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We find an explicit decomposition for the metaplectic representation restricted to either member of the dual reductive pair $(U(1), U(1))$ in $\widetilde{\mathrm{SL}}(2, F)$, where F is a p -adic field, with p odd.

1. Introduction and preliminaries. Let F be a p -adic field of odd residual characteristic with q being the order of the residue class field. Let \mathcal{O} be the ring of integers, \mathcal{P} the prime ideal, \mathcal{U} the units, π a prime element, and ν the valuation on F . Let $G = \mathrm{SL}(2, F)$.

For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, let $x(\sigma) = c$ if $c \neq 0$, and let $x(\sigma) = d$ if $c = 0$. Define a 2-cocycle on G by

$$\alpha(g_1, g_2) = (x(g_1), x(g_2))(-x(g_1)x(g_2), x(g_1g_2)).$$

This cocycle determines a nontrivial 2-sheeted covering group \widetilde{G} of G [G1].

Let E be a quadratic extension of F , and $x \mapsto \bar{x}$ the Galois action. The group $U(1)$ which preserves the Hermitian form $(x, y) \mapsto x\bar{y}$ on E is isomorphic to the group N^1 of norm one elements in E . The pair of subgroups $(U(1), U(1))$ of $\mathrm{SL}(2)$ form a dual reductive pair [H]. This dual pair is one of the simplest examples over a p -adic field. Some other basic examples of dual reductive pairs are discussed in [G2]. In this paper we determine the decomposition of the oscillator representation of \widetilde{G} upon restriction to $U(1) \subset \widetilde{G}$.

The results in this paper have recently been applied by Rogawski to the problem of calculating the multiplicities of certain automorphic representations π of $U(\mathbf{A})$ in the discrete spectrum of $L^2(U(k) \backslash U(\mathbf{A}))$, where U is a unitary group in 3 variables defined relative to a quadratic extension of number fields K/k [R1, R2]. I would like to thank Rogawski for several stimulating conversations and for encouraging me to publish this paper.

Let τ be a character of F . Choose a normalized measure μ so that $\mu(\mathcal{O}) = q^{\frac{\omega(\tau)}{2}}$, where $\omega(\tau)$ is the conductor of τ . Denote this measure by $d_\tau x$. Then if we define the Fourier transform on $S(F)$, the space of locally compact functions on F with compact support,

by

$$\hat{f}(x) = \int f(y)\tau(2xy) d_\tau y,$$

we have $\hat{\hat{f}}(x) = f(-x)$. For $a \in F$, we set $\tau_a(x) = \tau(ax)$. Let

$$\kappa(\tau) = \lim_{m \rightarrow -\infty} \int_{\mathcal{O}^m} \tau(x^2) d_\tau x.$$

Recall [Sh] that $\kappa(\tau) = 1$ if $\omega(\tau)$ even, and

$$\kappa(\tau) = G(\tau) = q^{-\frac{1}{2}} \sum_{x \in \mathcal{O}/\mathcal{P}} \tau(\pi^{n-1}x^2)$$

if $n = \omega(\tau)$ is odd. For $u \in \mathcal{U}$, let $\left(\frac{u}{\mathcal{P}}\right) = 1$ if u is a square, and $\left(\frac{u}{\mathcal{P}}\right) = -1$ otherwise. Then we have $G(\tau)^2 = \left(\frac{-1}{\mathcal{P}}\right)$ and $G(\tau_u) = \left(\frac{u}{\mathcal{P}}\right)G(\tau)$ for $u \in \mathcal{U}$.

We now define the metaplectic representation $W = W^\tau$ of \tilde{G} associated to the quadratic form $Q(x) = x^2$ by specifying the action on generators [G1]. Here $\zeta = \pm 1$, and $|a|$ is the absolute value on F .

$$\begin{aligned} W \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \zeta \right) f(x) &= \zeta \tau(bx^2) f(x), \\ W \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \zeta \right) f(x) &= \zeta |a|^{\frac{1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} f(ax), \\ W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta \right) f(x) &= \zeta \kappa(\tau) \hat{f}(x). \end{aligned}$$

The cocycle defining \tilde{G} splits on the compact subgroup $K = \text{SL}_2(\mathcal{O})$ by a function $s : K \rightarrow \mathbb{Z}_2$. K thus lifts as a subgroup of \tilde{G} by $k \mapsto (k, s(k))$, and we may thus restrict W to obtain a representation of K on $S(F)$. Note that $U(1) \subset K$. Our goal is to find the characters of $U(1)$ which appear in the restriction of W to $U(1)$.

Let $S(\mathcal{P}^r, \mathcal{P}^s)$ be the space of functions on F which have support on \mathcal{P}^r and which are constant on cosets of \mathcal{P}^s in \mathcal{P}^r . Suppose $\omega(\tau) = n \geq 1$. Then $S(\mathcal{O}, \mathcal{P}^n)$ is invariant under W^τ restricted to K , and the group

$$K_n = \left\{ k \in K \mid k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{P}^n} \right\}$$

acts trivially on $S(\mathcal{O}, \mathcal{P}^n)$. We thus obtain a representation $W_n = W_n^\tau$ of $K/K_n \cong \text{SL}_2(\mathcal{O}/\mathcal{P}^n)$ on $S(\mathcal{O}, \mathcal{P}^n)$. Note that we may consider τ as a character of $\mathcal{O}/\mathcal{P}^n$.

2. Calculation of the trace. In this section we calculate the trace of $W_n(t)$, where t denotes either an element of T or its image in $Sl_2(\mathcal{O}/\mathcal{P}^n)$, and

$$T = \left\{ \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \mid a^2 - b^2\alpha = 1 \right\}$$

is the torus in G corresponding to the quadratic extension $E = F(\sqrt{\alpha})$. It will suffice to let $\alpha = \tau$ or $\alpha = \varepsilon$, a primitive $(q - 1)$ st root of unity in \mathcal{O} .

LEMMA 1. For $t = \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \in T$, we have the decomposition

$$(1) \quad (t, s(t)) = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & 0 \\ b\alpha\alpha & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, 1 \right) \\ \times \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma(t) \right),$$

where $\gamma(t) = (a, b)$ if $\alpha = \varepsilon$, $b \in \mathcal{U}$, and $a \notin \mathcal{U}$, and $\gamma(t) = 1$ otherwise. Also,

$$(2) \quad \left(\begin{pmatrix} 1 & 0 \\ b\alpha\alpha & 1 \end{pmatrix}, 1 \right) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & -b\alpha\alpha \\ 0 & 1 \end{pmatrix}, 1 \right) \\ \times \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right).$$

Proof. Both statements are clearly true if $b=0$, so we suppose $b \neq 0$. A calculation shows that the right side of (1) equals $(t, (a, b\alpha\alpha)\gamma(t))$. We must therefore show that $s(t) = (a, b\alpha\alpha)\gamma(t)$ for $t \neq \pm I$. Recall [G] that $s(t) = (b\alpha, a)$ if $b \neq 0$ and $b\alpha \notin \mathcal{U}$, and $s(t) = 1$ otherwise. First suppose $\alpha = \pi$. In this case $a \in \mathcal{U}$, so $(a, b\pi\alpha) = (a, b\pi) = s(t)$. Now suppose $\alpha = \varepsilon$. If $b \notin \mathcal{U}$, then $b^2\varepsilon \in \mathcal{P}^2 \Rightarrow a^2 = 1 + b^2\varepsilon \in 1 + \mathcal{P}^2 \subset \mathcal{U} \Rightarrow a \in \mathcal{U}$. Then $\gamma(t) = 1$, so $(a, b\varepsilon\alpha)\gamma(t) = (a, b\varepsilon) = s(t)$. If $b \in \mathcal{U}$, then $s(t) = 1$, so we must show $(a, b\varepsilon\alpha)\gamma(t) = 1$. If $a \in \mathcal{U}$, then $\gamma(t) = 1$ so we need $(a, b\varepsilon\alpha) = 1$, which is true since $a \in \mathcal{U}$ and $b \in \mathcal{U}$. If $a \notin \mathcal{U}$, then $\gamma(t) = (a, b)$, so we must show $(a, b\varepsilon\alpha)(a, b) = 1 \Leftrightarrow (a, \varepsilon a) = 1$. But $a \notin \mathcal{U} \Rightarrow a^2 \in \mathcal{P}^2 \Rightarrow 1 + b^2\varepsilon \in \mathcal{P}^2 \Rightarrow -b^2\varepsilon \in 1 + \mathcal{P}^2$. This shows $-\varepsilon \in \mathcal{U}^2$, so $(a, \varepsilon a) = (a, (-\varepsilon)(-a)) = (a, -\varepsilon)(a, -a) = (a, -a) = 1$.

LEMMA 2. Suppose $t = \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \in T$ and $a \in \mathcal{U}$. Then for $f \in S(\mathcal{O}, \mathcal{P}^n)$,

$$(W_n(t, s(t))f)(x) = \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathcal{O}/\mathcal{P}^n} K_{b\alpha a}(ax, s) \tau \left(\frac{b}{a} s^2 \right) f(s),$$

where, for $c \in \mathcal{O}$,

$$K_c(x, s) = q^{-n} \sum_{\mathcal{O}/\mathcal{P}^n} \tau(-cr^2)\tau(-2xr)\tau(2rs).$$

Proof. For any $\phi \in S(\mathcal{O}, \mathcal{P}^n)$, we have, for $c \in \mathcal{O}$,

$$\begin{aligned} & \left(W \left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1 \right) \phi \right) (x) \\ &= \left(W \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1-c & \\ 0 & 1 \end{pmatrix}, 1 \right) \right. \\ & \quad \left. \times W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \right) (x) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_{-1})} \left(W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1-c & \\ 0 & 1 \end{pmatrix}, 1 \right) \right. \\ & \quad \left. \times W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \right) (-x) \\ &= \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})} \left(W \left(\begin{pmatrix} 1-c & \\ 0 & 1 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \right)^\wedge (-x). \end{aligned}$$

But $\phi \in S(\mathcal{O}, \mathcal{P}^n) \Rightarrow \hat{\phi} \in S(\mathcal{O}, \mathcal{P}^n)$, so for $c \in \mathcal{O}$, we have

$$W \left(\begin{pmatrix} 1-c & \\ 0 & 1 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \in S(\mathcal{O}, \mathcal{P}^n).$$

For any $\psi \in S(\mathcal{O}, \mathcal{P}^n)$, we have

$$\begin{aligned} \hat{\psi}(x) &= \int \psi(y)\tau(2xy) d_\tau y = \int_{\mathcal{O}} \psi(y)\tau(2xy) d_\tau y \\ &= \sum_{r \in \mathcal{O}/\mathcal{P}^n} \int_{\mathcal{P}^n} \psi(r+y)\tau(2x(r+y)) d_\tau y \\ &= \sum_{r \in \mathcal{O}/\mathcal{P}^n} \psi(r)\tau(2xr) \int_{\mathcal{P}^n} \tau(2xy) d_\tau y. \end{aligned}$$

But $y \mapsto \tau(2xy)$ is trivial on $\mathcal{P}^n \Leftrightarrow x \in \mathcal{O}$, so $\hat{\psi}(x) = 0$ if $x \notin \mathcal{O}$, and if $x \in \mathcal{O}$, we have

$$\hat{\psi}(x) = q^{-\frac{n}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \psi(r)\tau(2xr).$$

Therefore,

$$\begin{aligned} & \left(W \left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1 \right) \phi \right) (x) \\ &= \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})} q^{-\frac{n}{2}} \sum_{r \in \mathcal{O} / \mathfrak{P}^n} \left(W \left(\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1 \right) \right. \\ & \quad \left. \times W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \right) (r) \tau(-2xr) \\ &= \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})} q^{-\frac{n}{2}} \sum_{r \in \mathcal{O} / \mathfrak{P}^n} \tau(-cr^2) \left(W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \right) (r) \tau(-2xr) \\ &= \frac{\kappa(\tau)^3}{\kappa(\tau_{-1})} q^{-n} \sum_{r \in \mathcal{O} / \mathfrak{P}^n} \tau(-cr^2) \tau(-2xr) \sum_{s \in \mathcal{O} / \mathfrak{P}^n} \phi(s) \tau(2rs). \end{aligned}$$

But

$$\frac{\kappa(\tau)^3}{\kappa(\tau_{-1})} = 1,$$

so we get

$$\left(W \left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1 \right) \phi \right) (x) = \sum_{s \in \mathcal{O} / \mathfrak{P}^n} K_c(x, s) \phi(s),$$

where for $c \in \mathcal{O}$,

$$K_c(x, s) = q^{-n} \sum_{r \in \mathcal{O} / \mathfrak{P}^n} \tau(-cr^2) \tau(-2xr) \tau(2rs).$$

Now we calculate the action of $W_n(t, s(t))$ for $a \in \mathcal{U}$. Note that in this case, $\gamma(t) = 1$. For $f \in S(\mathcal{O}, \mathfrak{P}^n)$, we have

$$\begin{aligned} & (W_n(t, s(t))f)(x) \\ &= \left(W \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1 & 0 \\ baa & 1 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, 1 \right) f \right) (x) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} \left(W \left(\begin{pmatrix} 1 & 0 \\ baa & 1 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, 1 \right) f \right) (ax) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathcal{O} / \mathfrak{P}^n} K_{baa}(ax, s) \tau \left(\frac{b}{a} s^2 \right) f(s). \end{aligned}$$

Here we used the fact that

$$a \in \mathcal{U} \Rightarrow W \left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, 1 \right) f \in S(\mathcal{O}, \mathfrak{P}^n).$$

This completes the proof of Lemma 2.

If $a \in \mathcal{U}$, the action of $W_n(t, s(t))$ is therefore given by the kernel

$$\frac{\kappa(\tau)}{\kappa(\tau_a)} K_{b\alpha a}(ax, s) \tau\left(\frac{b}{a}s^2\right).$$

We now use this kernel to calculate the trace of $W_n(t, s(t))$ when $a \in \mathcal{U}$. The kernel is a function defined on $\mathcal{O}/\mathcal{P}^n \times \mathcal{O}/\mathcal{P}^n$, so we have

$$\begin{aligned} (3) \text{ trace } W_n(t, s(t)) &= \sum_{s \in \mathcal{O}/\mathcal{P}^n} \frac{\kappa(\tau)}{\kappa(\tau_a)} K_{b\alpha a}(as, s) \tau\left(\frac{b}{a}s^2\right) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathcal{O}/\mathcal{P}^n} q^{-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-b\alpha ar^2) \tau(2rs(1-a)) \tau\left(\frac{b}{a}s^2\right) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-b\alpha ar^2) \sum_{s \in \mathcal{O}/\mathcal{P}^n} \tau\left(\frac{b}{a}s^2 + 2r(1-a)s\right). \end{aligned}$$

Suppose $\nu(b) = k$. The inner sum can be written

$$\begin{aligned} &\sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau\left(\frac{b}{a}(u+v)^2\right) \tau(2r(1-a)(u+v)) \\ &= \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right) \tau(2r(1-a)u) \sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau(2r(1-a)v) \end{aligned}$$

since $\frac{b}{a}uv \in \mathcal{P}^n$ and $\frac{b}{a}v^2 \in \mathcal{P}^n$.

Consider the sum

$$\sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau(2r(1-a)v).$$

Since $a \in \mathcal{U}$, we may have $\nu(a-1) = 0$ or $\nu(a-1) > 0$. Suppose first that $\nu(a-1) = 0$. Then $\tau_{2r(1-a)}$ is trivial on $\mathcal{P}^{n-k} \Leftrightarrow \omega(\tau_{2r(1-a)}) \leq n-k \Leftrightarrow r \in \mathcal{P}^k$. If $r \notin \mathcal{P}^k$, we have

$$\sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau(2r(1-a)v) = 0,$$

and (3) therefore equals

$$(4) \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} q^k \sum_{r \in \mathcal{P}^k/\mathcal{P}^n} \tau(-b\alpha ar^2) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2 + 2r(1-au)\right).$$

The inner sum in (4) equals

$$(5) \quad \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} \left(u^2 + \frac{2r(1-a)a}{b} u \right) \right) \\ = \tau \left(-\frac{r^2(1-a)^2 a}{b} \right) \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} \left(u + \frac{r(1-a)a}{b} \right)^2 \right).$$

Since $\nu(b) = k$ and $v \in \mathfrak{P}^k$, we have $\nu\left(\frac{r(1-a)a}{b}\right) = \nu(r) - \nu(b) \geq 0$, so $\left\{u + \frac{r(1-a)a}{b}\right\} = \mathcal{O}/\mathfrak{P}^{n-k}$ and (5) equals

$$\tau \left(-\frac{r^2(1-a)^2 a}{b} \right) \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} u^2 \right).$$

So if $a \in \mathcal{U}$, $a - 1 \in \mathcal{U}$, and $\nu(b) = k$, we have

$$(6) \quad \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathfrak{P}^k/\mathfrak{P}^n} \tau(-b\alpha r^2) \\ \times \tau \left(-\frac{r^2(1-a)^2 a}{b} \right) \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} u^2 \right) \\ = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathfrak{P}^k/\mathfrak{P}^n} \tau(cr^2) \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} u^2 \right),$$

where $c = -\frac{2a^2(a-1)}{b}$.

Now we consider the sum

$$\sum_{v \in \mathfrak{P}^{n-k}/\mathfrak{P}^n} \tau(2r(1-a)v)$$

in the case when $\nu(a-1) > 0$. We have $a^2 - 1 = b^2\alpha \Rightarrow \nu(a-1) + \nu(a+1) = 2\nu(b) + \nu(\alpha)$. Since $a-1 \in \mathfrak{P}$, we have $a+1 = (a-1) + 2 \in \mathcal{U}$, so $\nu(a-1) = 2\nu(b) + \nu(\alpha)$. We therefore have $\nu(a-1) > \nu(b) = k$. This shows that $\tau_{2r(1-a)}$ is trivial on \mathfrak{P}^{n-k} for all $r \in \mathcal{O}/\mathfrak{P}^n$, and so (3) implies

$$(7) \quad \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{O}/\mathfrak{P}^n} \tau(-b\alpha r^2) \\ \times \sum_{u \in \mathcal{O}/\mathfrak{P}^{n-k}} \tau \left(\frac{b}{a} u^2 + 2r(1-a)u \right).$$

Considering (5) again, we have $\nu(\frac{r(1-a)a}{b}) > 0$, so if $a \in \mathcal{U}$, $a - 1 \in \mathcal{P}$, and $\nu(b) = k$, we have

$$\begin{aligned}
 (8) \text{ trace } W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-b\alpha ar^2) \\
 &\quad \times \tau\left(-\frac{r^2(1-a)^2 a}{b}\right) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right) \\
 &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} q^k \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right),
 \end{aligned}$$

where $c = -\frac{2a^2(1-a)}{b}$.

We summarize (6) and (8) as follows

LEMMA 3. Suppose $a \in \mathcal{U}$ and $\nu(b) = k$. Let $c = -\frac{2a^2(1-a)}{b}$. Then

$$(9) \text{ trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{O}^l/\mathcal{P}^n} \tau(cr^2) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right),$$

where $l = k$ if $a - 1 \in \mathcal{U}$ and $l = 0$ if $a - 1 \in \mathcal{P}$.

To calculate these sums we need

LEMMA 4. If $\omega(\tau) = n$ then $\sum_{x \in \mathcal{O}/\mathcal{P}^n} \tau(x^2) = q^{\frac{n}{2}} \kappa(\tau)$.

Proof. Suppose n is even. Then

$$\begin{aligned}
 \sum_{x \in \mathcal{O}/\mathcal{P}^n} \tau(x^2) &= \sum_{u \in \mathcal{O}/\mathcal{P}^{\frac{n}{2}}} \sum_{v \in \mathcal{P}^{\frac{n}{2}}/\mathcal{P}^n} \tau((u+v)^2) \\
 &= \sum_{u \in \mathcal{O}/\mathcal{P}^{\frac{n}{2}}} \tau(u^2) \sum_{v \in \mathcal{P}^{\frac{n}{2}}/\mathcal{P}^n} \tau(2uv).
 \end{aligned}$$

But $v \mapsto \tau(2uv)$ is trivial on $\mathcal{P}^{\frac{n}{2}}/\mathcal{P}^n \Leftrightarrow u = 0$, so the sum is just $q^{\frac{n}{2}}$ in this case.

If n is odd, then

$$\sum_{x \in \mathcal{O}/\mathcal{P}^n} \tau(x^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{\frac{n+1}{2}}} \tau(u^2) \sum_{v \in \mathcal{P}^{\frac{n+1}{2}}/\mathcal{P}^n} \tau(2uv).$$

In this case, $v \mapsto \tau(2uv)$ is trivial on $\mathcal{P}^{\frac{n+1}{2}} \Leftrightarrow u \in \mathcal{P}^{\frac{n-1}{2}}$, so the sum equals

$$q^{\frac{n-1}{2}} \sum_{u \in \mathcal{P}^{\frac{n-1}{2}}/\mathcal{P}^{\frac{n+1}{2}}} \tau(u^2).$$

Writing $u = \pi^{\frac{n-1}{2}}$, with $r \in \mathcal{O}/\mathcal{P}$, the sum equals

$$q^{\frac{n-1}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}} \tau(\pi^{n-1}r^2) = q^{\frac{n-1}{2}} q^{\frac{1}{2}} G(\tau) = q^{\frac{n}{2}} G(\tau).$$

This completes the proof of Lemma 4.

Now we apply Lemma 4 to the sums in (9). First, $\omega(\tau_{\frac{b}{a}}) = \omega(\tau) - \nu(\frac{b}{a}) = n - k$, so

$$\sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right) = q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}).$$

Suppose $\nu(a - 1) = 0$. Then

$$\sum_{r \in \mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(c\pi^{2k}u^2).$$

Since $\nu(c) = \nu(\frac{a-1}{b}) = -\nu(b) = -k$, $\omega(\tau_{c\pi^{2k}}) = n - 2k - \nu(c) = n - k$, and we have

$$\sum_{r \in \mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = q^{\frac{n-k}{2}} \kappa(\tau_{c\pi^{2k}}) = q^{\frac{n-k}{2}} \kappa(\tau_c).$$

Now suppose $\nu(a - 1) > 0$ and consider

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2).$$

If $\alpha = \varepsilon$ then $\nu(a - 1) = 2\nu(b) = 2k$. We write

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) \sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau(2cuv).$$

But $\omega(\tau_{2cu}) = n - \nu(cu) \leq n - k \Leftrightarrow \nu(cu) \geq k$, which is true for all $u \in \mathcal{O}$, so

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = q^k \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) = q^k q^{\frac{n-k}{2}} \kappa(\tau_c),$$

where we used Lemma 4 since $\omega(\tau_c) = n - k$.

If $\alpha = \pi$, then $\nu(a - 1) = 2\nu(b) + 1 = 2k + 1$. We write

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k-1}} \sum_{v \in \mathcal{P}^{n-k-1}/\mathcal{P}^n} \tau(c(u+v)^2)$$

and argue as above to obtain

$$\sum_{r \in \mathcal{O} / \mathcal{P}^n} \tau(cr^2) = q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c).$$

Suppose that $a \in \mathcal{U}$ and $\nu(b) = k \geq 0$. We have now shown that if $\nu(a - 1) = 0$, then we have

$$\begin{aligned} (10) \quad \text{trace } W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

If $\nu(a - 1) > 0$ and $\alpha = \varepsilon$,

$$\begin{aligned} (11) \quad \text{trace } W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^k q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= q^k \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

If $\nu(a - 1) > 0$ and $\alpha = \pi$,

$$\begin{aligned} (12) \quad \text{trace } W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= q^{\frac{2k+1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

We can summarize (10), (11), and (12) as follows.

LEMMA 5. *If $a \in \mathcal{U}$ and $b \neq 0$, then*

$$\text{trace } W_n(t, s(t)) = q^{\frac{\nu(a-1)}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}),$$

where $c = -\frac{2a^2(1-a)}{b}$.

To calculate $\text{trace } W_n(t, s(t))$ when $a \in \mathcal{P}$ we need another decomposition. Note that since $a \in \mathcal{P}$, we have $\alpha = \varepsilon$ and $b \in \mathcal{U}$.

LEMMA 6.

$$(t, s(t)) = \left(\left(\begin{pmatrix} -\frac{1}{be} & 0 \\ 0 & \frac{1}{be} \end{pmatrix}, 1 \right), \left(\begin{pmatrix} 1 & ab\varepsilon \\ 0 & 1 \end{pmatrix}, 1 \right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right), \left(\begin{pmatrix} 1 & \frac{a}{be} \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Proof. A calculation shows that the right side equals $(t, 1)$. Noting that $s(t) = 1$ in this case completes the proof.

Suppose $\nu(a) = m \geq 1$ and $\omega(\tau) = n$. Choose $f \in S(\mathcal{O}, \mathcal{P}^n)$. Using Lemma 6, we see that

$$\begin{aligned} & (W_n(t, s(t))f)(x) \\ &= \left| -\frac{1}{b\varepsilon} \right|^{\frac{1}{2}} \frac{\kappa(\tau)}{\kappa(\tau - b\varepsilon)} \left(W \left(\begin{pmatrix} 1 & ab\varepsilon \\ 0 & 1 \end{pmatrix}, 1 \right) \right. \\ & \quad \times W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{pmatrix}, 1 \right) f \left(-\frac{1}{b\varepsilon}x \right) \\ &= \frac{\kappa(\tau)}{\kappa(\tau - b\varepsilon)} \tau \left(ab\varepsilon \left(-\frac{1}{b\varepsilon} \right)^2 \right) \\ & \quad \times \left(W \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left(\begin{pmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{pmatrix}, 1 \right) f \right) \left(-\frac{1}{b\varepsilon}x \right) \\ &= \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \tau \left(\frac{a}{b\varepsilon}x^2 \right) q^{-\frac{n}{2}} \sum_{s \in \mathcal{O}/\mathcal{P}^n} \left(W \left(\begin{pmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{pmatrix}, 1 \right) f \right) (s) \tau \left(-\frac{2sx}{b\varepsilon} \right) \\ &= \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \tau \left(\frac{a}{b\varepsilon}x^2 \right) q^{-\frac{n}{2}} \sum_{s \in \mathcal{O}/\mathcal{P}^n} \tau \left(\frac{a}{b\varepsilon}s^2 \right) \tau \left(-\frac{2sx}{b\varepsilon} \right) f(s) \\ &= \sum_{s \in \mathcal{O}/\mathcal{P}^n} K(x, s) f(s), \end{aligned}$$

where

$$K(x, s) = q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \tau \left(\frac{a}{b\varepsilon}x^2 \right) \tau \left(\frac{a}{b\varepsilon}s^2 \right) \tau \left(-\frac{2sx}{b\varepsilon} \right).$$

Since $\frac{a}{b\varepsilon} \in \mathcal{O}$ and $-\frac{2s}{b\varepsilon} \in \mathcal{O}$,

$$\begin{aligned} \text{trace } W_n(t, s(t)) &= \sum_{s \in \mathcal{O}/\mathcal{P}^n} K(s, s) \\ &= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \sum_{s \in \mathcal{O}/\mathcal{P}^n} \tau \left(\frac{a}{b\varepsilon}s^2 \right) \tau \left(\frac{a}{b\varepsilon}s^2 \right) \tau \left(-\frac{2s^2}{b\varepsilon} \right) \\ &= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \sum_{s \in \mathcal{O}/\mathcal{P}^n} \tau(cs^2), \end{aligned}$$

where $c = \frac{2(a-1)}{b\varepsilon}$. Since $\nu(c) = \nu(a-1) - \nu(b) = 0$, we have $\omega(\tau_c) = n$. Using Lemma 4, we have

LEMMA 7. *If $a \in \mathcal{P}$, then*

$$\text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)^2}{\kappa(\tau - b\varepsilon)} \kappa(\tau_c),$$

where $c = \frac{2(a-1)}{b\varepsilon}$.

3. Further calculation of the trace. We now refine the formulas in Lemma 5 and Lemma 7. Suppose $E = F(\sqrt{\varepsilon})$. Letting $T_n = T \cap K_n$, we have a filtration $T \supset T_1 \supset \dots$, with $[T : T_1] = q + 1$ and $[T_i : T_{i+1}] = q$ for $i \geq 1$. Let $n = \omega(\tau)$.

PROPOSITION 1. *Suppose E/F is unramified.*

- (1) For $t \in T_k - T_{k+1}$, $k \geq 1$, $\text{trace } W_n(t, s(t)) = (-1)^{n-k} q^k$.
- (2) For $t \notin T_1$, $\text{trace } W_n(t, s(t)) = \left(\frac{2(a-1)}{\mathfrak{P}}\right)^n$.

Proof. Assume first $t \in T_1$. Then $a - 1 \in \mathfrak{P}$ and $b \in \mathfrak{P}$. We have $\nu(a) = 0$. If $t \in T_k - T_{k+1}$, then $\nu(b) = k \geq 0$. We apply Lemma 5. We have $\nu(c) = \nu(\frac{b}{a}) = k$. Also, $\nu(a - 1) = 2\nu(b) = 2k$. If n is even, we have $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition k is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$ and so $\text{trace} = q^k$. If n is even and k is odd, $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$, so $\text{trace} = q^k G(\tau_c) G(\tau_{\frac{b}{a}})$. Letting $b = u\pi^k$ and $a - 1 = v\pi^{2k}$, we have $c = -\frac{2a^2v}{u}$, so $\text{trace} = q^k \left(\frac{-2vu}{\mathfrak{P}}\right) \left(\frac{ua}{\mathfrak{P}}\right) G(\tau)^2 = q^k \left(\frac{2vu^2a}{\mathfrak{P}}\right) = q^k \left(\frac{2va}{\mathfrak{P}}\right)$. But $a - 1 \in \mathfrak{P} \Rightarrow a \in \mathcal{U}^2 \Rightarrow \left(\frac{a}{\mathfrak{P}}\right) = 1$. Also, $a^2 = (1 + v\pi^{2k})^2 = 1 + 2v\pi^{2k} + v^2\pi^{4k}$, and $1 + b^2\varepsilon = 1 + u^2\pi^{2k}\varepsilon$. But $a^2 = 1 + b^2\varepsilon$, so $u^2\pi^{2k}\varepsilon = 2v\pi^{2k} + v^2\pi^{4k} \Rightarrow u^2\varepsilon = 2v + v^2\pi^{2k} \Rightarrow 2v = u^2\varepsilon - v^2\pi^{4k} = u^2\varepsilon(1 - \frac{v^2\pi^{2k}}{u^2\varepsilon}) \in u^2\varepsilon(1 + \mathfrak{P}) \subset \varepsilon\mathcal{U}^2$, which implies $2v$ is not a square $\Rightarrow \left(\frac{2v}{\mathfrak{P}}\right) = -1$, so $\text{trace} = -q^k$.

If n is odd then $\frac{\kappa(\tau)}{\kappa(\tau_a)} = \frac{G(\tau)}{G(\tau_a)} = \left(\frac{a}{\mathfrak{P}}\right)$. If k is even, then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$. Arguing as in the case of n even and k odd, we have $\text{trace} = q^k \left(\frac{a}{\mathfrak{P}}\right) G(\tau_c) G(\tau_{\frac{b}{a}}) = q^k \left(\frac{2v}{\mathfrak{P}}\right) = -q^k$. If k is odd, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1 \Rightarrow \text{trace} = q^k \left(\frac{a}{\mathfrak{P}}\right)$. But $a - 1 \in \mathfrak{P} \Rightarrow \left(\frac{a}{\mathfrak{P}}\right) = 1$, so $\text{trace} = q^k$. This completes the proof of (1) of Proposition 1.

Now assume $t \notin T_1$. Then $a - 1 \in \mathcal{U}$ or $b \in \mathcal{U}$. We consider various cases: (1) $a - 1 \in \mathcal{U}$, $b \in \mathcal{U}$; (2) $a - 1 \in \mathcal{U}$, $b \in \mathfrak{P}$; (3) $a - 1 \in \mathfrak{P}$, $b \in \mathcal{U}$. Case (3) cannot arise, since $a^2 - 1 = b^2\varepsilon \Rightarrow \nu(a - 1) + \nu(a + 1) = 2\nu(b)$. Then $\nu(a - 1) > 0 \Rightarrow \nu(b) > 0$, which is a contradiction.

We first consider case (1). In this case, we have $\nu(a - 1) = 0$, $\nu(b) = 0$, and we may have $a \in \mathcal{U}$ or $a \in \mathfrak{P}$. Suppose first $a \in \mathcal{U}$. We use Lemma 5. If n is even, $\kappa(\tau) = \kappa(\tau_a) = 1$. Also, $\nu(\frac{b}{a}) = \nu(c) = 0$, so $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$. Since $\nu(a - 1) = 0$, $\text{trace} = 1$. If n is odd, $\text{trace} = \frac{G(\tau)}{G(\tau_a)} G(\tau_c) G(\tau_{\frac{b}{a}}) = \left(\frac{a}{\mathfrak{P}}\right) \left(\frac{c}{\mathfrak{P}}\right) \left(\frac{ba}{\mathfrak{P}}\right) G(\tau)^2 = \left(\frac{2(a-1)}{\mathfrak{P}}\right)$. Now suppose $a \in \mathfrak{P}$. Then we must use Lemma 7. If n is even,

$\kappa(\tau) = \kappa(\tau_{-b\varepsilon}) = \kappa(\tau_c) = 1$, so trace = 1. If n is odd, trace = $\frac{G(\tau)}{G(\tau_{-b\varepsilon})}G(\tau_c) = \left(\frac{cb\varepsilon}{\mathcal{P}}\right) = \left(\frac{2(a-1)}{\mathcal{P}}\right)$.

We next consider case (2). Now we have $a-1 \in \mathcal{U}$ and $b \in \mathcal{P}$, so $a \in \mathcal{U}$ and we can use Lemma 5. If n is even, then $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $\nu(b)$ is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace = 1. If $\nu(b)$ is odd, then trace = $G(\tau_c)G(\tau_{\frac{b}{a}})$. Writing $b = u\pi^{2k+1}$, this equals $\left(\frac{-2(a-1)u}{\mathcal{P}}\right)\left(\frac{ua}{\mathcal{P}}\right)G(\tau)^2 = \left(\frac{2a(a-1)}{\mathcal{P}}\right)$. We claim $\left(\frac{2a(a-1)}{\mathcal{P}}\right) = 1$. We have $\nu(a-1) + \nu(a+1) = 2\nu(b) \geq 2$, so $a-1 \in \mathcal{U} \Rightarrow a+1 \in \mathcal{P} \Rightarrow a = -1 + d$, $d \in \mathcal{P}$. This shows $a-1 = -2 + d = -2(1 - \frac{1}{2}d) \in -2\mathcal{U}_1 \subset -2\mathcal{U}^2$, so $\left(\frac{a-1}{\mathcal{P}}\right) = \left(\frac{-2}{\mathcal{P}}\right)$. Also, $a = -1 + d \in (-1)\mathcal{U}_1 \Rightarrow \left(\frac{a}{\mathcal{P}}\right) = \left(\frac{-1}{\mathcal{P}}\right)$. Therefore, $\left(\frac{2a(a-1)}{\mathcal{P}}\right) = \left(\frac{2}{\mathcal{P}}\right)\left(\frac{a}{\mathcal{P}}\right)\left(\frac{a-1}{\mathcal{P}}\right) = \left(\frac{2}{\mathcal{P}}\right)\left(\frac{-1}{\mathcal{P}}\right)\left(\frac{-2}{\mathcal{P}}\right) = 1$, so in this case trace = 1.

Now suppose n is odd. Then $\kappa(\tau) = G(\tau)$ and $\kappa(\tau_a) = G(\tau_a)$, so trace = $\left(\frac{a}{\mathcal{P}}\right)\kappa(\tau_c)\kappa(\tau_{\frac{b}{a}})$. If $\nu(b)$ is even, $b = u\pi^{2k}$, then trace = $\left(\frac{a}{\mathcal{P}}\right)G(\tau_c)G(\tau_{\frac{b}{a}}) = \left(\frac{a}{\mathcal{P}}\right)\left(\frac{-2(a-1)u}{\mathcal{P}}\right)\left(\frac{ua}{\mathcal{P}}\right)G(\tau)^2 = \left(\frac{2(a-1)}{\mathcal{P}}\right)$. If $\nu(b)$ is odd, $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace = $\left(\frac{a}{\mathcal{P}}\right)$. But we saw above that $\left(\frac{2a(a-1)}{\mathcal{P}}\right) = 1$, so trace = $\left(\frac{a}{\mathcal{P}}\right) = \left(\frac{2(a-1)}{\mathcal{P}}\right)$. This finishes case (2) and thus completes the proof of Proposition 1.

Now we assume E/F is ramified, $E = F(\sqrt{\pi})$. We have a filtration $T \supset T_0 \supset T_1 \supset \dots$, where $T_n = \left\{ \begin{pmatrix} a & b \\ b\pi & a \end{pmatrix} \mid a \in 1 + \mathcal{P}^{2n+1}, b \in \mathcal{P}^n \right\}$. We have $[T : T_0] = 2$ and $[T_n : T_{n+1}] = q$ for $n \geq 1$. Recall that we have a bijection $\phi : \mathcal{O} \rightarrow T_0$, where we identify $\begin{pmatrix} a & b \\ b\pi & a \end{pmatrix} \in T_0$ with $a + b\sqrt{\pi} \in N^1$ [S]. ϕ is given by

$$\phi(x) = \frac{1 + \pi x^2}{1 - \pi x^2} + \sqrt{\pi} \frac{2x}{1 - \pi x^2},$$

$x \in \mathcal{O}$. Representatives for \mathcal{P}^n in \mathcal{O} can be taken to be $\{a_0 + a_1\pi + \dots + a_{n-1}\pi^{n-1} \mid a_i = 0 \text{ or } a_i = \varepsilon^j, 0 \leq j \leq q-2\}$.

PROPOSITION 2. *Suppose E/F is ramified.*

- (1) *Say $t \in T_i - T_{i+1}$, $t = \phi(x)$, $x = a_i\pi^i + \dots + a_{n-1}\pi^{n-1}$, with $a_i = \varepsilon^{j(t)}$, $0 \leq j(t) \leq q-2$. Then*

$$\text{trace } W_n(t, s(t)) = q^{\frac{2+i}{2}} (-1)^{j(t)} \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right)^{n+i+1} G(\tau).$$

- (2) *Say $t \in T - T_0$. Then $\text{trace } W_n(t, s(t)) = \left(\frac{-1}{\mathcal{P}}\right)^n$.*

Proof. We may use Lemma 5 in all cases. Assume first $t \in T_i - T_{i+1}$. Suppose that n and i are both even. With $x = a_i\pi^i + \dots + a_{n-1}\pi^{n-1}$, $\nu(x) = i$. If $\phi(x) = a + b\sqrt{\pi}$, then $\nu(b) = i$, $\nu(a-1) = 2i+1$, and $\nu(c) = i+1$, where $c = -\frac{2a^2(a-1)}{b}$. Then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = 1$. Therefore trace $W_n(t, s(t)) = q^{\frac{2i+1}{2}} G(\tau_c)$. But $G(\tau_c) = \left(\frac{-2}{\mathcal{P}}\right) G(\tau_{\frac{a-1}{b}})$. Now, $\frac{a-1}{b} = \pi x$, so $G(\tau_{\frac{a-1}{b}}) = G(\tau_{\pi x}) = \left(\frac{a_i + a_{i+1}\pi + \dots + a_{n-1}\pi^{n-i-1}}{\mathcal{P}}\right) G(\tau)$. With $a_i = \varepsilon^{j(t)}$, $a_i + a_{i+1}\pi + \dots + a_{n-1}\pi^{n-i-1} \in \varepsilon\mathcal{U}^2$, so $G(\tau_{\frac{a-1}{b}}) = \left(\frac{\varepsilon^{j(t)}}{\mathcal{P}}\right) G(\tau) = (-1)^{j(t)} G(\tau)$. So

$$\text{trace} = q^{\frac{2i+1}{2}} \left(\frac{-2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau) = q^{\frac{2i+1}{2}} (-1)^{j(t)} \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right)^{n+i+1} G(\tau).$$

If n is even and i is odd, then $\kappa(\tau_c) = 1$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$, so trace $= q^{\frac{2i+1}{2}} G(\tau_c) G(\tau_{\frac{b}{a}})$. We have

$$\begin{aligned} \frac{b}{a} &= \frac{2x}{1 + \pi x^2} \\ &= \frac{2a_i\pi^i}{1 + \pi x^2} \left[1 + \frac{a_{i+1}}{a_i}\pi + \dots + \frac{a_{n-1}}{a_i}\pi^{n-i-1} \right] \in \frac{2a_i\pi^i}{1 + \pi x^2} \mathcal{U}^2, \end{aligned}$$

so $G(\tau_{\frac{b}{a}}) = \left(\frac{2a}{\mathcal{P}}\right) G(\tau) = \left(\frac{2\varepsilon^{j(t)}}{\mathcal{P}}\right) G(\tau) = \left(\frac{2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau)$. Therefore, trace $= q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau)$.

If n is odd and i is even,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_{\frac{b}{a}}) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau).$$

If n is odd and i is odd,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_c) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau).$$

This completes the proof of (1).

Now suppose $t \notin T_0$. For elements of T/T_0 we use $\{t\} = \{-r\}$, $r \in T_0$. We therefore write $t = \begin{pmatrix} -a & -b \\ -b\pi & -a \end{pmatrix}$, with $a \in 1 + \mathcal{P}$, $b \in \mathcal{O}$, and $c = -\frac{2a^2(a+1)}{b}$. If n is even, then $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $\nu(b)$ is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace $= 1$. If $\nu(b)$ is odd, trace $= G(\tau_{-\frac{2(a+1)}{b}}) G(\tau_{\frac{b}{a}})$. Writing $b = u\pi^{2l+1}$, this equals $\left(\frac{-1}{\mathcal{P}}\right) \left(\frac{-2(a+1)u}{\mathcal{P}}\right) \left(\frac{ua}{\mathcal{P}}\right) = \left(\frac{2a(a+1)}{\mathcal{P}}\right)$. But $\nu(a-1) + \nu(a+1) = 2\nu(b) + 1$, with $\nu(a+1) = 0$ and $\nu(b) > 0$, so $a-1 \in \mathcal{P} \Rightarrow a+1 \in 2 + \mathcal{P} \subset 2\mathcal{U}^2 \Rightarrow$

$\left(\frac{a+1}{\mathcal{P}}\right) = \left(\frac{2}{\mathcal{P}}\right)$. Also, $a \in 1 + \mathcal{P} \Rightarrow \left(\frac{a}{\mathcal{P}}\right) = 1$, so $\left(\frac{2a(a+1)}{\mathcal{P}}\right) = \left(\frac{a}{\mathcal{P}}\right) = 1$, and therefore trace = 1.

If n is odd, trace = $\frac{G(\tau)}{G(\tau_{-a})} \kappa(\tau_{-\frac{2(a+1)}{b}}) \kappa(\tau_{\frac{b}{a}}) = \left(\frac{-1}{\mathcal{P}}\right) \kappa(\tau_{-\frac{2(a+1)}{b}}) \kappa(\tau_{\frac{b}{a}})$. If $\nu(b)$ is even, write $b = u\pi^{2k}$. Then trace = $\left(\frac{-1}{\mathcal{P}}\right) \left(\frac{-2(a+1)u}{\mathcal{P}}\right) G(\tau) \left(\frac{u}{\mathcal{P}}\right) G(\tau) = \left(\frac{2(a+1)}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right)$. But we still have $a + 1 \in 2\mathcal{U}^2$, so trace = $\left(\frac{-1}{\mathcal{P}}\right)$. If $\nu(b)$ is odd, $\kappa(\tau_{-\frac{2(a+1)}{b}}) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace = $\left(\frac{-1}{\mathcal{P}}\right)$. For $t \notin T_0$, therefore, trace = $\left(\frac{-1}{\mathcal{P}}\right)^n$. This completes the proof of Proposition 2.

4. Calculation of multiplicities. In this section we choose $\chi \in \widehat{T}$ with conductor $c(\chi)$ less than or equal to n , and we calculate $\langle \chi, W_n \rangle$, the multiplicity of χ in W_n , χ and W_n being considered as representations of T/T_n .

Assume first that E/F is unramified. Let us say that the conductor of the trivial character of T is zero, and we let θ_0 be the unique nontrivial character of conductor 1 such that $\theta_0^2 = 0$.

LEMMA 8. For $t \notin T_1$, $t = \begin{pmatrix} a & b \\ b\varepsilon & a \end{pmatrix}$, we have $\left(\frac{2(a-1)}{\mathcal{P}}\right) = -\theta_0(t)$.

Proof. We identify $t \in T$ with $\lambda = a + b\sqrt{\varepsilon} \in N^1$. Let $|x|_E$ be the valuation on E . If $|1 + \lambda|_E = 1$, we can write $\lambda = \frac{1+x\sqrt{\varepsilon}}{1-x\sqrt{\varepsilon}}$, $x \in \mathcal{O}$. Then $\lambda + \lambda^{-1} + 2 = \frac{4}{1-\varepsilon x^2}$, and $2(a - 1) = \lambda + \lambda^{-1} - 2 = \frac{4\varepsilon x^2}{1-\varepsilon x^2}$. It is proved in [S-Sh] that if $|1 + \lambda|_E = 1$, then $\left(\frac{\lambda + \lambda^{-1} + 2}{\mathcal{P}}\right) = \left(\frac{1-\varepsilon x^2}{\mathcal{P}}\right) = \theta_0(\lambda)$. Therefore, $\left(\frac{2(a-1)}{\mathcal{P}}\right) = \left(\frac{\lambda + \lambda^{-1} - 2}{\mathcal{P}}\right) = \left(\frac{4\varepsilon x^2(1-\varepsilon x^2)}{\mathcal{P}}\right) = -\left(\frac{1-\varepsilon x^2}{\mathcal{P}}\right) = -\theta_0(t)$. If $|1 + \lambda|_E > 0$, then $-\lambda \in 1 + \mathcal{P}_E$ (\mathcal{P}_E the prime ideal in E) and $\lambda = -s^2$, $s \in N^1$. Write $s = c + d\sqrt{\varepsilon}$. Then $\lambda = -s^2 \Rightarrow 2(a - 1) = -4c^2$, so $\left(\frac{2(a-1)}{\mathcal{P}}\right) = \left(\frac{-1}{\mathcal{P}}\right)$. But we also have $\lambda = -s^2 \Rightarrow \theta_0(\lambda) = \theta_0(-s^2) = \theta_0(-1)$, and it is proved in [S-Sh] that $\theta_0(-1) = -\left(\frac{-1}{\mathcal{P}}\right)$. Therefore, $\left(\frac{2(a-1)}{\mathcal{P}}\right) = \left(\frac{-1}{\mathcal{P}}\right) = -\theta_0(-1) = -\theta_0(\lambda)$. This completes the proof of Lemma 8.

PROPOSITION 3. Suppose E/F is unramified and $c(\chi) = i$.

- (1) If n is even and i is even, then $\langle \chi, W_n \rangle = 1$.
- (2) If n is even and i is odd, then $\langle \chi, W_n \rangle = 0$.
- (3) Say n is odd and i is even. Then $\langle \chi, W_n \rangle = 0$ if $\chi \neq 1$, and $\langle 1, W_n \rangle = 1$.
- (4) Say n is odd and i is odd. Then $\langle \chi, W_n \rangle = 1$ if $\chi \neq \theta_0$, and $\langle \theta_0, W_n \rangle = 0$.

Proof. Suppose $n = \omega(\tau)$ is even and $c(\chi) = i > 1$. Then

$$\langle \chi, W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} \bar{\chi}(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} \bar{\chi}(t) (-1)^m q^m \right].$$

But $\sum_{t \notin T_1} \bar{\chi}(t) = \sum_{t \in T} \bar{\chi}(t) - \sum_{t \in T_1} \bar{\chi}(t) = 0$, so

$$\begin{aligned} \langle \chi, W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \\ &\quad \times \left[q^n + \sum_{m=1}^{i-2} \left[(-1)^m q^m \sum_{t \in T_m} \bar{\chi}(t) - (-1)^m q^m \sum_{t \in T_{m+1}} \bar{\chi}(t) \right] \right. \\ &\quad \left. + \left[(-1)^{i-1} q^{i-1} \sum_{t \in T_{i-1}} \bar{\chi}(t) - (-1)^{i-1} q^{i-1} \sum_{t \in T_i} 1 \right] \right. \\ &\quad \left. + \sum_{m=i}^{n-1} \left[(-1)^m q^m q^{n-m} - (-1)^m q^m q^{n-m-1} \right] \right] \\ &= \frac{1}{(q+1)q^{n-1}} \left[q^n - (-1)^{i-1} q^{i-1} q^{n-i} \right. \\ &\quad \left. + \sum_{m=i}^{n-1} \left[(-1)^m q^n - (-1)^m q^{n-1} \right] \right] \\ &= \frac{1}{(q+1)q^{n-1}} \left[q^n - (-1)^{i-1} q^{n-1} + (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^m \right]. \end{aligned}$$

If i is even, this equals one, and if i is odd, it equals zero.

If n is even and $c(\chi) = 1$, then

$$\begin{aligned} \langle \chi, W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} \bar{\chi}(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^m q^m \right] \\ &= \frac{1}{(q+1)q^{n-1}} \left[q^n - q^{n-1} - (q^n - q^{n-1}) \right] = 0. \end{aligned}$$

Also, if n is even, then

$$\langle 1, W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} 1 + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^m q^m \right] = 1.$$

This proves (1) and (2) of Proposition 3.

Now suppose n is odd. If $c(\chi) = i > 1$ then

$$\langle \chi, W_n \rangle = \frac{1}{(q+1)q^{n-1}} \times \left[q^n - \sum_{t \notin T_1} \bar{\chi}(t)\theta_0(t) + \sum_{m=1}^{i-1} \sum_{t \in T_m - T_{m+1}} \bar{\chi}(t)(-1)^{m+1}q^m + \sum_{m=i}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1}q^m \right].$$

But $\sum_{t \notin T_1} \bar{\chi}(t)\theta_0(t) = 0$ and

$$\sum_{m=1}^{i-2} \sum_{t \in T_m - T_{m+1}} \bar{\chi}(t)(-1)^{m+1}q^m = 0,$$

so

$$\langle \chi, W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + (-1)^i q^{i-1} q^{n-i} + (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^{m+1} q^m \right].$$

If i is even, this equals zero and if i is odd, it equals one.

If $c(\chi) = 1$ or $\chi = 1$, then

$$\begin{aligned} \langle \chi, W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \left[q^n - \sum_{t \notin T_1} \bar{\chi}(t)\theta_0(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1}q^m \right] \\ &= \frac{1}{(q+1)q^{n-1}} \left[q^n - \sum_{t \in T} \bar{\chi}(t)\theta_0(t) + \sum_{t \in T_1} \bar{\chi}(t)\theta_0(t) \right] \\ &= \frac{q^n}{(q+1)q^{n-1}} - \langle \chi, \theta_0 \rangle + \frac{q^{n-1}}{(q+1)q^{n-1}} \\ &= 1 - \langle \chi, \theta_0 \rangle. \end{aligned}$$

This completes the proof of Proposition 3.

Now we assume E/F is ramified. Let θ_0 be the unique nontrivial character of T/T_0 .

PROPOSITION 4. *Let E/F be ramified. Then*

- (1) $\langle 1, W_n \rangle = 1$ if n is even or $\left(\frac{-1}{\mathcal{F}}\right) = 1$, and equals 0 otherwise.
- (2) $\langle \theta_0, W_n \rangle = 1 - \langle 1, W_n \rangle$.

Proof. We have

$$\langle 1, W_n \rangle = \frac{1}{2q^n} \left[q^n + \sum_{t \notin T_0} \left(\frac{-1}{\mathcal{P}} \right)^n + \sum_{i=0}^{n-1} \sum_{t \in T_i - T_{i+1}} q^{\frac{2i+1}{2}} (-1)^j \left(\frac{2}{\mathcal{P}} \right) \left(\frac{-1}{\mathcal{P}} \right)^{n+i+1} G(\tau) \right],$$

where j was defined in Proposition 2. Consider $\sum_{t \in T_i - T_{i+1}} (-1)^j$. Since $a_i = \varepsilon^j$, and $h \neq i \Rightarrow a_h$ can assume the values $0, 1, \varepsilon, \dots, \varepsilon^{q-2}$, this sum is zero, so $\langle 1, W_n \rangle = \frac{1}{2q^n} [q^n + (\frac{-1}{\mathcal{P}})^n q^n]$, which gives the result.

Similarly, $\langle \theta_0, W_n \rangle = \frac{1}{2q^n} [q^n + (\frac{-1}{\mathcal{P}})^n \sum_{t \notin T_0} \theta_0(t)]$. But $\sum_{t \notin T_0} \theta_0(t) = \sum_{t \in T} \theta_0(t) - \sum_{t \in T_0} \theta_0(t) = -q^n$, so $\langle \theta_0, W_n \rangle = \frac{1}{2} [1 - (\frac{-1}{\mathcal{P}})^n]$. This completes the proof of Proposition 4.

PROPOSITION 5. *Assume $c(\chi) = m > 0$. Then $\langle \chi, W_n \rangle$ equals 0 or 1, and exactly half of the characters χ of conductor m satisfy $\langle \chi, W_n \rangle = 1$.*

Proof. We have

$$\langle \chi, W_n \rangle = \frac{1}{2q^n} \left[q^n + \sum_{t \notin T_0} \bar{\chi}(t) \left(\frac{-1}{\mathcal{P}} \right)^n + \sum_{i=0}^{n-1} \sum_{t \in T_i - T_{i+1}} \bar{\chi}(t) q^{\frac{2i+1}{2}} \left(\frac{-1}{\mathcal{P}} \right)^{n+i+1} (-1)^{j(t)} G(\tau) \right],$$

where $j(t)$ is as in Proposition 2. Since χ is nontrivial on T_0 , $\sum_{t \notin T_0} \bar{\chi}(t) = 0$, so

$$\begin{aligned} \langle \chi, W_n \rangle = \frac{1}{2q^n} & \left[q^n + \left(\frac{2}{\mathcal{P}} \right) \left(\frac{-1}{\mathcal{P}} \right)^{n+1} G(\tau) \right. \\ & \times \left[\sum_{i=0}^{m-2} \left(\frac{-1}{\mathcal{P}} \right)^i q^{\frac{2i+1}{2}} \sum_{t \in T_i - T_{i+1}} \bar{\chi}(t) (-1)^{j(t)} \right. \\ & \quad \left. + \left(\frac{-1}{\mathcal{P}} \right)^{m-1} q^{\frac{2m-1}{2}} \sum_{t \in T_{m-1} - T_m} \bar{\chi}(t) (-1)^{j(t)} \right. \\ & \quad \left. \left. + \sum_{i=m}^{n-1} \left(\frac{-1}{\mathcal{P}} \right)^i q^{\frac{2i+1}{2}} \sum_{t \in T_i - T_{i+1}} (-1)^{j(t)} \right] \right]. \end{aligned}$$

As before, $\sum_{t \in T_i - T_{i+1}} (-1)^{j(t)} = 0$ for $m \leq i \leq n - 1$. Now consider $\sum_{t \in T_i - T_{i+1}} \bar{\chi}(t)(-1)^{j(t)}$ for $0 \leq i \leq m - 2$. Write this sum as

$$\sum_{S_1} \sum_{S_2} \bar{\chi}(\phi(a_i \pi^i + \dots + a_{n-1} \pi^{n-1}))(-1)^{j(t)},$$

where $S_1 = \{a_i, a_{i+1}, \dots, a_{m-2} | a_i \neq 0\}$, $S_2 = \{a_{m-1}, \dots, a_{n-1}\}$, and ϕ is the map on \mathcal{O} to T_0 which was recalled above. If $x \in \mathcal{P}^n$, then $\phi(x) \in T_n$. If $x, y \in \mathcal{O}$,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)} = \frac{a - b\sqrt{\pi}}{a + b\sqrt{\pi}} = c + d\sqrt{\pi},$$

where $a = 1 - \pi(x^2 + xy + y^2)$, $b = \pi xy(x + y)$, $c = \frac{a^2 + b^2 \pi}{a^2 - b^2 \pi}$, and $d = -\frac{2ab}{a^2 - b^2 \pi}$. Let $x = a_i \pi^i + \dots + a_{m-2} \pi^{m-2}$ and $y = a_{m-1} \pi^{m-1} + \dots + a_{n-1} \pi^{n-1}$. Then $\nu(x) = i$ and y either equals 0 or satisfies $\nu(y) \geq m - 1$. We need only consider the case $y \neq 0$. Then $\nu(x + y) \geq i$, so $\nu(c) \geq 2m + 1$ and $\nu(d) \geq m$. Therefore, $c + d\sqrt{\pi} \in T_m$. Since $\chi \equiv 1$ on T_m , we have $\chi(\phi(x))\chi(\phi(y)) = \chi(\phi(x + y))$. This shows that

$$\sum_{t \in T_i - T_{i+1}} \bar{\chi}(t)(-1)^{j(t)} = \sum_{S_1} \bar{\chi}(\phi(x))(-1)^{j(t)} \sum_{S_2} \bar{\chi}(\phi(y)).$$

But

$$\sum_{S_2} \bar{\chi}(\phi(y)) = \sum_{t \in T_{m-1}} \bar{\chi}(t) = 0$$

since $\chi \neq 1$ on T_{m-1} . Therefore,

$$\sum_{t \in T_i - T_{i+1}} \bar{\chi}(t)(-1)^{j(t)} = 0$$

for $0 \leq i \leq m - 2$.

Next, consider

$$\sum_{t \in T_m - T_{m+1}} \bar{\chi}(t)(-1)^{j(t)}.$$

Here, $t = \phi(a_{m-1} \pi^{m-1} + \dots + a_{n-1} \pi^{n-1})$, with $a_{m-1} = \varepsilon^{j(t)}$, $0 \leq j(t) \leq q - 2$. Let $x = a_{m-1} \pi^{m-1}$, $y = a_m \pi^m + \dots + a_{n-1} \pi^{n-1}$. As before,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)} \in T_m,$$

which makes

$$\begin{aligned}
 (13) \quad \sum_{t \in T_m - T_{m+1}} \bar{\chi}(t)(-1)^{j(t)} &= \sum_{S_2} \bar{\chi}(\phi(x))\bar{\chi}(\phi(y))(-1)^{j(t)} \\
 &= q^{n-m} \sum_{a_{m-1} \neq 0} \bar{\chi}(\phi(a_{m-1}\pi^{m-1}))(-1)^{j(t)},
 \end{aligned}$$

since $\phi(y) \in T_m$ and $\chi \equiv 1$ on T_m .

We have a map

$$\mathcal{P}^{m-1}/\mathcal{P}^m \xrightarrow{\phi} T_{m-1}/T_m \xrightarrow{\bar{\chi}} \mathbb{C}.$$

For $x, y \in \mathcal{P}^{m-1}$,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)} \in T_m,$$

so $\bar{\chi}\phi$ is an additive homomorphism on $\mathcal{P}^{m-1}/\mathcal{P}^m$ to \mathbb{C} . Letting $\psi = \bar{\chi}\phi$, (13) becomes

$$q^{n-m} \sum_{j=0}^{q-2} \psi(\varepsilon^j \pi^{m-1})(-1)^j = q^{n-m} \sum_{x \in \mathcal{O}/\mathcal{P}} \psi(\pi^{m-1}x^2) = q^{n-m} q^{\frac{1}{2}} G(\psi).$$

(Note that $\psi_{\pi^{m-1}}$ is a character of \mathcal{O}/\mathcal{P} .) We can now write

$$\langle \chi, W_n \rangle = \frac{1}{2q^n} \left[q^n + \left(\frac{2}{\mathcal{P}} \right) \left(\frac{-1}{\mathcal{P}} \right)^{n+m} q^n G(\tau)G(\psi) \right],$$

which equals 0 or 1. Notice that $\psi_{\pi^{m-1}} = \tau_{\pi^{n-1}\varepsilon^i u}$ for some $0 \leq i \leq q-2$, $u \in 1 + \mathcal{P}$. Then $G(\tau)G(\psi) = \left(\frac{-\varepsilon^i}{\mathcal{P}} \right) = \left(\frac{-1}{\mathcal{P}} \right) (-1)^i$, which takes on each value ± 1 for half the $q-1$ possible values of i . This completes the proof of Proposition 5.

If E/F is ramified, suppose that we replace τ by τ_u , $u \in \mathcal{U}$. Then the characters of a given conductor appearing in W_n^τ will be the same as those appearing in $W_n^{\tau_u}$ if $\left(\frac{u}{\mathcal{P}} \right) = 1$. If $\left(\frac{u}{\mathcal{P}} \right) = -1$, then the two sets of characters of a given conductor $m > 0$ appearing respectively in W_n^τ and $W_n^{\tau_u}$ are disjoint. By varying τ , we thus obtain all characters of conductor $m > 0$ in the restriction to T of some W^τ .

5. Decomposition of $W^\tau|_T$. In this section we use the results of the preceding section to determine the decomposition of $W^\tau|_T$.

LEMMA 9. For $2k > -n$, let $H_k = S(\mathcal{P}^{-k}, \mathcal{P}^{n+k})$. Then H_k is an invariant subspace for W^τ which is equivalent to $W_{n+2k}^{\tau_\alpha}$, where $\alpha = \pi^{-2k}$.

Proof. Recall that if $\beta \in F$ and $\alpha = \beta^2$, then $W^\tau = R^{-1}W^{\tau_\alpha}R$, where $(Rf)(x) = |\beta|^{\frac{1}{2}}f(\beta x)$. Let $\beta = \pi^{-k}$. Then $\omega(\tau_\alpha) = n + 2k$. Suppose $g \in K$. Then $f \in H_k \Rightarrow Rf \in S(\mathcal{O}, \mathcal{P}^{n+2k}) \Rightarrow W^{\tau_\alpha}(g)Rf \in S(\mathcal{O}, \mathcal{P}^{n+2k}) \Rightarrow R^{-1}W^{\tau_\alpha}(g)Rf \in H_k$. Thus H_k is invariant under W^τ . Also, $W^\tau(g)f = f$ if $f \in H_k$ and $g \in K_{n+2k}$. We thus have a representation of K/K_{n+2k} on H_k which is a subrepresentation of W^τ and which is equivalent to $W_{n+2k}^{\tau_\alpha}$. This completes the proof of Lemma 8.

Suppose $W^\tau(t)f = \chi(t)f$ for all $t \in T$. If $f \in S(\mathcal{P}^r, \mathcal{P}^s)$, choose k so that $-k \leq r$ and $n + k \geq s$. Then $S(\mathcal{P}^r, \mathcal{P}^s) \subset S(\mathcal{P}^{-k}, \mathcal{P}^{n+k}) = H_k$. Then the action of W^τ on H_k is equivalent to $W_{n+2k}^{\tau_\alpha}$, $\alpha = \pi^{-2k}$, by Lemma 9. This implies χ appears in $W_{n+2k}^{\tau_\alpha}$. We apply Proposition 3 to each of the representations $W_{n+2k}^{\tau_\alpha}$, $k \geq 0$, to obtain

PROPOSITION 6. Suppose E/F is unramified, $\omega(\tau) = n$, and $c(\chi) = i$.

- (1) If n is even and i is even, then $\langle \chi, W^\tau|_T \rangle = 1$.
- (2) If n is even and i is odd, then $\langle \chi, W^\tau|_T \rangle = 0$.
- (3) If n is odd and i is even, then $\langle \chi, W^\tau|_T \rangle = 0$ if $\chi \neq 1$, and $\langle 1, W^\tau|_T \rangle = 1$.
- (4) If n is odd and i is odd, then $\langle \chi, W^\tau|_T \rangle = 1$ if $\chi \neq \theta_0$, and $\langle \theta_0, W^\tau|_T \rangle = 0$.

We argue in a similar fashion if E/F is ramified. Applying Propositions 4 and 5, we obtain

PROPOSITION 7. Suppose E/F is ramified and $\omega(\tau) = n$.

- (1) $\langle 1, W^\tau|_T \rangle = 1$ if n is even or $\left(\frac{-1}{\mathcal{P}}\right) = 1$, and equals 0 otherwise.
- (2) $\langle \theta_0, W^\tau|_T \rangle = 1 - \langle 1, W^\tau|_T \rangle$.
- (3) If $c(\chi) = m > 0$, then

$$\langle \chi, W^\tau|_T \rangle = 1 \Leftrightarrow G(\tau)G(\psi) = \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right)^{n+m},$$

where $\psi = \bar{\chi}\phi$. Otherwise, $\langle \chi, W^\tau|_T \rangle = 0$.

(4) *Exactly half the characters χ of a given conductor satisfy $\langle \chi, W^\tau|_T \rangle = 1$.*

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PACIFIC JOURNAL OF MATHEMATICS

Volume 158 No. 2 April 1993

- On the extension of Lipschitz functions from boundaries of subvarieties to strongly pseudoconvex domains 201
K. ADACHI and HIROSHI KAJIMOTO
- On a nonlinear equation related to the geometry of the diffeomorphism group 223
DAVID DAI-WAI BAO, JACQUES LAFONTAINE and TUDOR S. RATIU
- Fixed points of boundary-preserving maps of surfaces 243
ROBERT F. BROWN and BRIAN SANDERSON
- On orthomorphisms between von Neumann preduals and a problem of Araki 265
L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT
- Primitive subalgebras of complex Lie algebras. I. Primitive subalgebras of the classical complex Lie algebras 273
I. V. CHEKALOV
- L^n solutions of the stationary and nonstationary Navier-Stokes equations in R^n 293
ZHI MIN CHEN
- Some applications of Bell's theorem to weakly pseudoconvex domains 305
XIAO JUN HUANG
- On isotropic submanifolds and evolution of quasiccaustics 317
STANISŁAW JANECZKO
- Currents, metrics and Moishezon manifolds 335
SHANYU JI
- Stationary surfaces in Minkowski spaces. I. A representation formula 353
JIANGFAN LI
- The dual pair $(U(1), U(1))$ over a p -adic field 365
COURTNEY HUGHES MOEN
- Any knot complement covers at most one knot complement 387
SHICHENG WANG and YING QING WU



0030-8730(1993)158:2;1-H