$L^n$ SOLUTIONS OF THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS IN $R^n$

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It is shown that the Navier-Stokes equations in the whole space $\mathbb{R}^n$ ($n \geq 3$) admit a unique small stationary solution which may be formed as a limit of a nonstationary solution as $t \to \infty$ in $L^n$-norms.

0. Introduction. As is well known, the existence of solutions to the exterior stationary Navier-Stokes equations was studied by Finn [2, 3], and small solutions from Finn [2, 3] may be formed as limits of nonstationary solutions as time $t \to \infty$ in local or global $L^2$-norms (cf. Heywood [9, 10], Galdi and Rionero [6], Miyakawa and Sohr [16], Borchers and Miyakawa [1]) and in the norms of other function spaces (cf. Heywood [11], Musuda [14]). However, it is still unknown even in the case of whole spaces whether or not

\begin{align}
\|v(t) - w\|_n + t^{1/2}\|Dv(t) - Dw\|_n + t^{1/2}\|v(t) - w\|_\infty \to 0 \quad \text{as } t \to \infty,
\end{align}

provided that $w$ and $v$ are, respectively, the solutions to the stationary Navier-Stokes equations

\begin{align}
-\Delta w + (w \cdot D)w + d\bar{p} = f, \quad D \cdot w = 0 \quad \text{in } \mathbb{R}^n
\end{align}

and the nonstationary Navier-Stokes equations

\begin{align}
v_t - \Delta v + (v \cdot D)v + D\bar{p} = f, \quad D \cdot v = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad v(0) = v_0 \quad \text{in } \mathbb{R}^n.
\end{align}

Here and in what follows, $n \geq 3$ denotes the space dimension, $\bar{p}$ and $\bar{p}$ represent the pressures associated with $w$ and $v$, respectively, $D$ is the gradient, $f = f(x)$ is a prescribed function, the dot $\cdot$ denotes the scalar product in $\mathbb{R}^n$, and $\| \cdot \|_r$ denotes the norm of the Lebesgue space $L^r = L^r(\mathbb{R}^n; \mathbb{R}^n)$.

The purpose of the paper is to show that (0.2) and (0.3) admit small regular solutions $w$ and $v(t)$ in $L^n$, respectively, such that (0.1) is valid. The problem above is, as usual, said to be a stability problem.
for \( w \), which has been studied by Kozono and Ozawa [13] in the case of bounded domains. From our view point, the global existence results of Kato [12] may be regarded as the stability theorems around the rest flow \( w = 0 \).

In this paper we shall use the following spaces.

\[ C_0^\infty = \text{the set of compactly supported solenoidal } u \in C^\infty(R^n; \mathbb{R}^n), \]

\[ J' = \text{the completion of } C_0^\infty \text{ in } L' \text{ for } 1 < r < \infty, \]

\[ W^{k,r} = \text{the Sobolev space } W^{k,r}(R^n; \mathbb{R}^n) \text{ for } 1 < r < \infty \text{ and } k = 1, 2, \]

\[ \tilde{W}^{1,r} = \{ u \in L^{nr/(n-r)}; Du \in L^r(R^n; \mathbb{R}^{n^2}) \} \text{ for } 1 < r < n, \]

\[ \tilde{W}^{2,r} = \{ u \in W^{1, nr/(n-r)}; D^2u \in L^r(R^n; \mathbb{R}^{n^3}) \} \text{ for } 1 < r < n/2, \]

where \( D^2 = \text{the Hessian matrix } [D_iD_j]_{n \times n} \text{ with } D_k = \partial/\partial x_k \). Moreover, we denote by \( P \) the linear bounded projection from \( L^r \) onto \( J' \) for \( 1 < r < \infty \) (cf. [15] for details), by \( A \) the Stokes operator \(-P\Delta\) associated with the domain \( W^{2,r} \cap J' \) for \( 1 < r < \infty \), by \((\cdot, \cdot)\) the duality pairing between \( L^r \) and \((L^r)^*\) for \( 1 \leq r < \infty \), and we set

\[ \|u\|_{-1,r} = \sup\{|(u, v)|; v \in C_0^\infty, \|Du\|_{r/(r-1)} = 1\} \text{ for } 1 < r < \infty. \]

Our main results read as follows.

**Theorem 0.1.** For \( n \geq 3 \) there is a small \( 0 < d < 1 \) such that (0.1) admits a unique solution

\[ w \in J^n \cap \tilde{W}^{1,2n/3} \cap \tilde{W}^{1,2n/5} \text{ with } \|Dw\|_{n/2} \leq d \]

satisfying

\[
\begin{align*}
\|Dw\|_{n/2} + \|w\|_{n} & \leq C\|f\|_{-1,n/2}, \\
\|Dw\|_{2n/5} + \|Dw\|_{2n/3} + \|w\|_{2n} + \|w\|_{2n/3} & \leq C(\|f\|_{-1,2n/5} + \|f\|_{-1,2n/3})
\end{align*}
\]

with \( C \) independent of \( f \) and \( w \), provided that

\[ f \in C_0^\infty \text{ and } \|f\|_{-1,n/2} \leq d^2. \]

**Theorem 0.2.** Let \( n \geq 3, f \in C_0^\infty, v_0 \in J^n \), and let \( \|v_0\|_{n} \) and \( \|f\|_{-1,2n/5} + \|f\|_{-1,2n/3} \) be sufficiently small. Then (0.3) admits a unique solution

\[ v \in BC([0, \infty); J^n) \text{ and } t^{1/2}D(v(t) - w) \in BC([0, \infty); L^n(R^n; R^{n^2})) \]
such that (0.1) is valid, where \( w \) is the solution of (0.2) from Theorem 0.1 and \( \text{BC} \) denotes the class of bounded and continuous functions.

Since there is no boundary to worry about in the whole space, our proof largely depends on the fact that \( P \) commutes with \( D \), and also based on the theory of analytic semigroups in various \( L^r \) spaces. Such an approach is developed from Fujita and Kato [5] and Kato [12].

In §1 we prove Theorem 0.1. In §2 we obtain resolvent estimates for the perturbed operator \( Au + P(u \cdot D)w + P(w \cdot D)u \) and therefore deduce decay estimates for the analytic semigroups generated by the perturbed operator. Theorem 0.2 is proved in §3.

1. Proof of Theorem 0.1. From the Sobolev inequality
\[
(1.1) \quad C^{-1}\|u\|_{nr/(n-2r)} \leq \|Du\|_{nr/(n-r)} \leq C\|D^2u\|_r \quad \text{for } 1 < r < n/2,
\]
the Calderon-Zygmund inequality (cf. [7])
\[
\|D^2u\|_r \leq C\|\Delta u\|_r \quad \text{for } 1 < r < \infty,
\]
the density of \( \{Au; u \in C_0^\infty\} \) in \( J^r \) for \( 1 < r < n/2 \), and the fact that \( P \) commutes with \( \Delta \), it follows that the Stokes operator \( A \) can be extended to a bounded and invertible operator from \( J^{nr/(n-2r)} \cap \hat{W}^{2,r} \) onto \( J^r \) for \( 1 < r < n/2 \). Consequently, we set the operator
\[
T: J^n \cap \hat{W}^{1,2n/5} \cap \hat{W}^{1,2n/3} \rightarrow \hat{W}^{2,r} \quad \text{for } n/3 < r < n/2
\]
such that
\[
Tw = Tfw = A^{-1}(f - P(w \cdot D)w).
\]
It is easy to see that to seek solutions of (0.2) means to seek fixed points of \( T \).

Let \( 2n/5 \leq r \leq 2n/3 \), \( w \in J^n \cap \hat{W}^{1,2n/5} \cap \hat{W}^{1,2n/3}, \) \( v \in C_0^\infty \). Then by the divergence condition \( D \cdot w = 0 \), we have
\[
(DTw, Dv) = (f, v) - ((w \cdot D)w, v) = (f, v) + (w, (w \cdot D)v) \leq (f, v) + \|w\|_n \|w\|_{nr/(n-r)} \|Dv\|_r/(r-1).
\]
Combining this with the inequality (cf. [17, 18])
\[
\|DTw\|_r \leq C \sup\{\|(DTw, Dv)\); v \in C_0^\infty, \|Dv\|_r/(r-1) = 1\}
\]
with \( C = C(n) \), we have, by (1.1),
\[
\|DTw\|_r \leq C(n)(\|f\|_{-1,r} + \|Dw\|_{n/2} \|Dw\|_r),
\]
and, similarly, for $u, w \in \mathcal{H}^n \cap \mathcal{W}^{1, 2n/5} \cap \mathcal{W}^{1, 2n/3}$
\[
\|DTw - DTu\|_r \leq C(n)(\|Dw\|_{n/2} + \|Du\|_{n/2})\|Dw - Du\|_r.
\]
Consequently, there is a small positive $d$ such that $T$ is a contraction mapping from the complete metric space
\[
\{w \in \mathcal{H}^n \cap \mathcal{W}^{1, 2n/5} \cap \mathcal{W}^{1, 2n/3}; \|Dw\|_{n/2} \leq d\}
\]
into itself provided that $f \in C_0^\infty$ with $\|f\|_{-1, n/2} < d^2$. We thus obtain the desired assertion by making use of the contraction mapping principle and (1.1). The proof is complete.

2. $L^p - L^q$ estimates. In the remainder of the paper we denote by $w$ the solution of (0.2) given in Theorem 0.1, and by $C$ the various constants which are always independent of the quantities $u, v, w, f, a, t, and z$. Moreover we set
\[
S = \{z \in \mathbb{C}; -3\pi/4 < \arg z < 3\pi/4\},
\]
\[
Lu = Au + Bu; \quad Bu = P(u \cdot D)w + P(w \cdot D)u,
\]
\[
L^*u = Au + B^*u; \quad B^*u = -P(w \cdot D)u + \sum_{i=1}^{n} Pu^i Dw^i
\]
for $u = (u_1, \ldots, u^n)$ and $w = (w_1, \ldots, w^n)$.

In arriving at $L^p - L^q$ estimates, we begin with the resolvent estimates for $L$ and $L^*$.

**Lemma 2.1.** Let $z \in S$ and $u \in C_0^\infty$. Then we have

(2.1) $|z| \|(L + z)^{-1}u\|_r \leq C\|u\|_r$ for $1 < r < \infty$,

(2.2) $|z| \|(L^* + z)^{-1}u\|_r \leq C\|u\|_r$ for $1 < r < \infty$,

(2.3) $|z|^{1/2}\|D(L + z)^{-1}u\|_r \leq C\|u\|_r$ for $1 < r < n$,

(2.4) $|z|^{1/2}\|D(L^* + z)^{-1}u\|_r \leq C\|u\|_r$ for $1 < r < \infty$,

provided that $\|Dw\|_{n/2}$ is sufficiently small;

(2.5) $|z|^{3/4}\|(L + z)^{-1}u\|_\infty \leq C\|u\|_{2n},$

(2.6) $|z|^{1/2}\|D(L + z)^{-1}u\|_n \leq C(\|u\|_n + |z|^{-1/4}\|u\|_{2n}),$

provided that $\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}$ is sufficiently small.

**Proof.** Let us recall the well-known resolvent estimates for the Stokes operator (cf. [15])

(2.7) $|z|\|(A + z)^{-1}u\|_r + |z|^{1/2}\|D(A + z)^{-1}u\|_r$

$+ \|D^2(A + z)^{-1}u\|_r \leq C\|u\|_r$
for \( z \in S, 1 < r < \infty \) and \( u \in J' \), and the Gagliardo-Nirenberg inequality (cf. [4])

\[
\|u\|_q \leq C\|u\|_r^{1-h}\|Du\|_p^h
\]

for \( 1 < r, p \leq q \leq \infty, 0 \leq h < 1, -n/q = h(1-n/p) - (1-h)n/r, \)
\( u \in C^\infty_0 \). Let us suppose \( z \in S \) and \( u \in J' \cap W^{1,r} \) for \( 1 < r < \infty \).

**Step 1.** We prove (2.1) and (2.2). From (2.7), (1.1), the Hölder inequality and the boundedness of \( P \) in \( L' \)-spaces it follows that for \( 1 < r < n/2, p = nr/(n-r) \) and \( q = nr/(n-2r) \),

\[
\|B(A + z)^{-1}u\|_r \leq C\|w\|_n \|D(A + z)^{-1}u\|_p
\]

\[+ C\|Dw\|_{n/2} \|D^2(A + z)^{-1}u\|_r
\]

\[\leq C\|Dw\|_{n/2} \|D^2(A + z)^{-1}u\|_r
\]

\[\leq (1/2)\|u\|_r, \text{ by setting } C\|Dw\|_{n/2} < 1/2.
\]

This is together with (2.7) and the identity

\[
L + z = (1 + B(A + z)^{-1})(A + z)
\]

implies

\[
|z| \|(L + z)^{-1}u\|_r \leq C\|u\|_r \text{ for } 1 < r < n/2.
\]

Similarly, we have

\[
|z| \|(L^* + z)^{-1}u\|_r \leq C\|u\|_r \text{ for } 1 < r < n/2.
\]

This yields for \( n < r < \infty, v \in L' \) with \( r' = r/(r-1) \),

\[
((L + z)^{-1}u, v) = (u, (L^* + z)^{-1}Pv) \leq C|z|^{-1}\|u\|_r\|v\|_{r'}
\]

and hence the validity of (2.1) with \( n < r < \infty \). Thus (2.1) with \( n/2 \leq r \leq n \) follows immediately from the Marcinkiewicz interpolation theorem (cf. [7]). (2.2) is verified in the same way.

**Step 2.** We prove (2.3). Observing that \( 1 < r < n \) and applying the condition \( D \cdot u = D \cdot w = 0 \) and the fact that \( D \) commutes with \( P \) yields

\[
(A + z)^{-1}Bu = \sum_{i=1}^n D_i(A + z)^{-1}P(u^iw + w^iu),
\]

we have, by (2.7) and (1.1),

\[
\|(A + z)^{-1}Bu\|_r \leq C|z|^{-1/2}\|w\|_n\|u\|_{nr/(n-r)}
\]

\[\leq C\|Dw\|_{n/2}\|Du\|_r|z|^{-1/2}
\]

\[\leq 2^{-1}|z|^{-1/2}\|Du\|_r, \text{ by setting } C\|Dw\|_{n/2} < 1/2
\]
and
\begin{equation}
\|D(A + z)^{-1}Bu\|_r \leq C\|w\|_n\|u\|_{nr/(n-r)} \\
\leq C\|Dw\|_{n/2}\|Du\|_r \\
\leq (1/2)\|Du\|_r, \quad \text{by setting } C\|Dw\|_{n/2} < 1/2.
\end{equation}

Consequently, we have
\begin{equation}
\|(A + z)^{-1}B)^k u\|_r \leq 2^{-k}\|z\|^{-1/2}\|Du\|_r, \quad \text{for integer } k > 0,
\end{equation}
and so
\[D(L + z)^{-1}u = \sum_{k=0}^{\infty} D((A + z)^{-1}B)^k(A + z)^{-1}u \quad \text{in } L'.\]

Applying (2.10) to the preceding identity repeatedly and using (2.7), we have
\[\|D(L + z)^{-1}u\|_r \leq 2\|D(A + z)^{-1}u\|_r \leq C\|z\|^{-1/2}\|u\|_r
\]
as required.

**Step 3.** We prove (2.4). Observing that
\[D_i(L^* + z)^{-1}u, v) = -(u, (L + z)^{-1}D_iPv) \\
\leq \|u\|_r\|(L + z)^{-1}D_iPv\|_{r'}
\]
for \(i = 1, \ldots, n, \ 1 < r < \infty, \ r' = r/(r-1)\) and \(v \in W^{1,r'}\), we need only to show the estimate
\begin{equation}
\|(L + z)^{-1}Du\|_r \leq C\|z\|^{-1/2}\|u\|_r, \quad \text{for } 1 < r < \infty.
\end{equation}

Indeed, taking (2.9), (1.1) and (2.7) into account, we have for \(n \leq r < \infty\),
\[\|(A + z)^{-1}Bu\|_r \leq C\sum_{i=1}^{n} \|D_i(A + z)^{-1}P(uw^i + wu^i)\|_{nr/(n+r)} \\
\leq C\|Dw\|_{n/2}\|u\|_r \leq (1/2)\|u\|_{r'}
\]
by setting \(C\|Dw\|_{n/2} < 1/2\), and hence for \(n \leq r < \infty\),
\[\|(L + z)^{-1}Du\|_r = \|(1 + (A + z)^{-1}B)^{-1}D(A + z)^{-1}u\|_r \\
\leq 2\|D(A + z)^{-1}u\|_r \\
\leq C\|z\|^{-1/2}\|u\|_r
\]
which arrives at (2.12) for \( n \leq r < \infty \). Moreover (2.12) with \( 1 < r < n \) is verified as follows:

\[
\|(L + z)^{-1}Du\|_r = \|(1 + (A + z)^{-1}B)^{-1}D(A + z)^{-1}u\|_r \\
\leq \|(D(A + z)^{-1}u\|_r + |z|^{-1/2}\|D^2(A + z)^{-1}u\|_r \\
\leq C|z|^{-1/2}\|u\|_r,
\]

where we have used (2.11) and (2.7).

**Step 4.** We prove (2.5). By (2.8) and (2.7), we obtain

\[
\|(A + z)^{-1}u\|_\infty \leq C\|(A + z)^{-1}u\|_{2n}^{1/2}\|D(A + z)^{-1}u\|_{2n}^{1/2} \\
\leq C|z|^{-3/4}\|u\|_{2n},
\]

and, by (2.7), (2.8), (1.1) and (2.9),

\[
\|(A + z)^{-1}Bu\|_\infty \leq C\|(A + z)^{-1}Bu\|_{2n}^{1/2}\|D(A + z)^{-1}Bu\|_{2n}^{1/2} \\
\leq C\|D(A + z)^{-1}Bu\|_{2n/3}^{1/2}\|D(A + z)^{-1}Bu\|_{2n}^{1/2} \\
\leq C\sum_{i=1}^n \|u^i w + w^i u\|_{2n/3}^{1/2}\|u^i w + w^i u\|_{2n}^{1/2} \\
\leq C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_\infty \\
\leq (1/2)\|u\|_\infty, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.
\]

We thus obtain

\[
\|(L + z)^{-1}u\|_\infty = \|(1 + (A + z)^{-1}B)^{-1}(A + z)^{-1}u\|_\infty \\
\leq 2\|(A + z)^{-1}u\|_\infty \leq C|z|^{-3/4}\|u\|_{2n}
\]

and hence the validity of (2.5).

**Step 5.** We prove (2.6). By (1.1), (2.9) and (2.7),

\[
\|(A + z)^{-1}Bu\|_n \leq C\|D(A + z)^{-1}Bu\|_{n/2}^{1/2}\|D(A + z)^{-1}Bu\|_{n/2}^{1/2} \\
\leq C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}\|u\|_n \\
\leq (1/2)\|u\|_n, \quad \text{by setting } C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2} < 1/2
\]

and, by (2.9), (2.7) and (1.1),

\[
\|D(A + z)^{-1}Bu\|_n \leq C\|w\|_n\|u\|_\infty \\
\leq C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_\infty \\
\leq (1/2)\|u\|_\infty, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.
\]
Hence, it is easy to see that
\[
\|D(L + z)^{-1}u\|_n \leq \sum_{k=0}^{\infty} \|D((A + z)^{-1}B)^k(A + z)^{-1}u\|_n
\]
\[
\leq \|D(A + z)^{-1}u\|_n + \sum_{k=0}^{\infty} \|(A + z)^{-1}B)^k(A + z)^{-1}u\|_{\infty}
\]
\[
\leq \|D(A + z)^{-1}u\|_n + \|(A + z)^{-1}u\|_{\infty}, \quad \text{by (2.14),}
\]
\[
\leq C(|z|^{1/2}\|u\|_n + |z|^{-3/4}\|u\|_{2n}), \quad \text{by (2.7), (2.13).}
\]

The proof is complete.

As an immediate consequence of (2.1) and (2.2), we conclude that \( L \) and \( L^* \) generate strongly continuous analytic semigroups \( e^{-tL} \) and \( e^{-tL^*} \) in \( J^r \) with \( 1 < r < \infty \), respectively, provided \( \|Dw\|_{n/2} \) is sufficiently small. What is more, we can now proceed to the proof of the following \( L^p - L^r \) estimates.

**Theorem 2.1.** Let \( t > 0 \), \( 1 < q < n \), \( v \in J^q \) and \( u \in C_0^\infty \). Then we have
\[
(2.15) \quad \|e^{-tL}u\|_p \leq Ct^{-(n/r-n/p)/2}\|u\|_r \quad \text{for } 1 < r \leq p < \infty,
\]
provided that \( \|Dw\|_{n/2} \) is sufficiently small;
\[
(2.16) \quad \|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n \leq Ct^{-n/2r}\|u\|_r \quad \text{for } 1 < r \leq n,
\]
\[
(2.17) \quad t^{n/2q}(t^{-1/2}\|e^{-tLv}\|_n + \|e^{-tL}v\|_\infty + \|De^{-tL}v\|_n) \to 0 \quad \text{as } t \to \infty,
\]
provided that \( \|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2} \) is sufficiently small.

**Proof.** By making use of the semigroup property of \( e^{-tL} \), Lemma 2.1, and the Dunford integral (cf. [8]) via a standard calculation, we have
\[
(2.18) \quad \|e^{-tL^*}u\|_r + t^{1/2}\|De^{-tL^*}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < \infty,
\]
\[
(2.19) \quad \|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n \leq Ct^{-1/2}\|e^{-t/2L}u\|_n + Ct^{-1/4}\|e^{-t/2L}u\|_{2n}
\]
under the assumptions of Theorem 2.1.
It follows from (2.8) and (2.18) that
\[ \|e^{-tL^*}u\|_p \leq C\|e^{-tL^*}u\|_r^{1-n/r+n/p}\|De^{-tL^*}u\|_r^{n/r-n/p} \]
\[ \leq Ct^{-(n/r-n/p)/2}\|u\|_r \]
for \( 1 < r \leq p < \infty \) and \( n/r - n/p < 1 \). Combining this with
the semigroup property of \( e^{-tL^*} \), we have for \( 1 < r \leq p < \infty \) and
\( a \in L^p/(p-1) \),
\[ \|e^{-tL^*}Pa\|_{r/(r-1)} \leq Ct^{-(n/r-n/p)/2}\|a\|_{p/(p-1)} \]
and hence
\[ (e^{-tL}u, a) = (u, e^{-tL^*}Pa) \leq Ct^{-(n/r-n/p)}\|u\|_r\|a\|_{p/(p-1)} \]
This gives (2.15). (2.16) follows from (2.19) and (2.15).

To prove (2.17), we note for \( a \in J^q \cap J^r \) with \( 1 < r < q \),
\[ t^{n/2q}(t^{-1/2}\|e^{-tL}v\|_n + \|e^{-tL}v\|_\infty + \|De^{-tL}v\|_n) \]
\[ \leq C\|v - a\|_q + Ct^{-(n/r-n/q)/2}\|a\|_r , \]
where we have used (2.15) and (2.16). Hence the density of \( J^q \cap J^r \)
in \( J^q \) implies (2.17). The proof is complete.

3. Proof of Theorem 0.2. From Theorem 0.1 we can suppose that
\( \|Dw\|_{n/2} + \|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2} \) is small such that (2.15)–(2.17) holds.

By using the projection \( P \) to (0.2)–(0.3), and setting \( u(t) = v(t) - w \)
and \( a = v_0 - w \), then (0.2)–(0.3) leads to the evolution equation
\[ (d/dt)u + Lu = -P(u \cdot D)u \quad (t > 0), \quad u(0) = a \]
in \( J^n \). Hence, our goal now remains to show that (3.1) has a unique
solution \( u \) belonging to the space
\[ U \equiv \{u \in BC([0, \infty); J^n); t^{1/2}Du(t) \in BC([0, \infty); L^n(R^n; \mathbb{R}^n))\} \]
such that
\[ Hu(t) \equiv \|u(t)\|_n + t^{1/2}\|u(t)\|_\infty + t^{1/2}\|Du(t)\|_\infty \to 0 \quad \text{as } t \to \infty \]
provided that \( a \in J^n \) with \( \|a\|_n \) small enough.

Let us impose the following notation.
\[ \|u\| = \sup_{t>0} Hu(t) , \]
\[ W = \{u \in U; \|u\| < \infty, \ Hu(t) \to 0 \text{ as } t \to \infty\} , \]
\[ Mu(t) = u_0(t) - \int_0^t e^{-(t-s)L}P(u \cdot D)u(s) \, ds ; \quad u_0(t) = e^{-tL}a . \]
Observing that for \( u \in C_0^\infty \),
\[
t^{1/2}(\|e^{-tL}P(u \cdot D)u\|_\infty + \|De^{-tL}P(u \cdot D)u\|_n) + \|e^{-tL}P(u \cdot D)u\|_n
\leq C t^{-1/4}(u \cdot D)u\|_{2n/3}, \quad \text{by (2.15)-(2.16)},
\]
\[
\leq C t^{-1/4}\|u\|_{2n}\|Du\|_n,
\]
and
\[
\|u\|_{2n} \leq \|u\|_{n}^{1/2}\|u\|_{\infty}^{1/2},
\]
we have for \( u \in W \),
\[
(3.2) \quad \|Mu(t)\|_n + t^{1/2}\|Mu(t)\|_\infty + t^{1/2}\|DMu(t)\|_n
\leq C\|a\|_n + C \int_0^t(t-s)^{-1/4}\|u(s)\|_{2n}\|Du(s)\|_n ds
\]
\[
+ C \int_0^t t^{1/2}(t-s)^{-3/4}\|u(s)\|_{2n}\|Du(s)\|_n ds,
\]
by (2.15)-(2.16),
\[
\leq C\|a\|_n + C\|u\|^2,
\]
and what is more, by using (2.17) and the property
\[
Hu_0(t) + Hu(t) \to 0 \quad \text{as} \quad t \to \infty
\]
via a calculation similar to (3.2), we have
\[
H(Mu)(t) \to 0 \quad \text{as} \quad t \to \infty.
\]
Moreover, by a standard calculation from [19] or [12], we have \( Mu \in U \) for \( u \in W \), and so \( M : W \to W \) and
\[
\|Mu\| \leq C\|a\|_n + C\|u\|^2.
\]
Additionally, similar to (3.2), we obtain for \( u_1, u_2 \in W \),
\[
\|Mu_1 - Mu_2\| \leq C(\|u_1\| + \|u_2\|)\|u_1 - u_2\|.
\]
From contraction mapping principle it follows that \( M \) has a fixed point \( u \) in \( W \) provided \( \|a\|_n \) is sufficiently small. As in [12, 5], we find that the fixed point \( u \) is the desired solution which exists uniquely in \( U \). The proof is complete.

References


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