherwise and $y = \rho_{\beta_k}$. Then

$$(3.4) \qquad \langle L * F - F, x \otimes y\rangle = \sum_{j=2^k}^{2^{k+1}-1} |L_{j,k} - 1| 2^{-k}$$
$$\leq K\|L * F - F\| \, \|x\|_2 \, \|y\|_2 \leq \frac{1}{2}.$$

Therefore, for at least half the terms in (3.4), $|L_{j,k} - 1| \leq \frac{1}{2}$. That means that

$$(3.5) \qquad |L_{j,k}| \geq \frac{1}{2}$$

for at least 2^{k-1} terms. For k sufficiently large, that contradicts Corollary 3.2. □

When G is a compact abelian group, $L^1(G)$ has a dense subset consisting of elements whose Fourier transforms have finite support. That is not possible for Bil^σ, since the characteristic function of any graph of a one-to-one function from $\widehat{G}$ to $\widehat{H}$ is the Fourier transform of an element of Bil^σ. In view of Corollary 3.2, one might hope that "finitely supported" could be replaced by "summable on sets of the form $\gamma \times \widehat{H}$, with uniform bound on the sums." That is not possible, as the next result asserts.

THEOREM 3.8. *Let G and H be compact abelian groups. Then the set of elements L in* Bil^σ *for which* $\sup_\rho \sum_\gamma |L(\gamma, \rho)| < \infty$ *is not dense.*

Proof. We adapt the proof of Theorem 3.7, using the same F as there.

Suppose that $L \in \mathrm{Bil}^\sigma$ is close to F. Then the Grothendieck measures for L must be close (in an L^2 sense) to those of F, that is they must be near to the respective Haar measures. That means that if $\|F-L\|$ is sufficiently small, then

$$|\langle F-L, x\otimes y\rangle| \le 2K\|F-L\|\,\|x\|_2\|u\|_2$$

for all $x \in L^2(G)$, $y \in l^2(G)$.

Suppose that $\|L-F\| < \frac{\varepsilon}{2K}$. Suppose also that $\sup_\rho \sum_\gamma |\widehat{L}(\gamma)| < \infty$. Then for sufficiently large k, $|L_{j,k}| < 2^{-1-k/2}$ for at least half the j in the range $2^k \le j \le 2^{k+1}-1$.

Then

$$\sum_{2^k\le j\le 2^{k+1}-1} |L_{j,k}-F_{j,k}|2^{-k/2} \ge 2^{-k/2}2^{-1-k/2}2^{k-1} = 2^{-2}$$

(evaluate at the same x, y as in the proof of Theorem 3.7). That implies that $|L_{j,k}-2^{-k/2}| < \varepsilon$, which is impossible for small ε. $\square$

4. Problems. We list in this section some open questions.

(1) What happens if L^∞ is replaced with $LUC(G)$? $C(G)$? [And one looks at the corresponding spaces defined via weak* limits?]

(2) What happens when we replace L^∞ with $VN(G)$?

(3) Does either Bil^σ or $\mathrm{Bil}(L^\infty(m_G), L^\infty(m_H))$ characterize the underlying groups? (Wendel's Theorem.)

(4) Same question for $\mathrm{BM}(G,H)$.

(5) What is the dual of $\mathrm{Bil}(L^\infty(m_G), L^\infty(m_H))$?

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