BILINEAR OPERATORS ON $L^\infty(G)$ OF LOCALLY COMPACT GROUPS

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Let $G$ and $H$ be compact groups. We study in this paper the space $\text{Bil}^\sigma = \text{Bil}^\sigma(L^\infty(G), L^\infty(H))$. That space consists of all bounded bilinear functionals on $L^\infty(G) \times L^\infty(H)$ that are weak* continuous in each variable separately. We prove, among other things, that $\text{Bil}^\sigma$ is isometrically isomorphic to a closed two-sided ideal in $\text{BM}(G, H)$. In the case of abelian $G$, $H$, we show that $\text{Bil}^\sigma$ does not have an approximate identity and that $\hat{G} \times \hat{H}$ is dense in the maximal ideal space of $\text{Bil}^\sigma$. Related results are given.

0. Introduction. Let $V$ and $W$ be Banach spaces over the complex numbers, and let $\text{Bil}(V, W)$ denote the space of bounded bilinear functions $F: V \times W \to C$. Then this is a Banach space under the usual vector space operators and the norm

$$\|F\| = \sup\{|F(x, y)| : x \in V, y \in W, \|x\| = \|y\| = 1\}.$$ 

Furthermore $\text{Bil}(V, W)$ may be identified with the dual space of $V \hat{\otimes} W$, the projective tensor product of $V$ and $W$. When $X$ and $Y$ are locally compact Hausdorff spaces, then elements in $\text{Bil}(C_0(X), C_0(Y))$, also denoted by $\text{BM}(X, Y)$, are called bimeasures (see Graham and Schreiber [7] and Gilbert, Ito and Schreiber [4]).

If $V$ and $W$ are dual Banach spaces, we let $\text{Bil}^\sigma(V, W)$ denote all $F \in \text{Bil}(V, W)$ such that $x \mapsto F(x, y)$ and $y \mapsto F(x, y)$ are continuous when $V$ and $W$ have the weak*-topology. Then, as readily checked, $\text{Bil}^\sigma(V, W)$ is a norm-closed subspace of $\text{Bil}(V, W)$. It is the purpose of this paper to study $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$ when $G$ and $H$ are compact groups.

In §1, we shall give some general results on

$$\text{Bil}^\sigma(L^\infty(X, \mu), L^\infty(Y, \nu))$$

when $X$, $Y$ are locally compact Hausdorff spaces and $\mu$, $\nu$ are positive regular Borel measures on $X$ and $Y$, respectively. In §2, we show that if $G$ and $H$ are compact groups, then $\text{Bil}^\sigma = \text{Bil}^\sigma(L^\infty(G), L^\infty(H))$.
is isometrically isomorphic to a closed ideal in $BM(G, H)$ with multiplication as defined in [2]. Furthermore, $Bil^\sigma$ has a dense subset consisting of bilinear functionals $F$ such that their Grothendieck measures $\mu_g$, $\nu_g$ are such that $d\mu_g/dm_G$ and $d\nu_g/dm_H$ are bounded away from 0 and from infinity (here $m_G$ and $m_H$ denote Haar measure on the respective groups). In §3, we shall concentrate on the case when $G$ and $H$ are both compact and abelian. We shall show that in this case $\widehat{G} \times \widehat{H}$ is dense in the maximal ideal space of $Bil^\sigma$ and that $Bil^\sigma$ is a symmetric Banach algebra. Furthermore $Bil^\sigma$ does not have an (even unbounded) approximate identity when $G$ and $H$ are infinite, compact. In §4, we shall list some open problems related to $Bil^\sigma$.

The space $Bil^\sigma(U, V)$ has been studied in a different context by Effros [3]. A consequence of Theorem 3.7 (below) is that $Bil^\sigma$ has no virtual diagonals; see the Remark following Theorem 3.7.

1. The space $Bil^\sigma$. If $X$ is a locally compact Hausdorff space, we let $L^\infty(X)$, $C(X)$, $C_0(X)$, and $C_{00}(X)$ be the spaces of bounded functions on $X$ which are, respectively, Borel measurable, continuous, continuous with limit zero at infinity and continuous with compact support. The supremum norm on each of those spaces will be denoted by $\| \cdot \|_\infty$. If $X$ and $Y$ are locally compact Hausdorff spaces, we write $V_0(X, Y) = C_0(X) \hat{\otimes} C_0(Y)$, the projective tensor product of $C_0(X)$ and $C_0(Y)$. Then the space $BM(X, Y)$ may be identified with the dual Banach space of $V_0(X, Y)$.

Throughout this section $X$ and $Y$ will denote locally compact Hausdorff spaces and $\mu, \nu$ will denote positive regular Borel measures on $X$ and $Y$ constructed from a fixed positive functional on $C_{00}(X)$ and $C_{00}(Y)$, respectively (see [9, §11]). We will write $L^\infty(\mu)$ and $L^\infty(\nu)$ for $L^\infty(X, \mu)$ and $L^\infty(Y, \nu)$ respectively. In this case, $L^\infty(\mu) = L^1(\mu)^*$, and $L^\infty(\nu) = L^1(\nu)^*$. We will write $Bil^\sigma$ for $Bil^\sigma(L^\infty(\mu), L^\infty(\nu))$. As usual, the norms for spaces $L^p$, $1 \leq p < \infty$, will be denoted by $\| \cdot \|_p$. When $G$ is a locally compact group, $L^p(G)$ will denote the $L^p$-space defined with respect to a fixed left Haar measure $m_G$ on $G$.

Proposition 1.1. $Bil^\sigma$ consists exactly of the bilinear functionals $F$ such that, for all $x \in L^\infty(\mu)$ and all $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$, for $f \in L^\infty(\mu)$, is given by integration against an element of $L^1(\mu)$ and $g \mapsto F(x, g)$, for $g \in L^\infty(\nu)$, is given by integration against an element of $L^1(\nu)$.
Proof. Let $F \in \text{Bil}^\sigma$. Fix $y \in L^\infty(\nu)$. Since $f \mapsto F(f, y)$ is weak* continuous in $f$, $f \mapsto F(f, y)$ must belong to the dual space of $L^\infty(\mu)$, when $L^\infty(\mu)$ is given the weak* topology, that is, $f \mapsto F(f, y)$ belongs to $L^1(\mu)$. The same argument applies to $g \mapsto F(x, g)$, for $g \in L^\infty(\nu)$.

On the other hand, suppose that, the bilinear functional $F$ is such that for all $x \in L^\infty(\mu)$, $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$, for $f \in L^\infty(\mu)$, is given by integration against an element of $L^1(\mu)$ and $g \mapsto F(x, g)$, for $g \in L^\infty(\nu)$, is given by integration against an element of $L^1(\nu)$. Then for each fixed $y \in L^\infty(\nu)$, $f \mapsto F(f, y)$ is weak* continuous in $f$, and for each fixed $x \in L^\infty(\mu)$, $g \mapsto F(x, g)$ is weak* continuous in $g$. Hence, $F \in \text{Bil}^\sigma$.

Proposition 1.2. Let $\omega$ be a non-negative, finite regular Borel measure on $X \times Y$. Then $\omega \in \text{Bil}^\sigma$ if and only if the projection of $\omega$ onto $X$ is absolutely continuous with respect to $\mu$ and the projection of $\omega$ onto $Y$ is absolutely continuous with respect to $\nu$.

Proof. If $\omega$ has the projection property, then it obviously has the weak* continuity property that is required for membership in $\text{Bil}^\sigma$.

On the other hand, suppose that $\omega \in \text{Bil}^\sigma$. Then $f \mapsto \int (f \otimes 1) \, d\omega$ is a non-negative, locally finite, regular Borel measure on $X$ that is the projection of $\omega$ on $X$. Also, $f \mapsto \int (f \otimes 1) \, d\omega$ is weak* continuous from $L^\infty(\mu)$ to $C$. If the projection of $\omega$ (let us call it $\omega'$) were not absolutely continuous with respect to $\mu$, then we could find a sequence of functions $f_n$ in $C(X)$ such that $0 \leq f_n \leq 1$, $f_n \to 0$ a.e. $d\mu$ and $\int f_n \, d\omega' \neq 0$. Of course, that sequence $f_n \to 0$ weak* in $L^\infty(\mu)$, so

$$\int (f \otimes 1) \, d\omega \to 0,$$

a contradiction. [More abstractly, we could just point out that any linear functional on $L^\infty(\mu)$ that is weak* continuous is necessarily given by integration against an element of $L^1(\mu)$, by general Banach space duality.]

A similar argument shows that the projection of $\omega$ on $Y$ is absolutely continuous with respect to $\mu$.

Lemma 1.3. Let $R, S$ be von Neumann algebras, and let $A, B$ be weak* dense $C^*$-subalgebras of $R, S$, respectively. Then the mapping given by restricting $\text{Bil}^\sigma(R, S)$ to $(A \hat{\otimes} B)$ is an isometry; that is, $\text{Bil}^\sigma(R, S)$ may be identified with a closed subspace of $(A \hat{\otimes} B)^*$.
Proof. Let \( F \in \text{Bil}^\sigma(R, S) \), \( \varepsilon > 0 \), and let \( x \in R \), \( y \in S \) be of norm one such that \( |F(x, y) - \|F\|| < \varepsilon/3 \). By the Kaplansky density theorem [14, Theorem 4.8], there exist nets \( x_\alpha \to x \) and \( y_\beta \to y \) with \( x_\alpha \) all belonging to the unit ball of \( A \) and \( y_\beta \) all in the unit ball of \( B \). By the weak*-weak* continuity of \( F \), \( F(x, y) = \lim_\alpha F(x_\alpha, y) \). Hence, for some \( \alpha_0 \) we have \( |F(x_\alpha_0, y) - F(x, y)| < \varepsilon/3 \). Similarly, there exists a \( \beta_0 \) such that \( |F(x_\alpha_0, y_\beta_0) - F(x_\alpha_0, y)| < \varepsilon/3 \). Hence \( |F(x_\alpha_0, y_\beta_0) - \|F\|| < \varepsilon \), and the result follows. \( \square \)

**Corollary 1.4.** The restriction of elements of \( \text{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu)) \) to the space \( C_0(X) \otimes C_0(Y) \) is an isometry. In particular, \( \text{Bil}^\sigma \) may be identified with a closed subspace of \( \text{BM}(X, Y) \).

We define \( L^\infty(X) \) to be the space of all bounded Borel functions on \( X \).

If \( \varphi_X \in L^\infty(X) \), and \( f_1 = f_2 \) locally \( \mu \)-a.e., then \( \varphi_X f_1 = \varphi_X f_2 \) locally \( \mu \)-a.e. In particular, for any \( f \in L^\infty(\mu) \), \( \varphi_X f \) defines an element in \( L^\infty(\mu) \), and the map \( f \mapsto \varphi_X f \) is weak*-weak* continuous.

Given \( \varphi_X \in L^\infty(X) \), \( \varphi_Y \in L^\infty(Y) \), and \( F \in \text{Bil}^\sigma \) we define a bounded bilinear functional \( \varphi \cdot F \) on \( L^\infty(\mu) \times L^\infty(\nu) \) by

\[
\langle \varphi \cdot F, (f, g) \rangle = \langle F, (\varphi_X f, \varphi_Y g) \rangle
\]

for \( f \in L^\infty(X) \) and \( g \in L^\infty(\nu) \). Then \( \varphi \cdot F \in \text{Bil}^\sigma \) and

\[
\|\varphi \cdot F\| \leq \|F\| \|\varphi_X\| \|\varphi_Y\|.
\]

We recall that the support of a bimeasure is the smallest closed subset \( Q \) in \( X \times Y \) such that \( \langle h, F \rangle = 0 \) for all \( h \in V_0(X, Y) \) for which \( h \equiv 0 \) in a neighborhood of \( Q \).

The following three results are variants (as indicated) of known facts. The proofs are essentially identical to those cited.

**Proposition 1.5** [7, Lemma 1.4]. The set of elements of \( \text{Bil}^\sigma \) that have compact support is norm dense in \( \text{Bil}^\sigma \).

**Proposition 1.6** [7, Lemma 1.5]. Let \( X' \) (resp. \( Y' \)) be a closed subspace of \( Y \) (resp. \( Y \)) and \( \mu', \nu' \) denote the restrictions of \( \mu, \nu \) to those closed subspaces. Then there is a projection of norm one from \( \text{Bil}^\sigma(L^\infty(\mu), L^\infty(\nu)) \) onto the space \( \text{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu')) \).

The image in \( \text{Bil}^\sigma(L^\infty(\mu'), L^\infty(\nu')) \) of a bimeasure is called the restriction of the bimeasure to \( X' \times Y' \) and is written \( F|_{X' \times Y'} \).
**COROLLARY 1.7.** Let $G$ (resp. $H$) be a locally compact group and $G'$ (resp. $H'$) an open subgroup. Then there is a norm one projection from $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$ onto $\text{Bil}^\sigma(L^\infty(G'), L^\infty(H'))$.

A bimeasure $F$ is *discrete* if there exist sequences of finite subsets $A_n$ of $X$ and $B_n$ of $Y$ such that $F = \lim_n F|_{A_n \times B_n}$ (norm limit). A bimeasure is *continuous* if its restriction to every product of finite sets is zero. Obviously, $\text{BM}_c$ and $\text{BM}_d$ are norm closed vector spaces. The set of discrete bimeasures is denoted $\text{BM}_d(X, Y)$ and the set of continuous bimeasures is denoted $\text{BM}_c(X, Y)$. Graham and Schreiber showed that topologically $\text{BM}(X, Y) = \text{BM}_d(X, Y) \oplus \text{BM}_c(X, Y)$ [7, Theorem 1.8].

**PROPOSITION 1.8.** If either $\mu$ or $\nu$ is a continuous measure, then $\text{Bil}^\sigma$ is contained in $\text{BM}_c(X, Y)$. In particular, $\text{Bil}^\sigma$ is a proper subset of $\text{BM}(X, Y)$.

**Proof.** Let $F \in \text{Bil}^\sigma$. By Proposition 1.5, we may assume that $F$ is supported on a compact set $X' \times Y'$, so we will not distinguish between $F$ and $F|_{X' \times Y'}$. We write $F = F_1 + F_2$, where $F_1$ is continuous and $F_2$ is discrete. Let $A_n \subset X'$ (resp. $B_n \subset Y'$) be increasing sequences of finite subsets such that $F_2 = \lim_n F|_{A_n \times B_n}$. Let $A = \bigcup A_n$. Suppose that $\mu$ is a continuous measure. Then $\mu(A) = 0$. By Lusin’s Theorem [12, p. 54], (and enlarging $A$ if necessary) there exists a sequence of continuous functions $\{f_j\}$ such that $0 \leq f_j \leq 1$ for all $j$, $f_j \to 0$ on $A$, $f_j \to 1$ on $X \setminus A$ (pointwise in both cases), and the $f_n$ are supported in a common compact superset of $X' \times Y'$. It follows that for each integer $n$, every $f \in C_0(X)$ and every $g \in C_0(Y)$,

$$F(f, g) = \lim_j F(f_j f, g) = \lim_j (F_1 + F_2)(f_j f, g),$$

and

$$F_2|_{A_n \times B_n}(f, g) = \lim_j F_2(f_j f, g) = 0.$$

The first equality above follows from the weak* continuity of $F$ and the second from the fact that $f_j f \to 0$ on $A_n$ combined with the dominated convergence theorem. Thus, $F_2(f, g) = 0$ for all $f, g$, so $F_2 = 0$. \hfill $\Box$

**LEMMA 1.9.** Let $\mu$ and $\nu$ be non-negative, locally finite, regular Borel measures on the locally compact spaces $X, Y$, respectively. Then
for any $F \in \text{Bil}^\sigma$ there exist $p \in L^1(\mu)$ and $q \in L^1(\nu)$ such that $p \geq 0$, $q \geq 0$, $\|p\|_1 = \|q\|_1 = 1$ and

$$(1.1) \quad |F(f, g)| \leq K\|F\| \left( \int |f|^2 p \, d\mu \right)^{1/2} \left( \int |g|^2 q \, d\nu \right)^{1/2}$$

for all $f \in L^\infty(\mu)$ and $g \in L^\infty(\nu)$, where $K$ is a universal constant.

**Proof.** Suppose that $F \in \text{Bil}^\sigma$. By Proposition 1.5, we know that $F$ has $\sigma$-compact support. We thus may assume that $\mu$ and $\nu$ are $\sigma$-finite (since they are locally finite). [Indeed, let the support of $F$ be $\bigcup_{j=1}^\infty X_j \times Y_j$, where the $X_j$, $Y_j$ are compact. Let $\mu_j$ (resp. $\nu_j$) be the restriction of $\mu$ to $X_j$ (resp. $Y_j$). The assumption of local finiteness implies that $\mu_j$, $\nu_j$ are $\sigma$-finite measures.] Of course, $L^\infty(\mu)$ does not change if we replace $\mu$ by an equivalent probability measure. Also, weak* topologies on the $L^\infty$ space induced by the two measures (the probability measure and the original measure) are identical, by the uniqueness of the predual of $L^\infty(\mu)$ (see [14, p. 135]). Let the support of $F$ be $\bigcup_{j=1}^\infty X_j \times Y_j$, where the $X_j$, $Y_j$ are compact. Let $\mu_j$ (resp. $\nu_j$) be the restriction of $\mu$ to $X_j$ (resp. $Y_j$). The assumption of local finiteness implies that $\mu_j$, $\nu_j$ are finite measures. We may assume that $\mu_1$ and $\nu_1$ have norm $\frac{1}{2}$ and that $\|\mu_{j+1} - \mu_j\| = 2^{-j}$ and similarly for the $\nu_j$ for all $j$. Hence, $F \in \text{Bil}^\sigma(L^\infty(\sum \mu_j), L^\infty(\sum \nu_j))$. Thus, we may assume that $\mu$ and $\nu$ are probability measures.

Let a Grothendieck measure pair $\mu'$, $\nu'$ for $F$ be given. Then the pair $\mu'$, $\nu'$ has the property that

$$(1.2) \quad |F(f, g)| \leq K\|F\| \|f\|_{L^2(\mu')} \|g\|_{L^2(\nu')}$$

for all $f \in C(X)$, $g \in C(Y)$, where $K$ is the usual complex Grothendieck constant. Furthermore, $\mu'$ is a probability measure on $X'$ and $\nu'$ is a probability measure on $Y'$.

Let $\mu' = \mu_a + \mu_s$, where $\mu_a$ is absolutely continuous with respect to $\mu$ and $\mu_s$ is singular with respect to $\mu$. Let $A$, $B$ be a partition of $X$ into two disjoint Borel sets such that $\mu(B) = 0$, and

$$\mu_a(E) = \mu'(A \cap E) \quad \text{and} \quad \mu_s(E) = \mu'(B \cap E) \quad \text{for all Borel } E \subset X.$$ 

Let $f \in L^\infty(\mu)$ have norm one. By Lusin's Theorem [12, p. 54], there exists a sequence $\{f_n\}$ in $C(X)$ such that $\|f_n\| \leq 1$ for all $n$ and $f(x)\chi_A(x) = \lim_{n \to \infty} f_n$ pointwise a.e. $d(\mu + \mu_s)$. We note that $f\chi_A = f$ $\mu$-a.e. and $f\chi_A = 0$ $d\mu_s$-a.e. Hence, for each $h \in L^1(\mu)$,
$f_n \cdot h \to f \cdot h$ pointwise $d\mu$-a.e. and $|f_n \cdot h| \leq |f \cdot h|d\mu$-a.e. for all $n$. By the dominated convergence theorem (and here we need the actual finiteness of $\mu$), $\int f_n \cdot h \, d\mu \to \int f \cdot h \, d\mu$. That is,

(1.3) \quad f_n \to f$ in the weak* topology of $L^\infty(\mu)$.

Since $f_n \to 0$ pointwise a.e. $d\mu_s$, $|f_n| \to 0$ pointwise a.e. $d\mu_s$. Since $|f_n|^2 \leq 1$, the dominated convergence theorem again implies that

(1.4) \quad \int |f_n|^2 \, d\mu_s \to \int |f|^2 \, d\mu_s = 0 \quad \text{and} \quad \int |f_n|^2 \, d\mu_a \to \int |f|^2 \, d\mu_a.$

Hence $\int |f_n|^2 \, d\mu' \to \int |f_n|^2 \, d\mu_a$. Also, by (1.2),

(1.5) \quad |F(f_n, g)| \leq K\|F\| \|f_n\|_{L^2(\mu')} \|g\|_{L^2(\nu')} \quad \text{for all } g \in C(Y).

Now, $F(f_n, g) \to F(f, g)$ by (1.3) and

\[
\|f_n\|_{L^2(\mu)}^2 = \int |f_n \chi_A|^2 \, d\mu' \\
\quad \quad \quad \quad \quad \to \int |f|^2 \, d\mu' \\
\quad \quad \quad \quad \quad = \int |f|^2 \, d\mu_a + \int |f|^2 \, d\mu_s \\
\quad \quad \quad \quad \quad = \int |f|^2 \, d\mu_a,
\]

by (1.4). Therefore,

\[
|F(f, g)| \leq K\|F\| \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu')},
\]

by (1.5).

A similar argument applied to $g \in L^\infty(\nu)$ gives

\[
|F(f, g)| \leq K\|F\| \|f\|_{L^2(\mu_a)} \|g\|_{L^2(\nu_a)}. \quad \Box
\]

Let $f$ be a Borel function on the locally compact space $X$, and $\omega$ be a non-negative, locally finite, regular Borel measure on $X$. We say that $f$ is bounded away from 0 and $\infty$ if there exist constants $0 < c < C < \infty$ such that $c \leq f(x) \leq C$ a.e. $d\omega$.

**Lemma 1.10.** Let $\mu$ and $\nu$ denote regular Borel locally measures on the locally compact spaces $X$ and $Y$. Then $\text{Bil}^\sigma$ has a dense subset consisting of the bilinear functionals $F$ such that their Grothendieck
measures $\mu_g, \nu_g$ are such that $d\mu_g/d\mu$ and $d\nu_g/d\nu$ are bounded away from zero and away from $\infty$.

Proof. Let $F \in \text{Bil}^\sigma$. We may assume that $\mu$ and $\nu$ are probability measures and that we have a Grothendieck measure pair $\mu_g, \nu_g$ for $F$ with $\mu_g \ll \mu$ and $\nu_g \ll \nu$. The validity of this second assumption follows from Lemma 1.9.

Now, by (1.1, and using the notation of Lemma 1.9.), if $A$ is a Borel subset of $X$ and $B$ is a Borel subset of $Y$, then

$$|\langle f \chi_A \otimes g \chi_B, F \rangle| \to 0 \quad \text{as } \mu(A) \to 0, \text{ and/or } \nu(B) \to 0,$$

by the Lebesgue dominated convergence theorem.

Thus, given $n > 0$, define the Borel sets $A_n, B_n$ by

$$A_n = \{x \in X : p(x) \notin [1/n, n]\}$$

and

$$B_n = \{y \in Y : q(x) \notin [1/n, n]\},$$

where $p, q$ are as in Lemma 1.9.

Then $\mu(A_n) \to 0$ and $\nu(B_n) \to 0$ as $n \to \infty$.

Let $\delta > 0$ be given. Then there exists $n > 0$ such that

$$|\langle f \chi_{A_n} \otimes g \chi_{B_n}, u \rangle| \leq \frac{\delta}{4} \|f\|_{\infty} \|g\|_{\infty} \quad \text{for all } f \in L^\infty(\mu), \ g \in L^\infty(\nu).$$

We let $F_1 = (\chi_{A_n} \otimes \chi_{B_n})\mu \times \nu + ((1 - \chi_{A_n}) \otimes (1 - \chi_{B_n}))F$. It is then clear that $\|F - F_1\| \leq \delta$.

2. Locally compact groups. In this section, $G$ and $H$ will be locally compact groups, not both discrete. We now write $\text{Bil}^\sigma$ in place of $\text{Bil}^\sigma(L^\infty(G), L^\infty(H))$. We study the properties of the particular space $\text{Bil}^\sigma$, where we are already using the group structure to define $\text{Bil}^\sigma$. We remind the reader that we continue the identification of $\text{Bil}^\sigma$ with a closed subspace of $\text{BM}_c(G, H)$ (see Corollary 1.4 and Proposition 1.8).

Furthermore, by Proposition 1.1, $\text{Bil}^\sigma$ consists of the bilinear functionals $F$ such that, for all $x \in L^\infty(m_G), y \in L^\infty(m_H), f \mapsto F(f, y)$ ($f \in L^\infty(m_G)$) is given by integration against an element of $L^1(\mu)$ and $g \mapsto F(x, g)$ ($g \in L^\infty(m_H)$) is given by integration against an element of $L^1(\nu)$.

We note that $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$ is a $(L^\infty(m_G), L^\infty(m_H))$ module in the sense that the (obviously bounded) operations $(g \cdot F)$
and $F \cdot f$ are defined by
$$(g \cdot F)(h, k) = F(h, gk) \quad \text{and} \quad (F \cdot f)(h, k) = F(fh, k)$$
for all $F \in \text{Bil}(L^\infty(m_G), L^\infty(m_H))$, $f, h \in L^\infty(m_G)$ and $g, k \in L^\infty(m_H)$.

Also, $\text{Bil}^\sigma$ is a closed submodule of the $\text{Bil}(L^\infty(m_G), L^\infty(m_H))$.

We define $L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu) = \text{def} (\text{Bil}^\sigma)^*$. Then $L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu)$ is
a dual $(L^\infty(m_G), L^\infty(m_H))$-module when the operations are defined by
$$\langle g \cdot M, F \rangle = \langle M, g \cdot F \rangle \quad \text{and} \quad \langle M \cdot f, F \rangle = \langle M, F \cdot f \rangle,$$
where $M \in L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu)$, $F \in \text{Bil}^\sigma$, $f \in L^\infty(\mu)$ and $g \in L^\infty(\nu)$.

A dual module is normal if the mappings
$$f \mapsto f \cdot M \text{ from } L^\infty(\mu) \to L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu) \quad \text{and}$$
$$g \mapsto M \cdot g \text{ from } L^\infty(\nu) \to L^\infty(\mu) \hat{\otimes}^\sigma L^\infty(\nu)$$
are both weak*—weak* continuous.

**Theorem 2.1.** Let $G$ and $H$ be locally compact groups. Then $\text{Bil}^\sigma$ is an ideal in $BM(G, H)$. Also, $\text{Bil}^\sigma$ is a normal $(L^\infty(G), L^\infty(H))$ module.

**Proof.** Immediate from Lemma 1.9 and the facts that (i) $BM(G, H)$ is an algebra under convolution (see [7, 2.5] or [4, 2.4]) and (ii) that the Grothendieck measures for a convolution product may be taken to be the convolutions of the Grothendieck measures of the factors [4, loc. cit].

The last assertion is a consequence of [3, Lemma 2.2] and Lemma 1.9 above. \[\square\]

**Remarks 2.2.** (a) Note that the mapping
$$\theta : L^\infty(G) \otimes L^\infty(H) \to L^\infty(G) \hat{\otimes}^\sigma L^\infty(H)$$
defined by $\theta(f \otimes g)(F) = F(f, g)$ is one-to-one. Hence, we may identify the space $L^\infty(G) \otimes L^\infty(H)$ with its image in $L^\infty(G) \hat{\otimes}^\sigma L^\infty(H)$. That image is weak* dense.

Furthermore, if $M \in L^\infty(G) \hat{\otimes}^\sigma L^\infty(H)$ of norm one, then there is a net $M_\alpha = \sum \lambda_i^\alpha (f_i^\alpha \otimes g_i^\alpha)$, with the $f_i$'s and $g_i$'s in their respective unit balls, the $\lambda_i^\alpha$'s nonnegative with sum one, such that $M_\alpha \to M$ in the weak* topology. (See [3, p. 139 and p. 141].)

(b) There is a unique weak*-continuous extension to $L^\infty(G) \otimes L^\infty(H)$ of the multiplication map
$$\pi : L^\infty(G) \otimes L^\infty(G) \to L^\infty(G)$$
given by $f \otimes g \mapsto f \cdot g$ (see [3, p. 142]).
**Theorem 2.3.** Let $G$ and $H$ be compact groups. Then $\text{Bil}^\sigma$ has a dense subset consisting of the bilinear functionals $F$ such that their Grothendieck measures $\mu$, $\nu$ are such that $d\mu/dm_G$ and $d\nu/dm_H$ are bounded away from zero and away from $\infty$.

*Proof.* Immediate from Lemma 1.10. □

**Lemma 2.4.** Let $\mu$ and $\nu$ be continuous probability measures on the locally compact spaces $X$ and $Y$ respectively. Then there is a projection of norm one from $\text{BM}(X, Y)$ onto $\text{Bil}^\sigma$.

*Proof.* It is well-known (and easy to see) that $\text{BM}(X, Y)$ may be imbedded isometrically in $\text{Bil}(M(X)^*, M(Y)^*)$. Let $f_0 \in M(X)^*$, be such that $f_0$ is one a.e. with respect to (the image of) $\mu$ and zero with respect to (the image of) all measures on $X$ that are singular with respect to $\mu$. Define $g_0 \in M(X)^*$ analogously. Then the composition of $F \mapsto (f_0 \times g_0)F$ with the restriction of the resulting element to $C(Y) \times C(Y)$ is a linear norm-reducing mapping $P$ of $\text{BM}(X, Y)$. Furthermore, $PF = F$ for all $F \in \text{Bil}^\sigma$. Finally, (straightforward computations show that) $f \mapsto PF(f, g)$ is absolutely continuous with respect to $\mu$ for all $g \in C(Y)$ and that $g \mapsto PF(f, g)$ is absolutely continuous with respect to $\mu$ for all $f \in C(X)$. That is, $PF \in \text{Bil}^\sigma$. It follows that $P$ is the required projection. □

**Theorem 2.5.** Let $G$ and $H$ be locally compact groups. Then there is no projection from $\text{Bil}^\sigma$ onto the closed subspace of $\text{Bil}^\sigma$ generated by $L^1(G \times H)$.

*Proof.* This is immediate from [7, Theorem 1] and Lemma 2.4 above. □

**Lemma 2.6.** Let $G$ be a compact group and $U$ an open subset of $G$. Then there exists an integer $n \geq 1$ such that $U^n$ is an open subgroup of $G$.

*Proof.* Let $y \in U$. The closed semigroup $H$ generated by $y$ is a compact semigroup. Therefore $H$ contains an idempotent [2, 1.8]; that idempotent is necessarily the identity of $G$. (Alternatively, we can apply the fact [2, 1.10] that a compact subsemigroup of a group is a subgroup, so $e \in H$, which is, in fact, a group.) In any case, $e$ is in the closure of $\{y^l\}$.
Let $V$ be any symmetric neighborhood of $e$. We may assume that $V$ is so small that $yV \subseteq U$. Then

$$y^l V \subseteq (yV)^l \subseteq U^l.$$  

Since $\{y^l\}$ accumulates at $e$, there are large $l$'s such that $y^l \in V^{-1} = V$. Therefore $y^{-l} \in V$, so

$$e = y^l y^{-l} \in y^l V \subseteq U^l.$$

That is, $e \in U^l$. Thus, we may assume $e \in U^{lm}$ for all $m > 0$. In particular, the sets $U^{lm}$ are increasing. Again, consider the closed subgroup $H$ generated by $y \in U^{lm_0}$ for some $m_0 > 0$. ($H$ is a subgroup by [2, 1.10].) If that closed subgroup is finite, then eventually it is contained in $U^{lm}$ for some $m \geq m_0$. Otherwise, every element of it is an accumulation point of the set $\{y^n : n > 0\}$. (That also follows from the fact that a compact semigroup in a compact group is necessarily a group.) Hence, every element of $H$ belongs to some $U^{lm}$. This argument applies to every element of $\bigcup_{m \geq 1} U^{lm}$. That is, the group $K = \bigcup_{m \geq 1} U^{lm}$.

Since $\bigcup_{m \geq 1} U^{lm}$ is a group, and open, it is also a closed subgroup, and therefore it is compact. Therefore $\bigcup_{m \geq 1} U^{lm} = U^{lm(0)}$ for some $m(0)$.

By the monotonicity of the $U^{lm}$, $K = U^{lm(0)}$.\qed

We can now give a variant of Lemma 1.10.

**Theorem 2.7.** (1) Let $G, H$ be compact and connected groups. Then the set of those $u \in \text{Bil}^\sigma$ for which there is an $n \geq 1$ for which the Grothendieck measures for $u^n$ are Haar measure is a dense subset of $\text{Bil}^\sigma$.

(2) Let $G, H$ be compact groups. Then the set of those $F \in \text{Bil}^\sigma$ for which there is an $n \geq 1$ for which the Grothendieck measures for $F^n$ are Haar measure on an open subgroup of $G$ is a dense subset of $\text{Bil}^\sigma$.

**Proof.** Let $\mu, \nu$ be Grothendieck measures for $F$. Then $\mu = (f + g)m_G$, where $f$ is continuous and $\|g\|$ is small. Similarly for $\nu$. Then the Grothendieck measures for $F^n$ are $\mu^n$ and $\nu^n$. By Lemma 2.6, $f^n > 0$ on an open subgroup of $G$. We may throw away the terms involving $g$ in $(f + g)^n$, thus obtaining the required conclusion for both (i) and (ii).\qed
3. Compact abelian groups. Suppose that $G$ and $H$ are compact abelian groups with character groups $\widehat{G}$ and $\widehat{H}$, respectively.

Let $u \in \text{BM}(G, H)$. The Fourier transform $\hat{u}$ of $u$ is defined by

$$\hat{u}(\gamma, \rho) = \langle \gamma \otimes \rho, u \rangle, \quad \text{for all } \gamma \in \widehat{G}, \ \rho \in \widehat{H}.$$ 

Then $\hat{u}$ is well-defined and $\|\hat{u}\|_\infty \leq \|u\|$ (see [7, p. 97]).

**Remark.** The multipliers of $\text{Bil}^\sigma$ are exactly the elements of $\text{BM}(G, H)$.

This is immediate upon taking weak* limits, since the unit ball of $\text{Bil}^\sigma$ is dense in the unit ball of $\text{BM}$, even though (see below) $\text{Bil}^\sigma$ does not have an approximate identity. Here are some details.

We first note that the measures in the unit ball of $\text{Bil}^\sigma$ are weak* dense in that ball (one proof of that is known as Riemann sums for double integrals; another is known as “bounded spectral synthesis” for sets whose union is a Kronecker set [13, Corollary 4]). The argument in the “bounded spectral synthesis” form easily adapts to the case of approximation by measures belonging to a fixed $L$-space that is weak* dense in $M(G \times H)$. Hence, the measures in the unit ball of $\text{Bil}^\sigma$ are dense in the unit ball of $\text{BM}(G, H)$.

Suppose that $\phi$ is a function defined on $\widehat{G} \times \widehat{H}$ such that $\phi \hat{u}$ is the Fourier transform (see below) of an element of $\text{Bil}^\sigma$ for all $u \in \text{Bil}^\sigma$. Then $\|\phi \hat{u}\| \leq C \|u\|$ for all $u$ and some constant $C$. We note that the set Fourier-Stieltjes transform of $\text{BM}(G, H)$ is closed under bounded pointwise convergence (that follows from a diagonalization argument and the fact that the unit ball of $\text{BM}(G, H)$ is compact in the weak* topology). By taking weak* limits (within the unit ball), we conclude that $\phi$ is a multiplier of $\text{BM}(G, H)$. Since $\text{BM}(G, H)$ has an identity, the remark follows. □

Suppose that we have a $u$ whose Grothendieck measures $\mu$, $\nu$ are such that $d\mu/dm_G$ and $d\nu/dm_H$ are bounded away from zero and away from $\infty$. Then, by using that and the Plancherel Theorem, we can identify $L^2(\mu)$ with $L^2(\widehat{G})$ and $L^2(\nu)$ with $L^2(\widehat{H})$. Using those identifications, we can explicitly compute the linear mapping $T: L^2(\widehat{G}) \to L^2(\widehat{H})$. Here, $T$ is the mapping associated with the Grothendieck measures. Of course, we have lost information about the constant in the Grothendieck inequality. The new mapping $T$ is given by:

$$(T \hat{f})(\rho) = \sum_\gamma \hat{u}(\gamma, \rho) \hat{f}(\gamma),$$
where \( \hat{f} \in L^2(\hat{G}) \). That follows at once from the fact that
\[
\langle u, f \otimes g \rangle = \sum_{\gamma, \rho} \hat{u}(\gamma, \rho) \hat{f}(\gamma) \hat{g}(\rho),
\]
for all \( f \in C(G) \), \( g \in C(H) \),
which, in turn, is an easy calculation from
\[
\langle u, f \otimes g \rangle = \left\langle u, \left( \sum \hat{f}(\gamma) \hat{g}(\rho) \right) \otimes \left( \sum \hat{g}(\rho) \hat{p} \right) \right\rangle.
\]
The norm of the new \( T \) is now bounded by the product of three numbers: the norm of the old \( T \), the supremum of \( d\mu/dm_G \), and the reciprocal of the infimum of \( d\nu/dm_H \).

**Proposition 3.1.** Let \( G \) and \( H \) be compact abelian groups. Let \( u \in \text{Bil}^\sigma \). Suppose that \( u \) has \( \mu, \nu \) for its Grothendieck measures with \( d\mu/dm_G \) and \( d\nu/dm_H \) both bounded away from zero and away from \( \infty \). Then there exists a constant \( C > 0 \) such that \( \sum_{\gamma} |\hat{u}(\gamma, \rho)|^2 < C \) for every fixed \( \rho \in \hat{H} \) and \( \sum_{\rho} |\hat{u}(\gamma, \rho)|^2 < C \) for every fixed \( \gamma \in \hat{G} \).

**Proof.** By the discussion preceding the statement of Proposition 3.1, we see that there is a linear transformation \( T: L^2(G) \to L^2(H) \) such that
\[
\langle u, f \otimes g \rangle = \langle Tf, g \rangle \quad \text{for all } f \in C(G) \text{ and all } g \in C(H).
\]
(This transformation is the composition of the transformation discussed above with two Plancherel transformations.) Then
\[
\sum_{\gamma} |\hat{u}(\gamma, \rho)|^2 \leq \|T^*(\rho)\|_2^2 \leq \|T\|.
\]

**Corollary 3.2.** Let \( u \in \text{Bil}^\sigma \), where \( G, H \) are compact abelian groups. Then for every \( \varepsilon > 0 \) there exists \( N > 0 \) such that for each \( \rho \in \hat{H} \),
\[
\text{Card}\{\gamma : |\hat{u}(\gamma, \rho)| > \varepsilon\} \leq N.
\]

**Proof.** We fix \( \varepsilon > 0 \). Let \( v \) be such that \( \|u - v\| < \varepsilon/3 \) and such that \( v \) satisfies the hypotheses of Proposition 3.1. We let \( N \) be any integer greater than \( 9C/\varepsilon^2 \) (the \( C \) is from Proposition 3.1 applied to \( v \)). Then \( |\hat{u}(\gamma, \rho)| > \varepsilon \) implies \( |\hat{v}(\gamma, \rho)| > \varepsilon/3 \), and that can occur at most \( 9C/\varepsilon^2 \) times.

**Theorem 3.3.** Let \( u \in \text{Bil}^\sigma \), where \( G, H \) are compact abelian groups. Then the spectral radius of \( u \) is
\[
\sup_{\gamma \in \hat{G}, \rho \in \hat{H}} |\hat{u}(\gamma, \rho)|.
\]
Proof. By Theorem 2.3, we may assume that there is a Grothendieck measure pair $\mu, \nu$ for $u$ such that $d\mu/dm_G$ and $d\nu/dm_H$ are both bounded away from zero and infinity. Thus, we may assume that there is bounded linear transformation $T: L^2(G) \to L^2(H)$ such that $\langle u, f \otimes g \rangle = \langle Tf, g \rangle$ for all $f \in C(G)$ and $g \in C(H)$. Furthermore, for all continuous $f$ on $G$, $g$ on $H$,

$$\sum_{\gamma, \rho} |\hat{u}(\gamma, \rho)||\hat{f}(\gamma)||\hat{g}(\rho)| = |\langle u \ast \hat{u}, f \otimes g \rangle| \leq C'\|u\|^2\|f\|_2\|g\|_2,$$

where $C'$ is the product of four numbers: $K$ (the Grothendieck constant), the norm of $u$, the supremum of $d\mu/dm_G$, and the reciprocal of the infimum of $d\nu/dm_H$.

Let $f_1$ denote the Radon-Nikodym derivative $d\mu/dm_G$. Then $f_1$ has $L^1$-norm 1 and is bounded away from zero and infinity. Therefore, the $n$th convolution powers of $f_1$ converge to 1 uniformly, by Lemma 3.4 below. The same applies to $g_1 = d\nu/dm_H$.

That means that the Grothendieck measures (call them $\mu_n$, $\nu_n$) for $u^n$ become closer and closer to Haar measures, so the norm of the isomorphisms (and of their inverses) between $L^2(\mu_n)$ and $L^2(\hat{G})$ on the one hand, and $L^2(\nu_n)$ and $L^2(\hat{H})$ on the other hand, approach one. Thus, for sufficiently large $n$, we may assume that

$$\|u^n\|_{\text{Bil}^r} \leq C \sup\{|\langle u^n, f \otimes g \rangle| : \|f\|_2\|g\|_2 \leq 1\},$$

where $C$ does not depend on $n$, and the supremum is taken over all $f, g$ of uniform norm one.

But

$$\langle u^n, f \otimes g \rangle = \sum_{\gamma, \rho} \hat{u}^n(\gamma, \rho)\hat{f}(\gamma)\hat{g}(\rho).$$

Therefore

$$|\langle u^n, f \otimes g \rangle| \leq \|\hat{u}^{n-2}\|_\infty \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)||\hat{f}(\gamma)||\hat{g}(\rho)|.$$

It follows that

$$\|u^n\|_{\text{Bil}^r} \leq C \|\hat{u}^{n-2}\|_\infty \sup_{\gamma, \rho} \sum |\hat{u}^2(\gamma, \rho)||\hat{f}(\gamma)||\hat{g}(\rho)|,$$

where $C$ does not depend on $n$, and the supremum is taken over all $f, g$ of uniform norm one. By (3.1),

$$\sup_{\|f\|_\infty \leq 1, \|g\|_\infty \leq 1} \sum_{\gamma, \rho} |\hat{u}^2(\gamma, \rho)||\hat{f}(\gamma)||\hat{g}(\rho)| = \sup |\langle u \ast \hat{u}, f \otimes g \rangle| \leq C'\|u\|^2\|f\|_2\|g\|_2,$$

so $\|u^n\|_{\text{Bil}^r} \leq C''\|\hat{u}^{n-2}\|_\infty$ for all $n$. 

The conclusion about the spectral radius now follows easily. □

**Lemma 3.4.** Let \( f \) be a bounded non-negative Borel function on the compact group \( G \) that is bounded away from zero and has \( L^1 \)-norm one. Then the sequence of convolution powers of \( f \) converges uniformly to 1.

**Proof.** Since \( f \) is bounded, \( f \in L^2(G) \) and \( \hat{f} \in L^2(\hat{G}) \). Therefore \( f^2 = f*f \) has an absolutely convergent Fourier series, so, in particular, \( \hat{f} \in c_0(\hat{G}) \). Since \( f > 0 \) and \( \|f\|_1 = 1 \), \( \hat{f}(0) = 1 \). We apply the Lebesgue Dominated Convergence Theorem to \( \hat{f}^n \) (with \( |\hat{f}|^2 \) being the dominating function and \( n > 2 \)) to conclude that \( \hat{f}^n \) converges in \( l^1 \)-norm to a function \( f' \) that is equal to the characteristic function of a finite subset of \( \hat{G} \) (finite because \( \hat{f} \in c_0(\hat{G}) \)). Of course, that means that \( f^n \) converges uniformly to a function \( f_1 \) that is non-zero everywhere (the infimum of \( f^n \) is increasing with \( n \)). Thus, \( f_1 m_G \) is an idempotent probability measure. By [11, 3.2.4], \( f_1 m_G \) is Haar measure on a compact subgroup of \( G \). Since \( \hat{f}_1 = f' \) has finite support, that subgroup has finite index. If the index were greater than 1, \( f_1 \) would be zero somewhere, a contradiction. Therefore \( f_1 = 1 \) everywhere. □

**Corollary 3.5.** Let \( G \) and \( H \) be compact abelian groups. Then \( \hat{G} \times \hat{H} \) is dense in the maximal ideal space of \( \text{Bil}^\sigma \) and \( \text{Bil}^\sigma \) is a symmetric Banach algebra

**Proof.** This is a standard argument: the result is more or less immediate from Theorem 3.3. Here are the details.

We first note that \( \text{Bil}^\sigma \) is self-adjoint. For if \( S \in \text{Bil}^\sigma \) is such that its Gelfand transform \( \hat{S} \) is real on \( \hat{G} \times \hat{H} \), but not real on all of \( \Delta \text{Bil}^\sigma \) (the maximal ideal space), then for an appropriate \( k > 1 \), \( \exp(ikS) \) has Gelfand transform larger than one at that non-real value, but has Fourier-Stieltjes transform at most one, thus contradicting Theorem 3.3.

Since the space of Gelfand transforms \( \text{Bil}^{\sigma-} \) is self-adjoint and separating, it is uniformly dense in \( C_0(\Delta \text{Bil}^\sigma) \). If \( \hat{G} \times \hat{H} \) were not dense in \( \Delta \text{Bil}^\sigma \), then there would be a continuous function \( f \) on \( \Delta \text{Bil}^\sigma \) such that \( \|f\|_\infty = 1 \) and \( |f| < 1/2 \) on \( \hat{G} \times \hat{H} \). By estimating \( f \) uniformly by an element of \( \text{Bil}^\sigma \), we again contradict Theorem 3.3. □
We now give an example of an element of $\text{Bil}^\sigma$. The example is simple; we use it to show that $\text{Bil}^\sigma$ does not have approximate identities, even unbounded ones.

Let $\mu$ and $\nu$ denote regular Borel probability measures on the locally compact spaces $X$ and $Y$. Suppose that $\{\gamma_\alpha\}$ is an orthonormal basis for $L^2(\mu)$, and that $\{\rho_\beta\}$ is an orthonormal basis for $L^2(\nu)$. Let subsequences of those bases be chosen. Let $F(\gamma_\alpha, \rho_\beta)$ be defined by

$$F(\gamma_\alpha, \rho_\beta) = \begin{cases} 2^{-k/2}, & 2^k \leq j \leq 2^{k+1} - 1 \text{ and } j \geq 1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F(\gamma_\alpha, \rho_\beta) = 0 \text{ if there is no pair } j, k \text{ with } \alpha = \alpha_j \text{ and } \beta = \beta_k.$$

**Proposition 3.6.** With the above hypotheses,

1. $F$ is a bilinear functional on $L^2(\mu) \times L^2(\nu)$ that is bounded by 1;
2. $F$ represents an element of $\text{Bil}^\sigma$; and
3. Grothendieck measures for $F$ are given by $\mu, \nu$.

**Proof.** For the first part, let $x, y \in L^2(\mu) \times L^2(\nu)$, and let $x_j = \langle x, \gamma_\alpha \rangle$ for all $j$ and $y_k = \langle y, \rho_\beta \rangle$ for all $k$. Let also $F_{j,k} = F(\gamma_\alpha, \rho_\beta)$. Then

$$F(x, y) = \sum_{k} \sum_{j=2^k}^{2^{k+1}-1} F_{j,k} x_j y_k.$$

We may assume that the $x_j$ and $y_k$ are non-negative. For each $k$,

$$\sum_{j=2^k}^{2^{k+1}-1} F_{j,k} x_j \leq \left( \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2},$$

by the Cauchy-Schwarz inequality. Therefore,

$$|F(x, y)| \leq \sum_k \left( \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} y_k \leq \left( \sum_k \sum_{j=2^k}^{2^{k+1}-1} x_j^2 \right)^{1/2} \left( \sum_k y_k^2 \right)^{1/2}.$$
That is,

\[ F(x, y) \leq \|x\|_{L^2(\mu)} \|y\|_{L^2(\nu)}. \]

For the second assertion, by the first part and the fact that \( \mu, \nu \) are probability measures, \( |F(x, y)| \leq \|x\|_{\infty} \|y\|_{\infty} \) for all \( x \in L^\infty(\mu), y \in L^\infty(\nu) \). Hence \( F \) represents an element of \( \text{Bil}(L^\infty(\mu), L^\infty(\nu)) \). We must show that \( F \) is weak* continuous in each variable separately. Suppose that \( x_\lambda \to x \) weak* in \( L^\infty(\mu) \) and that \( y \in L^\infty(\nu) \). Note that \( L^\infty(\mu) \subseteq L^2(\mu) \subseteq L^1(\mu) \). By the latter containment, \( x_\lambda \) converges weak* in \( L^2(\mu) \) Since \( L^\infty(\mu) \) is dense in \( L^2(\mu) \),

\[ x_\lambda \to x \text{ weakly in } L^2(\mu). \]

Let

\[ z = \sum_k \langle y, \rho_{\beta_k} \rangle \left( \sum_{j=2^k}^{2^{k+1}-1} \langle x, \gamma_{\alpha_j} \rangle \gamma_{\alpha_j} \right). \]

Then \( z \in L^2(\mu) \) and \( \langle w, z \rangle = F(w, y) \) for all \( w \in L^\infty(\mu) \). Since \( z \in L^2(\mu) \),

\[ \lim_{\lambda} F(x_\lambda, y) = \lim_{\lambda} \langle x_\lambda, z \rangle = \langle x, z \rangle = F(x, y). \]

The weak* continuity in \( y \) is proved identically.

For the last assertion, we just apply (3.2) that \( \mu \) and \( \nu \) have the required property.

**Theorem 3.7.** Let \( G \) and \( H \) be infinite compact abelian groups. Then \( \text{Bil}^\sigma \) does not have an (even unbounded) approximate identity.

**Remark.** A virtual diagonal for a Banach algebra \( A \) is a bounded net \( \{m_\alpha\} \) in \( A \otimes A \) such that \( \lim_\alpha (m_\alpha a - \alpha a) = 0 \) and \( \lim \pi(m_\alpha)a = a \) for each \( a \in A \), where \( \pi(a \otimes b) = ab \). The Banach algebra \( A \) is amenable if and only if \( A \) has a virtual diagonal. If \( A \) is amenable, then \( A \) has a bounded approximate identity. Hence, \( \text{Bil}^\sigma \) is never amenable when \( G, H \) are compact abelian groups. See [1, p. 243] and [10, p. 50, Ex. 36].

**Proof.** Let the elements of \( \widehat{G} \) be denoted by \( \gamma_\alpha \) and the elements of \( \widehat{H} \) be denoted by \( \rho_\beta \). We apply the example of Proposition 3.6, only replacing \( \mu \) with \( m_G \) and \( \nu \) with \( m_H \). Suppose that \( L \in \text{Bil}^\sigma \) were such that \( \|L \ast F - F\| \leq \frac{1}{2K} \), where \( K \) is the usual complex Grothendieck constant.
By [4, 2.4], Grothendieck measures for a convolution of bimeasures are the convolution of Grothendieck measures of the factors. Combining that with the third item of Proposition 3.6, we see that Grothendieck measures for \( L \ast F - F = (L - \delta_0) \ast F \) are exactly Haar measure. That is, for all \( x \in L^2(G) \) and \( y \in L^2(H) \),

\[
(3.3) \quad |\langle L \ast F - F, x \otimes y \rangle| \leq K \|L \ast F - F\| \|x\|_2 \|y\|_2.
\]

For simplicity, denote \( F(\gamma_{\alpha_j}, \rho_{\beta_k}) \) by \( F_{j,k} \) and \( L(\gamma_{\alpha_j}, \rho_{\beta_k}) \) by \( L_{j,k} \). For each \( k \), let us compare the values of \( L \ast F \) and \( F \) at \( \gamma_{\alpha_j}, \rho_{\beta_k} \), for \( 2^k \leq j \leq 2^{k+1} - 1 \).

We will apply when \( x \) is the element of \( L^2(G) \) such that the Fourier transform of \( x \) is \( 2^{-k/2} e^{-\theta(j,k)} \), where \( \theta(j,k) \) is the argument of \( L_{j,k} - 1 \) if that difference is non-zero, and zero otherwise and \( y = \rho_{\beta_k} \). Then

\[
(3.4) \quad \langle L \ast F - F, x \otimes y \rangle = \sum_{j=2^k}^{2^{k+1}-1} |L_{j,k} - 1| 2^{-k} \\
\leq K \|L \ast F - F\| \|x\|_2 \|y\|_2 \leq \frac{1}{2}.
\]

Therefore, for at least half the terms in (3.4), \( |L_{j,k} - 1| \leq \frac{1}{2} \). That means that

\[
(3.5) \quad |L_{j,k}| \geq \frac{1}{2}
\]

for at least \( 2^{k-1} \) terms. For \( k \) sufficiently large, that contradicts Corollary 3.2.

When \( G \) is a compact abelian group, \( L^1(G) \) has a dense subset consisting of elements whose Fourier transforms have finite support. That is not possible for \( \text{Bil}_G \), since the characteristic function of any graph of a one-to-one function from \( \hat{G} \) to \( \hat{H} \) is the Fourier transform of an element of \( \text{Bil}_G \). In view of Corollary 3.2, one might hope that “finitely supported” could be replaced by “summable on sets of the form \( \gamma \times \hat{H} \), with uniform bound on the sums.” That is not possible, as the next result asserts.

**Theorem 3.8.** Let \( G \) and \( H \) be compact abelian groups. Then the set of elements \( L \) in \( \text{Bil}_G \) for which \( \sup_{\rho} \sum_{\gamma} |L(\gamma, \rho)| < \infty \) is not dense.

**Proof.** We adapt the proof of Theorem 3.7, using the same \( F \) as there.
Suppose that \( L \in \text{Bil}^\sigma \) is close to \( F \). Then the Grothendieck measures for \( L \) must be close (in an \( L^2 \) sense) to those of \( F \), that is they must be near to the respective Haar measures. That means that if \( \| F - L \| \) is sufficiently small, then

\[
|\langle F - L, x \otimes y \rangle| \leq 2K\|F - L\|\|x\|_2\|u\|_2
\]

for all \( x \in L^2(G), y \in l^2(G) \).

Suppose that \( \| L - F \| < \frac{\varepsilon}{2K} \). Suppose also that \( \sup_{\rho} \sum_{\gamma} |L(\gamma)| < \infty \). Then for sufficiently large \( k \), \( |L_j, k| < 2^{-1-k/2} \) for at least half the \( j \) in the range \( 2^k \leq j \leq 2^{k+1} - 1 \). Then

\[
\sum_{2^k \leq j \leq 2^{k+1} - 1} |L_j, k - F_j, k|2^{-k/2} \geq 2^{-k/2} \frac{1}{2} 2^{-k/2} 2^{k-1} = 2^{-2} \]

(evaluate at the same \( x, y \) as in the proof of Theorem 3.7). That implies that \( |L_j, k - 2^{-k/2}| < \varepsilon \), which is impossible for small \( \varepsilon \).

\( \square \)

4. Problems. We list in this section some open questions.

(1) What happens if \( L^\infty \) is replaced with \( LUC(G) \) or \( C(G) \)? [And one looks at the corresponding spaces defined via weak* limits?]

(2) What happens when we replace \( L^\infty \) with \( VN(G) \)?

(3) Does either \( \text{Bil}^\sigma \) or \( \text{Bil}(L^\infty(m_G), L^\infty(m_H)) \) characterize the underlying groups? (Wendel’s Theorem.)

(4) Same question for \( \text{BM}(G, H) \).

(5) What is the dual of \( \text{Bil}(L^\infty(m_G), L^\infty(m_H)) \)?

References


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