$L^n$ SOLUTIONS OF THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

ZHI MIN CHEN
It is shown that the Navier-Stokes equations in the whole space \( \mathbb{R}^n \) \((n \geq 3)\) admit a unique small stationary solution which may be formed as a limit of a nonstationary solution as \( t \to \infty \) in \( L^n \)-norms.

\[ (0.1) \quad \|v(t) - w\|_n + t^{1/2}\|Dv(t) - Dw\|_n + t^{1/2}\|v(t) - w\|_\infty \to 0 \]

provided that \( w \) and \( v \) are, respectively, the solutions to the stationary Navier-Stokes equations

\[ (0.2) \quad -\Delta w + (w \cdot D)w + d\bar{p} = f, \quad D \cdot w = 0 \quad \text{in} \quad \mathbb{R}^n \]

and the nonstationary Navier-Stokes equations

\[ (0.3) \quad v_t - \Delta v + (v \cdot D)v + D\bar{p} = f, \quad D \cdot v = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \]

\[ v(0) = v_0 \quad \text{in} \quad \mathbb{R}^n. \]

Here and in what follows, \( n \geq 3 \) denotes the space dimension, \( \bar{p} \) and \( \bar{p} \) represent the pressures associated with \( w \) and \( v \), respectively, \( D = \) the gradient, \( f = f(x) \) is a prescribed function, the dot \( \cdot \) denotes the scalar product in \( \mathbb{R}^n \), and \( \| \cdot \|_r \) denotes the norm of the Lebesgue space \( L^r = L^r(\mathbb{R}^n; \mathbb{R}^n) \).

The purpose of the paper is to show that (0.2) and (0.3) admit small regular solutions \( w \) and \( v(t) \) in \( L^n \), respectively, such that (0.1) is valid. The problem above is, as usual, said to be a stability problem.
for $w$, which has been studied by Kozono and Ozawa [13] in the case of bounded domains. From our viewpoint, the global existence results of Kato [12] may be regarded as the stability theorems around the rest flow $w \equiv 0$.

In this paper we shall use the following spaces.

$C_0^\infty = \text{the set of compactly supported solenoidal } u \in C^\infty(R^n; R^n)$,

$J^r = \text{the completion of } C_0^\infty \text{ in } L^r \text{ for } 1 < r < \infty$,

$W^{k,r} = \text{the Sobolev space } W^{k,r}(R^n; R^n) \text{ for } 1 < r < \infty \text{ and } k = 1, 2$,

$\overline{W}^{1,r} = \{u \in L^{nr/(n-r)}; Du \in L^r(R^n; R^n^3)\} \text{ for } 1 < r < n$,

$\overline{W}^{2,r} = \{u \in W^{1,nr/(n-r)}; D^2u \in L^r(R^n; R^n^3)\} \text{ for } 1 < r < n/2$,

where $D^2 = \text{the Hessian matrix } [D_iD_j]_{n \times n}$ with $D_k = \partial/\partial x_k$. Moreover, we denote by $P$ the linear bounded projection from $L^r$ onto $J^r$ for $1 < r < \infty$ (cf. [15] for details), by $A$ the Stokes operator $-P\Delta$ associated with the domain $W^{2,r} \cap J^r$ for $1 < r < \infty$, by $(\cdot, \cdot)$ the duality pairing between $L^r$ and $(L^r)^*$ for $1 \leq r < \infty$, and we set

$$\|u\|_{-1,r} = \sup\{|(u, v)|; v \in C_0^\infty, \|Du\|_{r/(r-1)} = 1\} \text{ for } 1 < r < \infty.$$  

Our main results read as follows.

**Theorem 0.1.** For $n \geq 3$ there is a small $0 < d < 1$ such that (0.1) admits a unique solution

$$w \in J^n \cap \overline{W}^{1,2n/3} \cap \overline{W}^{1,2n/5} \text{ with } \|Dw\|_{n/2} \leq d$$

satisfying

$$\|Dw\|_{n/2} + \|w\|_n \leq C\|f\|_{-1,n/2},$$

$$\|Dw\|_{2n/5} + \|Dw\|_{2n/3} + \|w\|_{2n} + \|w\|_{2n/3} \leq C(\|f\|_{-1,2n/5} + \|f\|_{-1,2n/3})$$

with $C$ independent of $f$ and $w$, provided that

$$f \in C_0^\infty \text{ and } \|f\|_{-1,n/2} \leq d^2.$$  

**Theorem 0.2.** Let $n \geq 3, f \in C_0^\infty, v_0 \in J^n$, and let $\|v_0\|_n$ and $\|f\|_{-1,2n/5} + \|f\|_{-1,2n/3}$ be sufficiently small. Then (0.3) admits a unique solution

$$v \in BC([0, \infty); J^n) \text{ and } t^{1/2}D(v(t) - w) \in BC([0, \infty); L^n(R^n; R^n^2))$$

where $D(v(t) - w)$ denotes the difference quotient.
such that (0.1) is valid, where \( w \) is the solution of (0.2) from Theorem 0.1 and \( BC \) denotes the class of bounded and continuous functions.

Since there is no boundary to worry about in the whole space, our proof largely depends on the fact that \( P \) commutes with \( D \), and also based on the theory of analytic semigroups in various \( L' \) spaces. Such an approach is developed from Fujita and Kato [5] and Kato [12].

In §1 we prove Theorem 0.1. In §2 we obtain resolvent estimates for the perturbed operator \( Au + P(u \cdot D)w + P(w \cdot D)u \) and therefore deduce decay estimates for the analytic semigroups generated by the perturbed operator. Theorem 0.2 is proved in §3.

1. Proof of Theorem 0.1. From the Sobolev inequality

\[
C^{-1}\|u\|_{\frac{nr}{n-2r}} \leq \|Du\|_{\frac{nr}{n-2r}} \leq C\|D^2u\|_r \quad \text{for } 1 < r < n/2,
\]

the Calderon-Zygmund inequality (cf. [7])

\[
\|D^2u\|_r \leq C\|\Delta u\|_r \quad \text{for } 1 < r < \infty,
\]

the density of \( \{Au; u \in C_0^\infty\} \) in \( J' \) for \( 1 < r < n/2 \), and the fact that \( P \) commutes with \( \Delta \), it follows that the Stokes operator \( A \) can be extended to a bounded and invertible operator from \( J^{n/(n-2r)} \cap \widehat{W}^{2,r} \) onto \( J' \) for \( 1 < r < n/2 \). Consequently, we set the operator

\[
T: J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3} \rightarrow \widehat{W}^{2,r} \quad \text{for } n/3 < r < n/2
\]
such that

\[
Tw = T_fw = A^{-1}(f - P(w \cdot D)w).
\]

It is easy to see that to seek solutions of (0.2) means to seek fixed points of \( T \).

Let \( 2n/5 \leq r \leq 2n/3 \), \( w \in J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3} \), \( v \in C_0^\infty \). Then by the divergence condition \( D \cdot w = 0 \), we have

\[
(DTw, Dv) = (f, v) - ((w \cdot D)w, v)
\]

\[
= (f, v) + (w, (w \cdot D)v)
\]

\[
\leq (f, v) + \|w\|_n\|w\|_{\frac{nr}{n-2r}}\|Dv\|_{\frac{r}{r-1}}.
\]

Combining this with the inequality (cf. [17, 18])

\[
\|DTw\|_r \leq C\sup\{(DTw, Dv); v \in C_0^\infty, \|Dv\|_{\frac{r}{r-1}} = 1\}
\]

with \( C = C(n) \), we have, by (1.1),

\[
\|DTw\|_r \leq C(n)(\|f\|_{-1,r} + \|Dw\|_{n/2}\|Dw\|_r),
\]
and, similarly, for $u$, $w \in J^n \cap \tilde{W}^{1, 2n/5} \cap \tilde{W}^{1, 2n/3}$
\[ \|DTw - DTu\|_r \leq C(n)(\|Dw\|_{n/2} + \|Du\|_{n/2})\|Dw - Du\|_r. \]
Consequently, there is a small positive $d$ such that $T$ is a contraction mapping from the complete metric space
\[ \{w \in J^n \cap \tilde{W}^{1, 2n/5} \cap \tilde{W}^{1, 2n/3}; \|Dw\|_{n/2} \leq d \} \]
into itself provided that $f \in C_0^\infty$ with $\|f\|_{-1, n/2} < d^2$. We thus obtain the desired assertion by making use of the contraction mapping principle and (1.1). The proof is complete.

2. $L^p - L^q$ estimates. In the remainder of the paper we denote by $w$ the solution of (0.2) given in Theorem 0.1, and by $C$ the various constants which are always independent of the quantities $u$, $v$, $w$, $f$, $a$, $t$, and $z$. Moreover we set
\[ S = \{z \in \mathbb{C}; -3\pi/4 < \arg z < 3\pi/4\}, \]
\[ Lu = Au + Bu; \quad Bu = P(u \cdot D)w + P(w \cdot D)u, \]
\[ L^*u = Au + B^*u; \quad B^*u = -P(w \cdot D)u + \sum_{i=1}^n Pu^i Dw^i \]
for $u = (u^1, \ldots, u^n)$ and $w = (w^1, \ldots, w^n)$.

In arriving at $L^p - L^q$ estimates, we begin with the resolvent estimates for $L$ and $L^*$.

**Lemma 2.1.** Let $z \in S$ and $u \in C_0^\infty$. Then we have
\begin{align*}
& (2.1) \quad |z| \|(L + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < \infty, \\
& (2.2) \quad |z| \|(L^* + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < \infty, \\
& (2.3) \quad |z|^{1/2}\|D(L + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < n, \\
& (2.4) \quad |z|^{1/2}\|D(L^* + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < \infty, 
\end{align*}
provided that $\|Dw\|_{n/2}$ is sufficiently small;
\begin{align*}
& (2.5) \quad |z|^{3/4}\|(L + z)^{-1}u\|_\infty \leq C\|u\|_{2n}, \\
& (2.6) \quad |z|^{1/2}\|D(L + z)^{-1}u\|_n \leq C(\|u\|_n + |z|^{-1/4}\|u\|_{2n}),
\end{align*}
provided that $\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}$ is sufficiently small.

**Proof.** Let us recall the well-known resolvent estimates for the Stokes operator (cf. [15])
\begin{align*}
& (2.7) \quad |z| \|(A + z)^{-1}u\|_r + |z|^{1/2}\|D(A + z)^{-1}u\|_r \\
& \quad \quad \quad \quad \quad \quad + \|D^2(A + z)^{-1}u\|_r \leq C\|u\|_r
\end{align*}
for $z \in S$, $1 < r < \infty$ and $u \in J'$, and the Gagliardo-Nirenberg inequality (cf. [4])

\begin{equation}
\|u\|_q \leq C\|u\|_r^{1-h}\|Du\|_p^h
\end{equation}

for $1 < r$, $p \leq q \leq \infty$, $0 \leq h < 1$, 
$-n/q = h(1 - n/p) - (1 - h)n/r$, 
$u \in C_0$. Let us suppose $z \in S$ and $u \in J' \cap W^{1,r}$ for $1 < r < \infty$.

**Step 1.** We prove (2.1) and (2.2). From (2.7), (1.1), the Hölder inequality and the boundedness of $P$ in $L^r$-spaces it follows that for $1 < r < n/2$, $p = nr/(n - r)$ and $q = nr/(n - 2r),$

\[
\|B(A + z)^{-1}u\|_r \leq C\|w\|_n\|D(A + z)^{-1}u\|_p
+ C\|Dw\|_{n/2}\|(A + z)^{-1}u\|_q
\leq C\|Dw\|_{n/2}\|D^2(A + z)^{-1}u\|_r
\leq C\|Dw\|_{n/2}\|u\|_r
\leq (1/2)\|u\|_r,
\]

by setting $C\|Dw\|_{n/2} < 1/2$.

This is together with (2.7) and the identity

\[
L + z = (1 + B(A + z)^{-1})(A + z)
\]

implies

\[
|z|\|(L + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < n/2.
\]

Similarly, we have

\[
|z|\|(L^* + z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < n/2.
\]

This yields for $n < r < \infty$, $v \in L^{r'}$ with $r' = r/(r - 1),$

\[
((L + z)^{-1}u, v) = (u, (L^* + z)^{-1}Pv) \leq C|z|^{-1}\|u\|_r \|v\|_{r'}
\]

and hence the validity of (2.1) with $n < r < \infty$. Thus (2.1) with $n/2 \leq r \leq n$ follows immediately from the Marcinkiewicz interpolation theorem (cf. [7]). (2.2) is verified in the same way.

**Step 2.** We prove (2.3). Observing that $1 < r < n$ and applying the condition $D \cdot u = D \cdot w = 0$ and the fact that $D$ commutes with $P$ yields

\begin{equation}
(A + z)^{-1}Bu = \sum_{i=1}^{n} D_i(A + z)^{-1}P(u^i w + w^i u),
\end{equation}

we have, by (2.7) and (1.1),

\[
\|(A + z)^{-1}Bu\|_r \leq C|z|^{-1/2}\|w\|_n\|u\|_{nr/(n-r)}
\leq C\|Dw\|_{n/2}\|Du\|_r|z|^{-1/2}
\leq 2^{-1}|z|^{-1/2}\|Du\|_r,
\]

by setting $C\|Dw\|_{n/2} < 1/2$.
and
\[ \|D(A + z)^{-1}Bu\|_r \leq C\|w\|_n\|u\|_{nr/(n-r)} \leq C\|Dw\|_{n/2}\|Du\|_r \leq (1/2)\|Du\|_r, \text{ by setting } C\|Dw\|_{n/2} < 1/2. \]

Consequently, we have
\[ \|(A + z)^{-1}B\|_r \leq 2^{-k}|z|^{-1/2}\|Du\|_r, \text{ for integer } k > 0, \]
and so
\[ D(L + z)^{-1}u = \sum_{k=0}^{\infty} D((A + z)^{-1}B)^k(A + z)^{-1}u \text{ in } L^r. \]

Applying (2.10) to the preceding identity repeatedly and using (2.7), we have
\[ \|D(L + z)^{-1}u\|_r \leq 2\|D(A + z)^{-1}u\|_r \leq C|z|^{-1/2}\|u\|_r \]
as required.

**Step 3.** We prove (2.4). Observing that
\[ (D_i(L^* + z)^{-1}u, v) = -(u, (L + z)^{-1}D_iPv) \leq \|u\|_r\|(L + z)^{-1}D_iPv\|_{r'}, \]
for \(i = 1, \ldots, n, 1 < r < \infty, r' = r/(r-1)\) and \(v \in W^{1,r'}\), we need only to show the estimate
\[ \|(L + z)^{-1}Du\|_r \leq C|z|^{-1/2}\|u\|_r, \text{ for } 1 < r < \infty. \]

Indeed, taking (2.9), (1.1) and (2.7) into account, we have for \(n \leq r < \infty\),
\[ \|(A + z)^{-1}Bu\|_r \leq C\sum_{i=1}^{n}\|DD_i(A + z)^{-1}P(uw^i + wu^i)\|_{nr/(n+r)} \leq C\|Dw\|_{n/2}\|u\|_r \leq (1/2)\|u\|_r, \]
by setting \(C\|Dw\|_{n/2} < 1/2\), and hence for \(n \leq r < \infty\),
\[ \|(L + z)^{-1}Du\|_r = \|(1 + (A + z)^{-1}B)^{-1}D(A + z)^{-1}u\|_r \leq 2\|D(A + z)^{-1}u\|_r \leq C|z|^{-1/2}\|u\|_r. \]
which arrives at (2.12) for $n \leq r < \infty$. Moreover (2.12) with $1 < r < n$ is verified as follows:

$$
\|(L + z)^{-1}Du\|_r = \|(1 + (A + z)^{-1}B)^{-1}(A + z)^{-1}u\|_r \\
\leq \|D(A + z)^{-1}u\|_r + |z|^{-1/2}\|D^2(A + z)^{-1}u\|_r \\
\leq C|z|^{-1/2}\|u\|_r,
$$

where we have used (2.11) and (2.7).

**Step 4.** We prove (2.5). By (2.8) and (2.7), we obtain

$$
(A + z)^{-1}u|_\infty \leq C|(A + z)^{-1}u|_{2n}^{1/2}\|D(A + z)^{-1}u\|_{2n}^{1/2} \\
\leq C|z|^{-3/4}\|u\|_{2n},
$$

and, by (2.7), (2.8), (1.1) and (2.9),

$$
(A + z)^{-1}Bu|_\infty \leq C|(A + z)^{-1}Bu|_{2n}^{1/2}\|D(A + z)^{-1}Bu\|_{2n}^{1/2} \\
\leq C\|D(A + z)^{-1}Bu\|_{2n/3}^{1/2}\|D(A + z)^{-1}Bu\|_{2n}^{1/2} \\
\leq C\sum_{i=1}^n\|u^iw + w^iu\|_{2n/3}^{1/2}\|u^iw + w^iu\|_{2n}^{1/2} \\
\leq C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_{\infty} \\
\leq (1/2)\|u\|_{\infty}, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.
$$

We thus obtain

$$
\|(L + z)^{-1}u\|_\infty = \|(1 + (A + z)^{-1}B)^{-1}(A + z)^{-1}u\|_\infty \\
\leq 2\|(A + z)^{-1}u\|_{\infty} \leq C|z|^{-3/4}\|u\|_{2n}
$$

and hence the validity of (2.5).

**Step 5.** We prove (2.6). By (1.1), (2.9) and (2.7),

$$
\|(A + z)^{-1}Bu\|_n \leq C\|D(A + z)^{-1}Bu\|_{n/2} \\
\leq C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}\|u\|_n \\
\leq (1/2)\|u\|_n, \quad \text{by setting } C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2} < 1/2
$$

and, by (2.9), (2.7) and (1.1),

$$
\|D(A + z)^{-1}Bu\|_n \leq C\|w\|_n\|u\|_\infty \\
\leq C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_\infty \\
\leq (1/2)\|u\|_\infty, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.
$$
Hence, it is easy to see that

\[
\|D(L + z)^{-1}u\|_n \leq \sum_{k=0}^{\infty} \|D((A + z)^{-1}B)^k(A + z)^{-1}u\|_n
\]

\[
\leq \|D(A + z)^{-1}u\|_n + \sum_{k=0}^{\infty} \|(A + z)^{-1}B\|^k(A + z)^{-1}u\|_{\infty}
\]

\[
\leq \|D(A + z)^{-1}u\|_n + \|(A + z)^{-1}u\|_{\infty}, \text{ by (2.14)},
\]

\[
\leq C(|z|^{1/2}\|u\|_n + |z|^{-3/4}\|u\|_n), \text{ by (2.7), (2.13)}.
\]

The proof is complete.

As an immediate consequence of (2.1) and (2.2), we conclude that

\[L\quad \text{and} \quad L^*\]

generate strongly continuous analytic semigroups \(e^{-tL}\) and \(e^{-tL^*}\) in \(J^r\) with \(1 < r < \infty\), respectively, provided \(\|Dw\|_{n/2}\) is sufficiently small. What is more, we can now proceed to the proof of the following \(L^p - L^r\) estimates.

**Theorem 2.1.** Let \(t > 0, \, 1 < q \leq n, \, v \in J^q\) and \(u \in C_0^\infty\). Then we have

\[
(2.15) \quad \|e^{-tL}u\|_p \leq Ct^{-(n/r-n/p)/2}\|u\|_r \quad \text{for} \quad 1 < r \leq p < \infty,
\]

provided that \(\|Dw\|_{n/2}\) is sufficiently small;

\[
(2.16) \quad \|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n \leq Ct^{-n/2r}\|u\|_r \quad \text{for} \quad 1 < r \leq n,
\]

\[
(2.17) \quad t^{n/2d}(t^{-1/2}\|e^{-tL}v\|_n + \|e^{-tL}v\|_\infty + \|De^{-tL}v\|_n) \to 0
\]

as \(t \to \infty\),

provided that \(\|w\|_{1/2}^{1/n}\|w\|_{1/2}^{1/n/3}\) is sufficiently small.

**Proof.** By making use of the semigroup property of \(e^{-tL}\), Lemma 2.1, and the Dunford integral (cf. [8]) via a standard calculation, we have

\[
(2.18) \quad \|e^{-tL^*}u\|_r + t^{1/2}\|De^{-tL^*}u\|_r \leq C\|u\|_r \quad \text{for} \quad 1 < r < \infty,
\]

\[
(2.19) \quad \|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n
\]

\[
\leq Ct^{-1/2}\|e^{-(t/2)L}u\|_n + Ct^{-1/4}\|e^{-(t/2)L}u\|_{2n}
\]

under the assumptions of Theorem 2.1.
It follows from (2.8) and (2.18) that
\[ \|e^{-tL^*} u\|_p \leq C\|e^{-tL^*} u\|_r^{1-n/r+n/p} \|De^{-tL^*} u\|_r^{n/r-n/p} \leq Ct^{-\frac{(n/r-n/p)}{2}}\|u\|_r \]
for \( 1 < r \leq p < \infty \) and \( n/r - n/p < 1 \). Combining this with the semigroup property of \( e^{-tL^*} \), we have for \( 1 < r \leq p < \infty \) and \( a \in L^p(p^{-1}) \),
\[ \|e^{-tL^*} Pa\|_{r/(r-1)} \leq Ct^{-\frac{(n/r-n/p)}{2}}\|a\|_{p/(p-1)} , \]
and hence
\[ (e^{-tL}u, a) = (u, e^{-tL^*} Pa) \leq Ct^{-\frac{(n/r-n/p)}{2}}\|u\|_r\|a\|_{p/(p-1)} \]
This gives (2.15). (2.16) follows from (2.19) and (2.15).

To prove (2.17), we note for \( a \in J^q \cap J^r \) with \( 1 < r < q \),
\[ t^{n/2q}(t^{-1/2}\|e^{-tL}v\|_n + \|e^{-tL}v\|_{\infty} + \|De^{-tL}v\|_n) \leq C\|v - a\|_q + Ct^{-\frac{(n/r-n/q)}{2}}\|a\|_r , \]
where we have used (2.15) and (2.16). Hence the density of \( J^q \cap J^r \) in \( J^q \) implies (2.17). The proof is complete.

3. Proof of Theorem 0.2. From Theorem 0.1 we can suppose that \( \|Dw\|_{n/2} + \|w\|_{2n/3}^{1/2} \|w\|_{2n/3}^{1/2} \) is small such that (2.15)–(2.17) holds.

By using the projection \( P \) to (0.2)–(0.3), and setting \( u(t) = v(t) - w \)
and \( a = v_0 - w \), then (0.2)–(0.3) leads to the evolution equation
\[ \frac{d}{dt}u + Lu = -P(u \cdot D)u \quad (t > 0) , \quad u(0) = a \]
in \( J^n \). Hence, our goal now remains to show that (3.1) has a unique solution \( u \) belonging to the space
\[ U \equiv \{ u \in BC([0, \infty) ; J^n) ; t^{1/2}Du(t) \in BC([0, \infty) ; L^n(R^n ; R^n^3 )) \} \]
such that
\[ Hu(t) \equiv \|u(t)\|_n + t^{1/2}\|u(t)\|_{\infty} + t^{1/2}\|Du(t)\|_{\infty} \to 0 \quad \text{as} \quad t \to \infty \]
provided that \( a \in J^n \) with \( \|a\|_n \) small enough.

Let us impose the following notation.
\[ \|u\| = \sup_{t>0} Hu(t) , \]
\[ W = \{ u \in U ; \|u\| < \infty , \quad Hu(t) \to 0 \quad \text{as} \quad t \to \infty \} , \]
\[ Mu(t) = u_0(t) - \int_0^t e^{-(t-s)L}P(u \cdot D)u(s) \, ds ; \quad u_0(t) = e^{-tL}a . \]
Observing that for $u \in C_0^\infty$,
\[ t^{1/2}(||e^{-tL}P(u \cdot D)u||_\infty + ||De^{-tL}P(u \cdot D)u||_n) + ||e^{-tL}P(u \cdot D)u||_n \leq C t^{-1/4}||u \cdot D)u||_{2n/3}, \] by (2.15)-(2.16),
\[ \leq C t^{-1/4}||u||_{2n}||Du||_n, \]
and
\[ ||u||_{2n} \leq ||u||_n^{1/2}||u||_\infty^{1/2}, \]
we have for $u \in W$,
\[ (3.2) \quad ||Mu(t)||_n + t^{1/2}||Mu(t)||_\infty + t^{1/2}||DMu(t)||_n \]
\[ \leq C||a||_n + C \int_0^t (t-s)^{-1/4}||u(s)||_{2n}||Du(s)||_n \, ds \]
\[ + C \int_0^t t^{1/2}(t-s)^{-3/4}||u(s)||_{2n}||Du(s)||_n \, ds, \]
by (2.15)-(2.16),
\[ \leq C||a||_n + C||u||^2, \]
and what is more, by using (2.17) and the property
\[ Hu_0(t) + Hu(t) \to 0 \quad \text{as } t \to \infty \]
via a calculation similar to (3.2), we have
\[ H(Mu)(t) \to 0 \quad \text{as } t \to \infty. \]
Moreover, by a standard calculation from [19] or [12], we have $Mu \in U$ for $u \in W$, and so $M: W \to W$ and
\[ ||Mu|| \leq C||a||_n + C||u||^2. \]
Additionally, similar to (3.2), we obtain for $u_1, u_2 \in W$,
\[ ||Mu_1 - Mu_2|| \leq C(||u_1|| + ||u_2||)||u_1 - u_2||. \]
From contraction mapping principle it follows that $M$ has a fixed point $u$ in $W$ provided $||a||_n$ is sufficiently small. As in [12, 5], we find that the fixed point $u$ is the desired solution which exists uniquely in $U$. The proof is complete.

REFERENCES


Received July 11, 1991 and in revised form March 13, 1992.

**TIANJIN UNIVERSITY**
**TIANJIN 3000 72**
**PEOPLE'S REPUBLIC OF CHINA**
The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 Mathematics Subject Classification scheme which can be found in the December index volumes of Mathematical Reviews. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Julie Speckart, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author’s University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 75 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $200.00 a year (10 issues). Special rate: $100.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

This publication was typeset using \LaTeX, the American Mathematical Society's \TeX macro system.

Copyright © 1993 by Pacific Journal of Mathematics
On the extension of Lipschitz functions from boundaries of subvarieties to strongly pseudoconvex domains

K. ADACHI and HIROSHI KAJIMOTO

On a nonlinear equation related to the geometry of the diffeomorphism group

DAVID DAI-WAI BAO, JACQUES LAFONTAINE and TUDOR S. RATIU

Fixed points of boundary-preserving maps of surfaces

ROBERT F. BROWN and BRIAN SANDERSON

On orthomorphisms between von Neumann preduals and a problem of Araki

L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT

Primitve subalgebras of complex Lie algebras. I. Primitive subalgebras of the classical complex Lie algebras

I. V. CHEKALOV

$L^n$ solutions of the stationary and nonstationary Navier-Stokes equations in $R^n$

ZHIMIN CHEN

Some applications of Bell’s theorem to weakly pseudoconvex domains

XIAO JUN HUANG

On isotropic submanifolds and evolution of quasicaustics

STANISŁAW JANECZKO

Currents, metrics and Moishezon manifolds

SHANYU JI

Stationary surfaces in Minkowski spaces. I. A representation formula

JIANGFAN LI

The dual pair $(U(1), U(1))$ over a $p$-adic field

COURTNEY HUGHES MOEN

Any knot complement covers at most one knot complement

SHICHEM WANG and YING QING WU