ON THE EXTENSION OF LIPSCHITZ FUNCTIONS FROM BOUNDARIES OF SUBVARIETIES TO STRONGLY PSEUDOCONVEX DOMAINS

K. Adachi and Hiroshi Kajimoto
ON THE EXTENSION OF LIPSCHITZ FUNCTIONS FROM BOUNDARIES OF SUBVARIETIES TO STRONGLY PSEUDOCONVEX DOMAINS

KENZÔ ADACHI AND HIROSHI KAJIMOTO

In this paper, we study the principal value integral on boundaries of subvarieties in strongly pseudoconvex domains and using it, we give a condition for the extendability of Lipschitz functions.

Introduction. Let \( D \) be a strongly pseudoconvex domain in \( C^n \) with \( C^\infty \) boundary. Henkin [6] and Ramírez [12] obtained independently the support function \( g(\zeta, z) \) for \( D \) which depends holomorphically on \( z \), and then, using this support function, they obtained the integral formula for holomorphic functions in \( \overline{D} \). On the other hand, Stout [14], when \( p = 1 \), and then, Hatziafratis [5], when \( p \) is arbitrary, obtained the integral formula for a certain subvariety \( V \) of codimension \( p \) in \( D \). By using the support function \( g(\zeta, z) \) and the integral formula for \( V \), we can obtain the kernel \( \Omega(\zeta, z) \) for \( (\zeta, z) \in \partial V \times \overline{D} \). In this paper, we shall define the principal value integral \( \text{P.V.} \int_{\partial V} f(\zeta)\Omega(\zeta, z) \) for a Lipschitz function \( f \) on \( \partial V \) and \( z \in \partial V \). The definition of the principal value integral is the same as that of Alt [2] when \( V = D \) (cf. Dolbeault [4]). By using the principal value integral we can give the condition for a Lipschitz function on \( \partial V \) to be the boundary value of a function that is holomorphic in \( D \) and continuous on \( \overline{D} \). Finally we end the introduction by giving an example which shows that the Lipschitz continuity is necessary in order to define the principal value integral.

Example. Define \( \varphi \in C^\infty(0, \infty) \) such that

\[
\varphi(\theta) = \begin{cases} 
1 & \text{if } 0 < \theta \leq \frac{\pi}{4}, \\
0 & \text{if } \theta \geq \frac{\pi}{2}.
\end{cases}
\]

Extend \( \varphi \) to an odd function on \( R\{0\} \). Let \( D \) be the unit disc in \( C \) and \( f \) be a function on \( \partial D \) such that

\[
f(e^{i\theta}) = \begin{cases} 
\frac{\varphi(\theta)}{\log|\theta|} & \text{if } 0 < |\theta| \leq \pi, \\
0 & \text{if } \theta = 0.
\end{cases}
\]
Then \( f \) is continuous but not Lipschitz continuous on \( \partial D \). Compute the principal value integral at \( 1 \in \partial D \). We find

\[
P.V. \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) d\zeta}{\zeta - 1} = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\varepsilon \leq |\theta| \leq \pi} \frac{f(e^{i\theta}) e^{i\theta}}{e^{i\theta} - 1} d\theta = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\varepsilon}^{\pi} f(e^{i\theta}) e^{i\theta} + \frac{1}{e^{i\theta} - 1} d\theta = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\varepsilon}^{\pi/4} \frac{1}{\log \theta} e^{i\theta} + \frac{1}{e^{i\theta} - 1} d\theta + \text{(finite value)}.
\]

But we have

\[
\frac{1}{\log \theta} e^{i\theta} + \frac{1}{e^{i\theta} - 1} \sim \frac{-2i}{\theta \log \theta}
\]

when \( \theta \to 0^+ \). This shows that the principal value integral of \( f \) at \( 1 \) does not converge.

1. **The integral formula on subvarieties.** Let \( D \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^\infty \) boundary. Let \( \rho \) be a defining function of \( D \), i.e., \( D = \{ z : \rho(z) < 0 \} \). We set

\[
F(\zeta, z) = \sum_{i=1}^{n} 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta_i - \zeta_i) \zeta_i(z_i - \zeta_i) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta_i - \zeta_i)(z_j - \zeta_j).
\]

According to the construction of Henkin [6] (cf. Henkin-Leiterer [8], p. 108), there exists a pseudoconvex neighborhood \( \tilde{D} \) of \( \overline{D} \), a neighborhood \( W \) of \( \partial D \), and a \( C^\infty \) function \( g : W \times \tilde{D} \to C \) such that for each \( \zeta \in W \), \( g(\zeta, z) \) is holomorphic in \( \tilde{D} \). For \( r > 0 \), define \( \Delta_r = \{ (\zeta, z) \in W \times \tilde{D} : |\zeta - z| < r \} \). Then there exist a constant \( \sigma_1 > 0 \) and a non-vanishing \( C^\infty \) function \( Q(\zeta, z) \) on \( \Delta_{\sigma_1} \) such that \( g(\zeta, z) = F(\zeta, z)Q(\zeta, z) \) for \( (\zeta, z) \in \Delta_{\sigma_1} \), and \( g(\zeta, z) \neq 0 \) for \( (\zeta, z) \in W \times \tilde{D}\Delta_{\sigma_1} \). Moreover \( g(\zeta, z) \) admits a division

\[
g(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, z)(z_j - \zeta_j)
\]

with \( g_j : W \times \tilde{D} \to C \) of class \( C^\infty \) and holomorphic in the second variable. Let \( h_1, \ldots, h_p \) \( (p < n) \) be holomorphic functions in \( \tilde{D} \).
Define
\[ \widetilde{V} = \{ z \in \tilde{D} : h_1(z) = \cdots = h_p(z) = 0 \}, \quad V = \widetilde{V} \cap D. \]
Assume that
(1) \[ \partial h_1 \wedge \cdots \wedge \partial h_p \wedge \partial \rho \neq 0 \text{ on } \partial V. \]
By Hefer's theorem, there exist holomorphic functions \( h_{ij}(\zeta, z) \) for \( (\zeta, z) \in \tilde{D} \times \tilde{D} \) such that
\[ h_i(\zeta) - h_i(z) = \sum_{j=1}^{n} h_{ij}(\zeta, z)(\zeta_j - z_j), \quad i = 1, \ldots, p. \]
Define
\[ \alpha^h(\zeta, z) = \begin{vmatrix} g_1 & h_{11} & \cdots & h_{p1} & \bar{\partial}_\zeta g_1 & \cdots & \bar{\partial}_\zeta g_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_n & h_{1n} & \cdots & h_{pn} & \bar{\partial}_\zeta g_n & \cdots & \bar{\partial}_\zeta g_n \end{vmatrix}, \]
\[ |\nabla h(\zeta)|^2 = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \left| \frac{\partial (h_1, \ldots, h_p)}{\partial (\zeta_{i_1}, \ldots, \zeta_{i_p})}(\zeta) \right|^2, \]
\[ \beta^h(\zeta) = (-1)^{p(p+1)/2} |\nabla h(\zeta)|^{-2} \begin{vmatrix} \frac{\partial h_1}{\partial \zeta_{i_1}} & \cdots & \frac{\partial h_1}{\partial \zeta_{i_p}} & d\zeta_1 & \cdots & d\zeta_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_p}{\partial \zeta_{i_1}} & \cdots & \frac{\partial h_p}{\partial \zeta_{i_p}} & d\zeta_n & \cdots & d\zeta_n \end{vmatrix} \]
and
\[ K_V(\zeta, z) = c(n, p)\alpha^h(\zeta, z) \wedge \beta^h(\zeta), \]
where
\[ c(n, p) = (-1)^p(n+1)(-1)^{n-1}/(n-p)!(2\pi i)^{n-p}. \]
Let \( n - p = k. \) We define the kernel \( \Omega(\zeta, z) \) by
\[ \Omega(\zeta, z) = \frac{K_V(\zeta, z)}{g(\zeta, z)^k}. \]
Let \( A(D) \) (resp. \( A(V) \)) be the space of functions that are holomorphic in \( D \) (resp. \( V \)) and continuous on \( \overline{D} \) (resp. \( \overline{V} \)). Then Hatzi-afritas [5] proved the following.

**Theorem 1.** For \( f \in A(V) \) and \( z \in V, \) the integral formula
(2) \[ f(z) = \int_{\partial V} f(\zeta)\Omega(\zeta, z) \]
holds.

Now we begin by proving the following lemma.
LEMMA 1. If \( h_1(z) = z_{k+1}, \ldots, h_p(z) = z_n \), then we have

\[
K_V(\zeta, z) = (-1)^{k(k-1)/2} \frac{(k - 1)!}{(2\pi i)^k} \sum_{j=1}^{k} (-1)^{j-1} g_j \bigwedge_{i=1 \atop i \neq j} \partial \zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k.
\]

Proof. By the definition of \( \alpha^h, \beta^h \), we have

\[
\alpha^h(\zeta, z) = \begin{vmatrix} g_1 & 0 & \cdots & 0 & \partial \zeta g_1 & \cdots & \partial \zeta g_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_p & 0 & \cdots & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ g_n & 0 & \cdots & 1 & \cdots & \cdots & 1 \end{vmatrix} = (-1)^p \begin{vmatrix} g_1 & \partial \zeta g_1 & \cdots & \partial \zeta g_1 \\ \vdots & \vdots & \ddots & \vdots \\ g_k & \partial \zeta g_k & \cdots & \partial \zeta g_k \end{vmatrix}
\]

\[
\beta^h(\zeta) = (-1)^{n-1} \left( \frac{n(n+1)}{2} \right) k! d\zeta_1 \wedge \cdots \wedge d\zeta_k.
\]

Therefore we have

\[
K_V(\zeta, z) = (-1)^{(n-k)(n+1)} (-1)^{n(n-1)/2} \frac{1}{k!(2\pi i)^k} \alpha^h(\zeta, z) \wedge \beta^h(\zeta, z)
\]

\[
= (-1)^{k(k-1)/2} \frac{(k - 1)!}{(2\pi i)^k} \sum_{j=1}^{k} (-1)^{j-1} g_j \bigwedge_{i=1 \atop i \neq j} \partial \zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k.
\]

This completes the proof of Lemma 1.

For \( \sigma > 0 \), we set \( S_{\zeta, \sigma} = \{ z : |z - \zeta| < \sigma \} \). Then we have the following.
**Lemma 2.** Let $\sigma$, $0 < \sigma < \sigma_1$, be sufficiently small. Then for $\zeta \in \partial V$, $z \in \tilde{D} \cap S_{\zeta, \sigma}$, we can choose a local coordinate system in such a way that

$$
\Omega(\zeta, z) = \frac{2^k}{(2\pi i)^k} \frac{\partial \rho(\zeta) \wedge (\overline{\partial} \partial \rho)^{k-1}(\zeta) + \epsilon(\zeta, z)}{F(\zeta, z)^k},
$$

where $\epsilon(\zeta, z)$ is a $(k, k-1)$ form satisfying $\epsilon(\zeta, z) = O(|\zeta - z|)$.

**Proof.** By a local holomorphic change of coordinates, we may assume that $h_1 = z_{k+1}, \ldots, h_p = z_n$. We have (cf. Anderson and Berndtsson [3], Lemma 3),

$$
\sum_{j=1}^{k} (-1)^{j-1} g_j \wedge_i (d^*_{\zeta}g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k)
\frac{g(\zeta, z)^{k}}{g(\zeta, z)^{6k}}
= c(k) \left( \sum_{j=1}^{k} g_j d\zeta_j \right) \wedge \left( \sum_{j=1}^{k} \overline{d}_{\zeta} \overline{g}_j \wedge d\zeta_j \right)^{k-1}
\frac{g(\zeta, z)^{k}}{g(\zeta, z)^{6k}},
$$

where $c(k) = (-1)^{k(k-1)/2} \frac{1}{(k-1)!}$.

If $|\zeta - z| < \sigma$, then we obtain

$$
\sum_{i=1}^{n} g_i(\zeta, z)(z_i - \zeta_i)
= \sum_{i=1}^{n} \left\{ \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_j - \zeta_j) \right) Q(\zeta, z) \right\} (z_i - \zeta_i).
$$

Therefore we have

$$
g_i(\zeta, \zeta) = 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) Q(\zeta, \zeta), \quad 1 \leq i \leq n.
$$

By Lemma 1, we obtain

$$
\Omega(\zeta, z) = \frac{1}{(2\pi i)^k} \frac{\left( \sum_{j=1}^{k} \frac{\partial \rho}{\partial \zeta_j}(\zeta) Q(\zeta, \zeta) d\zeta_j \right) \wedge \left( \sum_{j=1}^{k} \overline{\partial} \overline{\partial} \rho(\zeta) Q(\zeta, \zeta) d\zeta_j \right)^{k-1}}{F(\zeta, z)^k Q(\zeta, z)^k}
+ \frac{\epsilon(\zeta, z)}{F(\zeta, z)^k Q(\zeta, z)^k}
= \frac{2^k}{(2\pi i)^k} \frac{\partial \rho(\zeta) \wedge (\overline{\partial} \partial \rho)^{k-1}(\zeta) + \epsilon(\zeta, z)}{F(\zeta, z)^k}.
$$

This completes the proof of Lemma 2.
Now we prove the following lemma which will be used in the proof of Theorem 2 in order to calculate the principal value of the kernel function.

**Lemma 3.** For $\zeta, z \in \partial V$, it holds that

$$F(\zeta, z) - \overline{F(z, \zeta)} = O(|\zeta - z|^3).$$

**Proof.** We may assume that $h_1 = z_{k+1}, \ldots, h_p = z_n$. By the Taylor expansion, we have

$$\frac{1}{2}F(\zeta, z) = \sum_{i=1}^{k} \frac{\partial \rho}{\partial \xi_i}(\zeta)(z_i - \zeta_i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j)$$

$$= \sum_{i=1}^{k} \left( \frac{\partial \rho}{\partial \xi_i}(z) + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j}(z)(\zeta_j - z_j) \right)(z_i - \zeta_i)$$

$$+ \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \xi_i \partial \bar{\zeta}_j}(z)(\zeta_j - z_j)(\bar{\zeta}_j - \bar{z}_j) + O(|\zeta - z|^3).$$

On the other hand we have

$$\frac{1}{2}F(\zeta, z) = \sum_{i=1}^{k} \frac{\partial \rho}{\partial \bar{\xi}_i}(\zeta)(\bar{\zeta}_i - \bar{z}_i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \bar{\xi}_i \partial \bar{\xi}_j}(\zeta)(\bar{\zeta}_i - \bar{z}_i)(\bar{\zeta}_j - \bar{z}_j).$$

Thus we obtain
\[
\frac{1}{2}(F(\zeta, z) - F(z, \zeta))
\]

\[
= -\sum_{i=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_i - z_i)(\zeta_j - z_j)
\]

\[
- \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_i - z_i)(\zeta_j - z_j)
\]

\[
- \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_i - z_i)(\zeta_j - z_j) + O(|\zeta - z|^3)
\]

\[
= - \rho(\zeta) + \rho(z) + O(|\zeta - z|^3) = O(|\zeta - z|^3).
\]

This completes the proof of Lemma 3.

2. The principal value of the kernel function. Let \( z \in \partial V \) and \( f \) be a continuous function on \( \partial V \). If

\[
\lim_{\epsilon \to 0^+} \int_{\partial V \cap \{z: |g(\zeta, z)| > \epsilon\}} f(\zeta)\Omega(\zeta, z)
\]

exists, then we stand for the above limit by

\[
P.V. \int_{\partial V} f(\zeta)\Omega(\zeta, z).
\]

Now we are going to prove the following theorem which was obtained by Kerzman and Stein [9] when \( k = n \).

**Theorem 2.** For \( z \in \partial V \), it holds that

\[
P.V. \int_{\zeta \in \partial V} \Omega(\zeta, z) = \frac{1}{2}.
\]

**Proof.** Let \( q \in \partial V \) be fixed. We may assume that for \( \delta > 0 \) sufficiently small,

\[
\tilde{V} \cap S_{q, \delta} = \{z \in S_{q, \delta}: z_{k+1} = \cdots = z_n = 0\}.
\]
For $z \in \tilde{V} \cap S_{q, \delta}$, we have

$$\rho(z) = 2 \text{Re} \left( \sum_{i=1}^{k} \frac{\partial \rho}{\partial z_i}(q)(z_i - q_i) \right. + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(q)(z_i - q_i)(z_j - q_j) \bigg)$$

$$\left. + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(q)(z_i - q_i)(\bar{z}_j - \bar{q}_j) + O(|z - q|^3) \right).$$

For simplicity, we may assume that $q = 0$. By (1), we can find a new local coordinate system $w_1, \ldots, w_n$ by letting

$$w_1 = 2 \sum_{i=1}^{k} \frac{\partial \rho}{\partial z_i}(0)z_i + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0)z_iz_j,$$

and choosing $w_1, \ldots, w_n$ suitably such that $w(0) = 0$. Then we have

$$\rho(w) = \text{Re} w_1 + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial w_i \partial \bar{w}_j}(0)|w_i\bar{w}_j| + O(|w|^3).$$

By a unitary transformation $w'_i = (w'_1, \ldots, w'_n)$, we obtain

$$\rho(w') = \text{Re} \left( \sum_{j=1}^{k} a_j w'_j \right) + \sum_{i=1}^{k} \frac{\partial^2 \rho}{\partial w'_i \partial \bar{w}'_i}(0)|w'_i|^2 + O(|w'|^3),$$

where $(a_1, \ldots, a_k)$ is a non-zero vector. Again we can find a new local coordinate system $\zeta = (\zeta_1, \ldots, \zeta_n)$ such that

$$\rho(\zeta) = \text{Re} \zeta_1 + \sum_{i=1}^{k} |\zeta_i|^2 + O(|\zeta|^3).$$

We set $\zeta_j = x_j + iy_j$. Then we have

$$\partial \rho(0) \wedge (\bar{\partial} \partial \rho)^{k-1}(0)$$

$$= \left( \sum_{j=1}^{k} \frac{\partial \rho}{\partial \zeta_j}(0)d\zeta_j \right) \wedge \left( \sum_{i=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \bar{\zeta}_i}(0)d\bar{\zeta}_i \wedge d\zeta_i \right)^{k-1}.$$

Since $\partial \rho = 0$ on $\partial V$, we have

$$\sum_{j=1}^{k} \frac{\partial \rho}{\partial x_j}(0)dx_j + \sum_{j=1}^{k} \frac{\partial \rho}{\partial y_j}(0)dy_j = 0.$$
Therefore we have
\[
\partial \rho(0) \wedge (\overline{\partial} \partial \rho)^{k-1}(0) \\
= \frac{i}{2} \sum_{j=1}^{k} \left( \frac{\partial \rho}{\partial x_j}(0) dy_j - \frac{\partial \rho}{\partial y_j}(0) dx_j \right) \\
\wedge (2i)^{k-1} \left( \sum_{j=2}^{k} \frac{\partial^2 \rho}{\partial w_j \partial \overline{w}_j}(0) dx_j \wedge dy_j \right)^{k-1} \\
= 2^{k-2} i^k (k-1)! dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_k \wedge dy_k.
\]
Thus, by Lemma 2, we obtain
\[
\Omega(\zeta, z) = \frac{2^{k-2} (k-1)! dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_k \wedge dy_k + O(|\zeta| + |\zeta - z|)}{\pi^{k} F(\zeta, z)^k}.
\]
Let \( \nu \) be the unit inner normal of \( \partial V \) at 0. Then we have
\[
\nu = - \left( \frac{\partial \rho}{\partial x_1}(0), \frac{\partial \rho}{\partial y_1}(0), \ldots, \frac{\partial \rho}{\partial y_n}(0) \right) = (-1, 0, \ldots, 0).
\]
We set, for \( \delta > 0 \) sufficiently small, \( z = \nu \delta \). Then we have
\[
F(\zeta, z) - F(\zeta, 0) \\
= \sum_{i=1}^{k} (-\zeta_i + \nu_i \delta) \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (-\zeta_j + \nu_j \delta) \right) \\
- \sum_{i=1}^{k} (-\zeta_i) \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (-\zeta_j) \right) \\
= \sum_{i=1}^{k} 2 \nu_i \delta \frac{\partial \rho}{\partial \zeta_i}(\zeta) \\
+ \sum_{i,j=1}^{k} (\nu_i \nu_j \delta^2 - \nu_i \zeta_j \delta - \nu_j \zeta_i \delta) \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) \\
= -\delta + O(\delta |\zeta| + \delta^2).
\]
On the other hand we have
\[
F(0, \zeta) = \sum_{i=1}^{k} \zeta_i \left( 2 \frac{\partial \rho}{\partial \zeta_i}(0) + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(0) \zeta_j \right) = \zeta_1 = x_1 + iy_1,
\]
and

\[ 0 = \rho(\zeta) = x_1 + \sum_{i=2}^{k} |\zeta_i|^2 + O(|\zeta|^3). \]

Thus we obtain

\[ F(0, \zeta) = -\sum_{i=1}^{k} |\zeta_i|^2 + iy_1 + O(|\zeta|^3). \]

Taking account of Lemma 3, we have

\[ F(\zeta, z) = F(\zeta, 0) - \delta + O(|\zeta|\delta + \delta^2) \]

\[ = F(0, \zeta) - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2) \]

\[ = -iy_1 - \sum_{i=2}^{k} |\zeta_i|^2 - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2). \]

We set

\[ \beta(0, \varepsilon) = \lim_{\delta \to 0} \frac{K_{\nu}(\zeta, \nu\delta)}{\int_{\{\zeta \in \partial V: |F(\zeta, 0)| < \varepsilon\}} \frac{2^{k-2}(k-1)! \sigma(d\zeta)}{\pi^k F(\zeta, \nu\delta)^k}. \]

Then, by (4), we have

\[ \beta(0, \varepsilon) = \lim_{\delta \to 0} \frac{2^{k-2}(k-1)! \sigma(d\zeta)}{\pi^k F(\zeta, \nu\delta)^k} \]

where \( \sigma(d\zeta) \) is the surface element of \( \{ \zeta \in \partial V: |F(\zeta, 0)| < \varepsilon \} \). We set \( \zeta' = (\zeta_2, \ldots, \zeta_k) \). It holds from (3) that \( (\partial \rho/\partial x_1)(0) \neq 0 \). Therefore by the implicit function theorem, \( \rho = 0 \) can be represented by \( x_1 = \phi(y_1, \zeta') \), where \( \phi \) is a smooth function satisfying \( \phi(0, 0) = 0 \). Thus we have \( x_1 = O(|y_1| + |\zeta'|) \). Hence there exists a constant \( c_1 > 0 \) such that

\[ ||\zeta'|^2 + iy_1| - |F(\zeta, 0)|| \leq c_1(|\zeta'| + |y_1|)^3. \]

Therefore, for \( \varepsilon > 0 \) sufficiently small, we have

\[ \left\{ \zeta \in \partial V: ||\zeta'|^2 + iy_1| < \frac{\delta}{2} \right\} \]

\[ \subset \{ \zeta \in \partial V: |F(\zeta, 0)| < \varepsilon \} \subset \left\{ \zeta \in \partial V: ||\zeta'|^2 + iy_1| < \frac{3}{2} \varepsilon \right\}. \]

Thus we have

\[ \beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{\delta \to 0} \frac{\sigma(d\zeta)}{\int_{\{\zeta \in \partial V: |F(\zeta, 0)| < \varepsilon\}} F(\zeta, \nu\delta)^k}. \]
provided the limit of the right-hand side exists. We set \( A = \delta + |\zeta'|^2 + iy_1 \). Then we have \( F(\zeta, \nu \delta) = A + \mathcal{E} \) where \( \mathcal{E} = O(|\zeta|^3 + \delta |\zeta| + \delta^2) \). Then for some constant \( c_2 > 0 \), it holds that
\[
\left| \frac{\mathcal{E}}{A} \right| \leq c_2 (|\zeta'| + |y_1| + \delta).
\]
Thus if we choose \( \delta \) and \( \varepsilon \) sufficiently small, then we have
\[
\left| \frac{\mathcal{E}}{A} \right| \leq \frac{1}{2}.
\]
On the other hand we have
\[
\frac{1}{F(\zeta, \nu \delta)^k} = \frac{1}{(A + \mathcal{E})^k} = \frac{1}{A^k} + \mathcal{E}',
\]
where \( \mathcal{E}' \) satisfies, for some constant \( c_3 > 0 \),
\[
|\mathcal{E}'| \leq \frac{c_3 (|\zeta'| + |y_1| + \delta)}{|A|^k}.
\]
It holds that
\[
\int_{\{iy_1 + |\zeta'|^2 < \varepsilon\}} \frac{|\zeta'| + |y_1| + \delta}{|A|^k} \sigma(d\zeta) \to 0 \quad \text{(as } \delta \to 0, \varepsilon \to 0).\]
Thus we obtain
\[
\beta(0, \varepsilon) = \frac{2^{k-2}(k - 1)!}{\pi^k} \lim_{\delta \to 0} \int_{\{iy_1 + |\zeta'|^2 < \varepsilon\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + \delta)^k}.
\]
The dilation \((y - 1, \zeta') \to (\delta y_1, \sqrt{\delta} \zeta')\) gives
\[
\lim_{\varepsilon \to 0} \beta(0, \varepsilon) = \frac{2^{k-2}(k - 1)!}{\pi^k} \lim_{R \to \infty} \int_{\{iy_1 + |\zeta'|^2 < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}.
\]
The calculation of this limit is contained in Korányi and Vagi [10] as follows. Consider the integral
\[
I(\varepsilon, R) = \int_{\{\varepsilon < iy_1 + |\zeta'|^2 < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}, \quad 0 < \varepsilon < R.
\]
Introduce polar coordinates in \( C^{k-1} \),
\[
\rho = |\zeta'|, \quad \omega = \frac{\zeta'}{|\zeta'|}, \quad \sigma(d\zeta) = \rho^{2k-3} dy_1 d\rho d\omega
\]
where \( d\omega \) is the surface element in the unit sphere \( S^{2k-3} \) in \( C^{k-1} \). We denote the volume of the unit sphere \( S^{2k-3} \) by \( |S^{2k-3}| \). Next we make the variable change
\[
u = \rho^2, \quad d\nu = 2\rho d\rho
\]
and then introduce polar coordinates in the $u$, $y_1$ halfplane,

$$u + iy_1 = se^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \epsilon < s < R,$$

$$du\,dy_1 = s\,ds\,d\theta.$$  

We find

$$I(\epsilon, R) = \left| S^{2k-3} \right| \int_{\{\epsilon^2 + y_1^2 + \rho^4 > R^2\}} \frac{\rho^{2k-3} d\rho dy_1}{(1 + \rho^2 + iy_1)^k}$$

$$= \frac{|S^{2k-3}|}{2} \int_{\{\epsilon^2 + y_1^2 + u^2 < R, \ 0 \leq u\}} \frac{u^{k-1} du dy_1}{(1 + u + iy_1)^k}$$

$$= \frac{|S^{2k-3}|}{2} \int_{-\pi/2}^{\pi/2} \cos^{k-2} \theta \,d\theta \int_{\epsilon}^{R} \frac{s^{k-1} ds}{(1 + se^{i\theta})^k}.$$  

The variable change $\theta \to \theta - \frac{\pi}{2}, \ s \to \frac{1}{\rho}$ finally gives

$$I(\epsilon, R) = \frac{|S^{2k-3}|}{2} \int_{0}^{\pi} \sin^{k-2} \theta \,d\theta \int_{1/R}^{1/\epsilon} \frac{d\rho}{\rho(\rho - ie^{i\theta})}.$$  

Then Korányi and Vagi ([10], Lemma 6.2, p. 613) gives

$$\lim_{\epsilon \to 0^+, R \to \infty} I(\epsilon, R) = \frac{\pi^{k-1}}{(k - 2)!} \cdot \frac{\pi}{2^{k-1}(k - 1)!} = \frac{\pi^k}{2^{k-1}(k - 1)!}.$$  

Hence we obtain

$$\lim_{\epsilon \to 0^+} \beta(0, \epsilon) = \frac{1}{2},$$

which completes the proof of Theorem 2.

3. The continuous extension to the boundary. The following proposition is proved essentially by Adachi [1] (cf. Henkin [7]). But for the comparison with the principal value integral, we give the sketch of the proof.

**Proposition 1.** Define, for $z \in \overline{D}\partial V$,

$$H(z) = \int_{\zeta \in \partial V} \Omega(\zeta, z).$$

Then the function

$$\tilde{H}(z) = \begin{cases} H(z) & (z \in \overline{D}\partial V), \\ 1 & (z \in \partial V) \end{cases}$$

belongs to $A(D)$.
Proof. Since $H(z)$ is holomorphic in $\overline{D}\backslash \partial V$, it is sufficient to show that, for $z^0 \in \partial V$,
\[
\lim_{z \to z^0, z \in \overline{D}\backslash \partial V} H(z) = 1.
\]

We may assume that
\[
\tilde{V} \cap S_{z^0, \sigma_1} = \{ z \in S_{z^0, \sigma_1} : z_{k+1} = \cdots = z_n = 0 \}.
\]

By (1), we may assume, without loss of generality, that $(\partial \rho/\partial \zeta_1)(z^0) \neq 0$. For $z \in S_{z^0, \sigma_1}$, we consider the system of equations for $\zeta^0 = (\zeta_1^0, \ldots, \zeta_n^0)$ of the following form:
\[
\begin{cases}
\sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i} (\zeta^0_i - z_i) = 0, \\
\zeta_i^0 = z_i \quad (i = 2, \ldots, k), \quad \zeta_{k+1}^0 = \cdots = \zeta_n^0 = 0.
\end{cases}
\]

We set $\varepsilon = (|z_{k+1}|^2 + \cdots + |z_n|^2)^{1/2}$. Then by Adachi [1], there exist positive constants $\sigma_2$ ($< \sigma_1$), $\gamma_1$ and $\gamma_2$ such that for any $\sigma < \sigma_2$ and any $z \in S_{z^0, \sigma_2} \cap (\overline{D}\backslash \partial V)$, there exists a unique solution $\zeta^0 = \zeta^0(z)$ of the system (5) which belongs to the set $S_{z^0, \sigma} \cap V$ and satisfies the following.

(6) $\varepsilon \leq |z - \zeta^0| \leq \gamma_1 \varepsilon$,

(7) $\left| \sum_{i=1}^n \frac{\partial g}{\partial z_i}(\zeta, z)(\zeta_i^0 - z_i) \right| \leq \gamma_2 \varepsilon (|\zeta - z| + \varepsilon)$.

From the integral formula (2) we have
\[
H(\zeta^0) = 1.
\]

Hence it is sufficient to show that
\[
\lim_{z \to z^0, z \in \overline{D}\backslash \partial V} |H(z) - H(\zeta^0)| = 0.
\]

Let $z \in S_{z^0, \sigma_2} \cap (\overline{D}\backslash \partial V)$. Let $V'$ be an open subset in $\tilde{V}$ with smooth boundary such that $\overline{V} \subset V' \subset \overline{V}' \subset \tilde{V}$. By using Stokes' formula, we have
\[
H(z) = \int_{\partial V'} \Omega(\zeta, z) - \int_{V' - V} \overline{\partial}_z \Omega(\zeta, z).
\]

Define
\[
\Psi(z) = \int_{(V' - V) \cap S_{z^0, \sigma}} \overline{\partial}_z \Omega(\zeta, z).
\]
It is sufficient to show that
\[ \lim_{z \to z^0, z \in \bar{D} \setminus \partial V} |\Psi(z) - \Psi(z^0)| = 0. \]

We can write \( \Psi(z) \) in the following form
\[
\Psi(z) = \int_{(V' - V) \cap S_{z^0, \sigma}} \frac{A(\zeta, z)}{g(\zeta, z)^k} \\
+ \int_{(V' - V) \cap S_{z^0, \sigma}} \sum_{j=1}^{n} \frac{(\zeta - z_j)B_j(\zeta, z)}{g(\zeta, z)^{k+1}},
\]
where \( A(\zeta, z), B_j(\zeta, z) \) are \((k, k)\) forms that are smooth in \((\zeta, z)\) and holomorphic in \( z \). By using (6), (7), there are positive constants \( \gamma_3 \) and \( \gamma_4 \) such that
\[
\left| \frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_3 \int_{(V' - V) \cap S_{z^0, \sigma}} \frac{\epsilon dV}{|g(\zeta, z)|^{k+1}} \\
+ \gamma_4 \int_{(V' - V) \cap S_{z^0, \sigma}} \frac{|\zeta - z| \epsilon (|\zeta - z| + \epsilon) dV}{|g(\zeta, z)|^{k+2}}.
\]

By the estimates obtained by Henkin [7], we have for some constant \( \gamma_5 > 0 \)
\[
(8) \quad \left| \frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_5 (\epsilon |\log \epsilon| + \epsilon).
\]

Let
\[ z(\theta) = \zeta^0 + \theta(z - \zeta^0) \quad \text{for} \ \theta \in [0, 1]. \]
Then the uniqueness of the solution of the system of the system (5) implies \( \zeta^0(z(\theta)) = \zeta^0(z) \). Therefore from (8), we have, for some constant \( \gamma_6 > 0 \),
\[
\left| \frac{d\Psi(\zeta^0 + \lambda \theta(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} = \left| \frac{d\Psi(\zeta^0(z(\theta)) + \lambda(z(\theta) - \zeta^0(z(\theta))))}{d\lambda} \right|_{\lambda=1} \leq \gamma_6 (\theta \epsilon |\log(\theta \epsilon)| + \theta \epsilon).
\]
Thus we obtain for some constant $\gamma_7 > 0$,

$$|\Psi(z) - \Psi(\zeta^0)| = \left| \int_0^1 \frac{d}{d\theta} \Psi(\zeta^0 + \theta(z - \zeta^0)) \, d\theta \right|$$

$$= \left| \int_0^1 \frac{1}{\theta} \left( \frac{d\Psi(\zeta^0 + \lambda \theta(z - \zeta^0))}{d\lambda} \right) \bigg|_{\lambda=1} \, d\theta \right|$$

$$\leq \gamma_7 \int_0^1 (|\epsilon| \log \epsilon + |\epsilon| \log \theta + \epsilon) \, d\theta \to 0 \quad (\epsilon \to 0),$$

which completes the proof of Proposition 1.

4. The extension of Lipschitz functions from the boundary. In order to prove Lemma 5, we need the following lemma which is the modified version of Lemma 3.1 of Henkin [6].

**Lemma 4.** Let $t = (t_1, \ldots, t_{2k}) \in \mathbb{R}^{2k}$, $\epsilon > 0$, $0 < \delta < 1$, $t' = (t_2, \ldots, t_{2k})$. Then we have

$$I_1 = \int_{\{\delta^2 \leq |t'|^2 + \epsilon^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \epsilon^2)^2 + t_2^2]^{k/2}} \leq \gamma \log \frac{\gamma}{\delta},$$

where $\gamma$ is the constant which is independent of $\epsilon$ and $\delta$.

**Proof.** (a) In case $\epsilon^2 \leq \frac{1}{2} \delta^2$. Since $\delta^2 \leq |t'|^2 + \epsilon^2$, we have $|t'|^2 \geq \frac{1}{2} \delta^2$. Therefore we have

$$I_1 \leq \int_{\{\delta^2 \leq |t'|^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{(|t'|^2 + t_2^2)^{k/2}}.$$

If $k = 1$, then we have

$$I_1 \leq \int_{\delta/2}^{1} \frac{dt_2}{t_2} \leq \gamma \log \frac{\gamma}{\delta}.$$

If $k \geq 2$, by using polar coordinates, we have for some $\gamma_1 > 0$,
\[
I_1 \leq \gamma_1 \int_{\frac{\delta}{2}}^{1} dr \int_{0}^{\pi} \frac{r^{2k-2} \sin^{2k-3} \varphi \, d\theta}{(r^4 + r^2 \cos^2 \varphi)^{k/2}} \\
= \gamma_1 \int_{\frac{\delta}{2}}^{1} dr \int_{0}^{\pi} \frac{r^{k-2} \sin^{2k-3} \varphi \, d\varphi}{(r^2 + \cos^2 \varphi)^{k/2}} \\
\leq \gamma_1 \int_{\frac{\delta}{2}}^{1} dr \int_{0}^{\pi} \frac{\sin^{2k-3} \varphi \, d\varphi}{r^2 + \cos^2 \varphi} \\
\leq \gamma_1 \int_{\frac{\delta}{2}}^{1} dr \int_{-1}^{1} \frac{ds}{r^2 + s^2} = \gamma_1 \int_{\gamma/2}^{1} \frac{2}{r} \tan^{-1} \left( \frac{1}{r} \right) \, dr \\
\leq \pi \gamma_1 \int_{\frac{\delta}{2}}^{1} \frac{dr}{r} \leq \gamma \log \frac{\gamma}{\delta}.
\]

(b) In case \( \varepsilon^2 \geq \frac{1}{2} \delta^2 \). Then we have

\[
I_1 \leq \int_{\{|t'| \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \varepsilon^2)^2 + t_{2k}^2]^{k/2}}.
\]

If \( k = 1 \), then we have for some \( \gamma_2 > 0 \),

\[
I_1 \leq \gamma_2 \int_{0}^{1} \frac{dt_2}{t_2 + \delta^2} \leq \gamma \log \frac{\gamma}{\delta}.
\]

If \( k \geq 2 \), then we obtain for some \( \gamma_3 > 0 \),

\[
I_1 \leq \gamma_3 \int_{0}^{1} dr \int_{0}^{\pi} \frac{r^{2k-2} \sin^{2k-3} \varphi \, d\theta}{[(r^2 + \delta^2)^2 + r^2 \cos^2 \varphi]^{k/2}} \\
\leq \gamma_3 \int_{0}^{1} dr \int_{0}^{\pi} \frac{\sin^{2k-3} \varphi \, d\varphi}{(r + \delta^2/r)^2 + \cos^2 \varphi} \\
\leq \gamma_3 \int_{0}^{1} dr \int_{-1}^{1} \frac{ds}{(r + \delta^2/r)^2 + s^2} \\
\leq \pi \gamma_3 \int_{0}^{1} \frac{dr}{\delta^2/r + r} \leq \frac{\pi \gamma_3}{2} \int_{\delta^2}^{1+\delta^2} \frac{d\lambda}{\lambda} \leq \gamma \log \frac{\gamma}{\delta}.
\]

This completes the proof of Lemma 4.

Define, for \( \delta > 0 \),

\[(\partial V)_\delta = \{ w : |w - \zeta| < \delta \text{ for some } \zeta \in \partial V \} .\]

Then we have
Lemma 5. There exists $\delta_0 > 0$ such that for any $z \in \overline{D}$ and any $\delta$ $(0 < \delta < \delta_0)$, it holds that
\[
I_2 = \int_{(S_z, \delta_0)|S_z, \delta} \sigma(d\zeta) \frac{\sigma(d\zeta)}{|F(\zeta, z)|^k} \leq \gamma \log \frac{\gamma}{\delta}.
\]

where $\gamma$ is independent of $z$ and $\delta$.

Proof. There exist positive constants $\delta_0$ and $\gamma_1$ such that
\[-\operatorname{Re} F(\zeta, z) \geq \rho(\zeta) - \rho(z) + \gamma_1|\zeta - z|^2 \text{ for } z \in (\partial V)_{\delta_0}, \zeta \in S_z, \delta_0.\]

We may assume that $z \in (\partial V)_{\delta_0} \cap \overline{D}$, $d\rho \neq 0$ on $(\partial V)_{\delta_0}$, and that
\[
\tilde{V} \cap S_z, \delta_0 = \{w \in S_z, \delta_0: w_{k+1} = \cdots = w_n = 0\}.
\]

We can find a new local coordinate system $t = (t_1, \ldots, t_{2n})$ by letting
\[
t_1 + it_2 = \rho(\zeta) - \rho(z) + i \operatorname{Im} F(\zeta, z),
t_{2j-1} + it_{2j} = \zeta_j - z_j \text{ for } j = k + 1, \ldots, n,
\]
and choosing $t_3, \ldots, t_{2k}$ suitably such that $t(z) = 0$. Then there exist positive constants $\gamma_2$, $\gamma_3$ and $\gamma_4$ such that
\[
\gamma_2|\zeta - z|^2 \leq |t|^2 \leq \gamma_3|\zeta - z|^2,
\]
\[
|F(\zeta, z)| \geq \gamma_4[(t_1 + |t|^2)^2 + t_{2k}^2]^{1/2}.
\]

Define $\varepsilon^2 = |\rho(z)| + |z_{k+1}|^2 + \cdots + |z_n|^2$. Taking account of the relation
\[
(S_z, \delta_0)|S_z, \delta) \cap \partial V \subset \{t: t_1 = -\rho(z), \gamma_2\delta^2 \leq |t|^2 \leq \gamma_3\delta_0^2, t_{2j-1} + it_{2j} = -z_j (j = k + 1, \ldots, n)\},
\]
we have, together with Lemma 4, for some $\gamma_5 > 0$,
\[
I_2 \leq \gamma_5 \int_{\{\gamma_2\delta^2 \leq |t|^2 \leq \gamma_3\delta_0^2\}} \frac{dt_2 \cdots dt_{2k}}{[(t_1^2 + \cdots + t_{2k}^2 + \varepsilon^2)^2 + t_{2k}^2]^{k/2}} \leq \gamma \log \frac{\gamma}{\delta},
\]
which completes the proof of Lemma 5.

We set
\[
K_V(\zeta, z) = N_V(\zeta, z)\sigma(d\zeta).
\]

Then we have the following.
Lemma 6. Let $0 < \alpha \leq 1$. Then there exists a positive constant $\delta_0$ such that for any $z \in \overline{D}$ and any $\delta$ ($0 < \delta < \delta_0$),

$$I_3 = \int_{S_z,\delta \cap \partial V} \frac{|\zeta - z|^\alpha|N_V(\zeta, z)|}{|g(\zeta, z)|^k} \sigma(dz) \leq \gamma \delta^\alpha \log \frac{\gamma}{\delta},$$

where $\gamma$ is independent of $\delta$ and $z$.

Proof. There exists a positive constant $\gamma_1$ such that

$$I_3 \leq \gamma_1 \int_{S_z,\delta \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(dz)}{|F(\zeta, z)|^k}.$$

Thus, by Lemma 5, we have

$$I_3 \leq \gamma_1 \sum_{i=0}^{\infty} \int_{(S_z,\delta^2-i) \cap (S_z,\delta^2-(i+1)) \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(dz)}{|F(\zeta, z)|^k}$$

$$= \sum_{i=0}^{\infty} \left( \frac{i + 1}{2i\alpha} \log 2 \right) \gamma_1 \delta^\alpha + \gamma_1 \delta^\alpha \log \left( \frac{\gamma_2}{\delta} \right) \left( \sum_{i=0}^{\infty} \frac{1}{2i\alpha} \right)$$

$$\leq \gamma \delta^\alpha \log \frac{\gamma}{\delta},$$

which completes the proof of Lemma 6.

Let $F$ be a closed subset of $C^n$. According to the definition of Stein [13], we define the Lipschitz space for $0 < \alpha \leq 1$ such that

$$\text{Lip}(\alpha, F) = \{ f : |f(x)| \leq M, |f(x) - f(y)| \leq M|x - y|^\alpha, x, y \in F \}.$$  

From the extension theorem (Stein [13], Theorem 3, p. 174), $f \in \text{Lip}(\alpha, F)$ can be regarded as an element of $\text{Lip}(\alpha, C^n)$. We shall prove the following theorem which was proved by Martinelli in the case when the kernel is Bochner-Martinelli kernel (cf. Martinelli [11], Dolbeault [4]).

Theorem 3. Let $f \in \text{Lip}(\alpha, \partial V)$. Then it holds that for any $z \in \partial V$,

$$\lim_{t \to z, t \in \overline{D} \cap V} \int_{\xi \in \partial V} (f(\xi) - f(z)) \Omega(\xi, t)$$

$$= \int_{\xi \in \partial V} (f(\xi) - f(z)) \Omega(\xi, z).$$

Proof. Since $f \in \text{Lip}(\alpha, \partial V)$, the integral of the right-hand side converges. In view of Lemma 6, for $\varepsilon > 0$, there exists $\delta > 0$ such
that for any \( w \in \overline{D} \), we have

\[
\int_{S_{w,\delta} \cap \partial V} \frac{|f(\zeta) - f(w)| |N_V(\zeta, w)|}{|g(\zeta, z)|^k} \sigma(d\zeta) < \varepsilon.
\]

We set

\[
T(\zeta, w) = \frac{(f(\zeta) - f(w))N_V(\zeta, w)}{g(\zeta, w)}.
\]

\( A_\delta = \{(\zeta, w) \in \partial V \times \overline{D}: |w - \zeta| \geq \delta\} \).

Since \( T(\zeta, w) \) is continuous on \( A_\delta \), there exists \( \rho \) (\( 0 < \rho < \delta \)) such that

\[
|T(\zeta, z) - T(\zeta, w)| < \varepsilon, \quad |f(z) - f(w)| < \varepsilon
\]

for \( |w - z| < \rho, \quad |\zeta - z| \geq 2\rho, \quad \zeta \in \partial V \).

By Proposition 1, there exists a constant \( \gamma_1 \) such that

\[
|H(t)| \leq \gamma_1 \quad \text{for} \quad t \in \overline{D} |\partial V|.
\]

Thus we have for \( |t - z| < \rho, \quad t \in \overline{D} |\partial V| \),

\[
\left| \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, t) - \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \right|
\]

\[
= \left| \int_{\partial V} T(\zeta, t)\sigma(d\zeta) + (f(t) - f(z))H(t) - \int_{\partial V} T(\zeta, z)\sigma(d\zeta) \right|
\]

\[
\leq \gamma_1 |f(t) - f(z)| + \int_{\partial V |S_{t, \delta}} |T(\zeta, t) - T(\zeta, z)|\sigma(d\zeta)
\]

\[
+ \int_{S_{t, \delta} \cap \partial V} |T(\zeta, z)|\sigma(d\zeta) + \int_{S_{t, \delta} \cap \partial V} |T(\zeta, t)|\sigma(d\zeta)
\]

\[
\leq \gamma_1 \varepsilon + \varepsilon \int_{\partial V |S_{t, \delta}} \sigma(d\zeta) + \varepsilon + \varepsilon \leq \gamma_2 \varepsilon.
\]

This completes the proof of Theorem 3.

Now we shall prove the following theorem which shows that any Lipschitz function on \( \partial V \) is the continuous boundary value of a function of \( A(D) \) by adding \( \int_{\partial V} (f(\zeta) - f(\cdot))K_V(\zeta, \cdot) \).

**THEOREM 4.** Let \( f \in \text{Lip}(\alpha, \partial V), \quad 0 < \alpha \leq 1 \). Define

\[
\tilde{f}(z) = f(z) + \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \quad \text{for} \quad z \in \partial V,
\]

\[
f(z) = f(z) + \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \quad \text{for} \quad z \in \overline{D}.
\]
and
\[ F(z) = \int_{\zeta \in \partial V} f(\zeta) \Omega(\zeta, z) \quad \text{for} \ z \in \overline{D} \setminus \partial V. \]

Then the holomorphic function \( F(z) \) can be extended continuously to \( \overline{D} \) and it has the boundary value \( F|_{\partial V} = \hat{f} \).

**Proof.** For \( z \in \partial V \), we have from Theorem 3,
\[
\lim_{t \to z, t \in \overline{D} \setminus \partial V} (F(t) - \hat{f}(z)) = \lim_{t \to z, t \in \overline{D} \setminus \partial V} \left( \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, t) \right)
- \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) = 0.
\]

Since \( F(z) \) is holomorphic in \( \overline{D} \setminus \partial V \), \( F \in A(D) \). This completes the proof of Theorem 4.

The boundary value \( \hat{f} \) is also represented by
\[
\hat{f}(z) = \frac{1}{2} f(z) + \text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z)
\]
in view of the following.

**Lemma 7.** Let \( f \in \text{Lip}(\alpha, \partial V) \) and \( z \in \partial V \). Then we have
\[
\text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z) = \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) + \frac{1}{2} f(z).
\]

**Proof.** We set, for \( \varepsilon > 0 \), \( M(\varepsilon) = \partial V \cap \{ \zeta : |g(\zeta, z)| > \delta \} \). In view of Lemma 6 and Theorem 2, we obtain
\[
\lim_{\varepsilon \to 0} \int_{M(\varepsilon)} f(\zeta) \Omega(\zeta, z) = \lim_{\varepsilon \to 0} \int_{M(\varepsilon)} (f(\zeta) - f(z)) \Omega(\zeta, z)
+ f(z) \lim_{\varepsilon \to 0} \int_{M(\varepsilon)} \Omega(\zeta, z)
= \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) + \frac{1}{2} f(z).
\]

This completes the proof of Lemma 7.

Now we are going to prove the following which is a main theorem in this paper.
Theorem 5. Let \( f \in \text{Lip}(\alpha, \partial V) \), \( 0 < \alpha \leq 1 \). If \( f \) satisfies for any \( z \in \partial V \),

\[
P.V. \int_{\partial V} f(\zeta)\Omega(\zeta, z) = \frac{1}{2}f(z),
\]

then there exists a function \( F \in A(D) \) such that \( F|_{\partial V} = f \).

Proof. We set

\[
F(z) = \int_{\partial V} f(\zeta)\Omega(\zeta, z) \quad \text{for} \quad z \in \overline{D}\setminus\partial V.
\]

Then \( F \) can be extended continuously to \( \overline{D} \) and satisfies, for \( z \in \partial V \),

\[
F(z) = f(z) + \int_{\partial V} (f(\zeta) - f(z))\Omega(\zeta, z) = \frac{1}{2}f(z) + \text{P.V.} \int_{\partial V} f(\zeta)\Omega(\zeta, z) = f(z),
\]

which completes the proof of Theorem 5.

References


Received May 31, 1991.

NAGASAKI UNIVERSITY
NAGASAKI 852, JAPAN
On the extension of Lipschitz functions from boundaries of subvarieties to strongly pseudoconvex domains
K. Adachi and Hiroshi Kajimoto

On a nonlinear equation related to the geometry of the diffeomorphism group
David Dai-Wai Bao, Jacques Lafontaine and Tudor S. Ratiu

Fixed points of boundary-preserving maps of surfaces
Robert F. Brown and Brian Sanderson

On orthomorphisms between von Neumann preduals and a problem of Araki
L. J. Bunce and John David Maitland Wright

Primitive subalgebras of complex Lie algebras. I. Primitive subalgebras of the classical complex Lie algebras
I. V. Chekalov

$L^n$ solutions of the stationary and nonstationary Navier-Stokes equations in $R^n$
Zhi Min Chen

Some applications of Bell’s theorem to weakly pseudoconvex domains
Xiao Jun Huang

On isotropic submanifolds and evolution of quasicaustics
Stanisław Janeczko

Currents, metrics and Moishezon manifolds
Shanyu Ji

Stationary surfaces in Minkowski spaces. I. A representation formula
Jiangfan Li

The dual pair $(U(1), U(1))$ over a $p$-adic field
Courtney Hughes Moen

Any knot complement covers at most one knot complement
Shicheng Wang and Ying Qing Wu