THE DUAL PAIR \((U(1), U(1))\) OVER A \(p\)-ADIC FIELD

COURTNEY HUGHES MOEN
THE DUAL PAIR \((U(1), U(1))\) OVER A \(p\)-ADIC FIELD

COURTNEY MOEN

We find an explicit decomposition for the metaplectic representation restricted to either member of the dual reductive pair \((U(1), U(1))\) in \(\tilde{SL}(2, F)\), where \(F\) is a \(p\)-adic field, with \(p\) odd.

1. Introduction and preliminaries. Let \(F\) be a \(p\)-adic field of odd residual characteristic with \(q\) being the order of the residue class field. Let \(\mathcal{O}\) be the ring of integers, \(\mathcal{P}\) the prime ideal, \(\mathcal{U}\) the units, \(\pi\) a prime element, and \(v\) the valuation on \(F\). Let \(G = SL(2, F)\).

For \(\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\), let \(x(\sigma) = c\) if \(c \neq 0\), and let \(x(\sigma) = d\) if \(c = 0\). Define a 2-cocycle on \(G\) by

\[
\alpha(g_1, g_2) = (x(g_1), x(g_2))(-x(g_1)x(g_2), x(g_1g_2)).
\]

This cocycle determines a nontrivial 2-sheeted covering group \(\tilde{G}\) of \(G\) [G1].

Let \(E\) be a quadratic extension of \(F\), and \(x \mapsto \bar{x}\) the Galois action. The group \(U(1)\) which preserves the Hermitian form \((x, y) \mapsto x\bar{y}\) on \(E\) is isomorphic to the group \(N^1\) of norm one elements in \(E\). The pair of subgroups \((U(1), U(1))\) of \(SL(2)\) form a dual reductive pair [H]. This dual pair is one of the simplest examples over a \(p\)-adic field. Some other basic examples of dual reductive pairs are discussed in [G2]. In this paper we determine the decomposition of the oscillator representation of \(\tilde{G}\) upon restriction to \(U(1) \subset \tilde{G}\).

The results in this paper have recently been applied by Rogawski to the problem of calculating the multiplicities of certain automorphic representations \(\pi\) of \(U(A)\) in the discrete spectrum of \(L^2(U(k) \backslash U(A))\), where \(U\) is a unitary group in 3 variables defined relative to a quadratic extension of number fields \(K/k\) [R1, R2]. I would like to thank Rogawski for several stimulating conversations and for encouraging me to publish this paper.

Let \(\tau\) be a character of \(F\). Choose a normalized measure \(\mu\) so that \(\mu(\mathcal{O}) = q^{|\omega(\tau)|/2}\), where \(\omega(\tau)\) is the conductor of \(\tau\). Denote this measure by \(d_\tau x\). Then if we define the Fourier transform on \(S(F)\), the space of locally compact functions on \(F\) with compact support,
by
\[ \hat{f}(x) = \int f(y)\tau(2xy)\,d\tau y, \]
we have \( \hat{f}(x) = f(-x) \). For \( a \in F \), we set \( \tau_a(x) = \tau(ax) \). Let
\[ \kappa(\tau) = \lim_{m \to -\infty} \int_{\mathcal{O}^m} \tau(x^2)\,d\tau x. \]
Recall [Sh] that \( \kappa(\tau) = 1 \) if \( \omega(\tau) \) even, and
\[ \kappa(\tau) = G(\tau) = q^{-\frac{1}{2}} \sum_{x \in \mathcal{O}/\mathcal{P}} \tau(\pi^{n-1}x^2) \]
if \( n = \omega(\tau) \) is odd. For \( u \in \mathcal{U} \), let \( (\frac{u}{\mathcal{P}}) = 1 \) if \( u \) is a square, and \( (\frac{u}{\mathcal{P}}) = -1 \) otherwise. Then we have \( G(\tau)^2 = (\frac{1}{\mathcal{P}}) \) and \( G(\tau_u) = (\frac{u}{\mathcal{P}})G(\tau) \) for \( u \in \mathcal{U} \).

We now define the metaplectic representation \( W = W^\tau \) of \( \tilde{G} \) associated to the quadratic form \( Q(x) = x^2 \) by specifying the action on generators [G1]. Here \( \zeta = \pm 1 \), and \( |a| \) is the absolute value on \( F \).

\[ W\left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \zeta \right) f(x) = \zeta \tau(bx^2)f(x), \]
\[ W\left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \zeta \right) f(x) = \zeta |a|^{\frac{1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} f(ax), \]
\[ W\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta \right) f(x) = \zeta \kappa(\tau)\hat{f}(x). \]

The cocycle defining \( \tilde{G} \) splits on the compact subgroup \( K = \text{SL}_2(\mathcal{O}) \) by a function \( s : K \to \mathbb{Z}_2 \). \( K \) thus lifts as a subgroup of \( \tilde{G} \) by \( k \mapsto (k, s(k)) \), and we may thus restrict \( W \) to obtain a representation of \( K \) on \( S(F) \). Note that \( U(1) \subset K \). Our goal is to find the characters of \( U(1) \) which appear in the restriction of \( W \) to \( U(1) \).

Let \( S(\mathcal{P}^r, \mathcal{P}^s) \) be the space of functions on \( F \) which have support on \( \mathcal{P}^r \) and which are constant on cosets of \( \mathcal{P}^s \) in \( \mathcal{P}^r \). Suppose \( \omega(\tau) = n \geq 1 \). Then \( S(\mathcal{O}, \mathcal{P}^n) \) is invariant under \( W^\tau \) restricted to \( K \), and the group
\[ K_n = \left\{ k \in K | k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{P}^n} \right\} \]
acts trivially on \( S(\mathcal{O}, \mathcal{P}^n) \). We thus obtain a representation \( W_n = W_n^\tau \) of \( K/K_n \cong \text{SL}_2(\mathcal{O}/\mathcal{P}^n) \) on \( S(\mathcal{O}, \mathcal{P}^n) \). Note that we may consider \( \tau \) as a character of \( \mathcal{O}/\mathcal{P}^n \).
2. **Calculation of the trace.** In this section we calculate the trace of $W_n(t)$, where $t$ denotes either an element of $T$ or its image in $\text{Sl}_2(\mathcal{O}/\mathcal{P}^n)$, and

$$T = \left\{ \begin{pmatrix} a & b \\ b \alpha & a \end{pmatrix} \middle| a^2 - b^2 \alpha = 1 \right\}$$

is the torus in $G$ corresponding to the quadratic extension $E = F(\sqrt{\alpha})$. It will suffice to let $\alpha = \tau$ or $\alpha = \epsilon$, a primitive $(q - 1)$ st root of unity in $\mathcal{O}$.

**Lemma 1.** For $t = (\begin{pmatrix} a & b \\ b \alpha & a \end{pmatrix}) \in T$, we have the decomposition

$$t, s(t) = \left( \begin{pmatrix} 1 & 0 \\ b \alpha & 1 \end{pmatrix}, 1 \right) \left( \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \right) \gamma(t),$$

where $\gamma(t) = (a, b)$ if $\alpha = \epsilon$, $b \in \mathcal{U}$, and $a \not\in \mathcal{U}$, and $\gamma(t) = 1$ otherwise. Also,

$$\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) = \left( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right) \gamma(t),$$

Proof. Both statements are clearly true if $b = 0$, so we suppose $b \neq 0$. A calculation shows that the right side of (1) equals $(t, (a, b\alpha a)\gamma(t))$. We must therefore show that $s(t) = (a, b\alpha a)\gamma(t)$ for $t \neq \pm I$. Recall [G] that $s(t) = (b\alpha, a)$ if $b \neq 0$ and $b\alpha \not\in \mathcal{U}$, and $s(t) = 1$ otherwise. First suppose $\alpha = \pi$. In this case $a \in \mathcal{U}$, so $(a, b\pi \alpha) = (a, b\pi) = s(t)$. Now suppose $\alpha = \epsilon$. If $b \not\in \mathcal{U}$, then $b^2 \epsilon \in \mathcal{P}^2 \Rightarrow a^2 = 1 + b^2 \epsilon \in 1 + \mathcal{P}^2 \subset \mathcal{U} \Rightarrow a \in \mathcal{U}$. Then $\gamma(t) = 1$, so $(a, b\alpha a)\gamma(t) = (a, b\alpha a) = s(t)$. If $b \in \mathcal{U}$, then $s(t) = 1$, so we must show $(a, b\alpha a)\gamma(t) = 1$. If $a \in \mathcal{U}$, then $\gamma(t) = 1$ so we need $(a, b\alpha a) = 1$, which is true since $a \in \mathcal{U}$ and $b \in \mathcal{U}$. If $a \not\in \mathcal{U}$, then $\gamma(t) = (a, b)$, so we must show $(a, b\alpha a)(a, b) = 1 \Rightarrow (a, b\alpha a) = 1$. But a $\not\in \mathcal{U}$ so $a \neq \epsilon \Rightarrow a^2 = 1 + b^2 \epsilon \in \mathcal{P}^2 \Rightarrow a^2 \epsilon \in 1 + \mathcal{P}^2$. This shows $-\epsilon \in \mathcal{U}^2$, so $(a, b\alpha a)(a, -\epsilon a) = (a, -\epsilon)(a, a) = (a, -a) = 1$.

**Lemma 2.** Suppose $t = (\begin{pmatrix} a & b \\ b \alpha & a \end{pmatrix}) \in T$ and $a \in \mathcal{U}$. Then for $f \in S(\mathcal{O}, \mathcal{P}^n)$,

$$(W_n(t, s(t))f)(x) = \frac{K(t, s)}{K(t_a)} \sum_{s \in \mathcal{O} \cap \mathcal{P}^n} K_{b\alpha a}(ax, s) \tau \left( \frac{b}{a} \right) s^2 f(s),$$
where, for \( c \in \mathcal{O} \),

\[
K_c(x, s) = q^{-n} \sum_{\mathcal{O}/\mathcal{P}^n} \tau(-cr^2)\tau(-2xr)\tau(2rs).
\]

**Proof.** For any \( \phi \in S(\mathcal{O}, \mathcal{P}^n) \), we have, for \( c \in \mathcal{O} \),

\[
\begin{align*}
(W \left( \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1 \right) \phi)(x) & = (W \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1 \right) \\
& \quad \times W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi)(x) \\
& = \frac{\kappa(\tau)}{\kappa(\tau_{-1})} (W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1 \right) \\
& \quad \times W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi)(-x) \\
& = \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})} (W \left( \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi)(-x).
\end{align*}
\]

But \( \phi \in S(\mathcal{O}, \mathcal{P}^n) \Rightarrow \hat{\phi} \in S(\mathcal{O}, \mathcal{P}^n) \), so for \( c \in \mathcal{O} \), we have

\[
W \left( \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \phi \in S(\mathcal{O}, \mathcal{P}^n).
\]

For any \( \psi \in S(\mathcal{O}, \mathcal{P}^n) \), we have

\[
\hat{\psi}(x) = \int \psi(y)\tau(2xy) d_\tau y = \int_{\mathcal{O}} \psi(y)\tau(2xy) d_\tau y
\]

\[
= \sum_{r \in \mathcal{O}/\mathcal{P}^n} \int_{\mathcal{P}^n} \psi(r+y)\tau(2x(r+y)) d_\tau y
\]

\[
= \sum_{r \in \mathcal{O}/\mathcal{P}^n} \psi(r)\tau(2xr) \int_{\mathcal{P}^n} \tau(2xy) d_\tau y.
\]

But \( y \mapsto \tau(2xy) \) is trivial on \( \mathcal{P}^n \Leftrightarrow x \in \mathcal{O} \), so \( \hat{\psi}(x) = 0 \) if \( x \not\in \mathcal{O} \), and if \( x \in \mathcal{O} \), we have

\[
\hat{\psi}(x) = q^{-\frac{n}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \psi(r)\tau(2xr).
\]
Therefore,

\[
(W \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1) \phi)(x) = \frac{\kappa(\tau)^2}{\kappa(\tau - 1)} q^{-\frac{n}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}^n} (W \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1) \times W \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1) \phi)(r) \tau(-2xr)
\]

\[
= \frac{\kappa(\tau)^2}{\kappa(\tau - 1)} q^{-\frac{n}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-cr^2)(W \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1) \phi)(r) \tau(-2xr)
\]

\[
= \frac{\kappa(\tau)^3}{\kappa(\tau - 1)} q^{-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-cr^2) \tau(-2xr) \sum_{s \in \mathcal{O}/\mathcal{P}^n} \phi(s) \tau(2rs).
\]

But

\[
\frac{\kappa(\tau)^3}{\kappa(\tau - 1)} = 1,
\]

so we get

\[
(W \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1) \phi)(x) = \sum_{s \in \mathcal{O}/\mathcal{P}^n} K_c(x, s) \phi(s),
\]

where for \( c \in \mathcal{O} \),

\[
K_c(x, s) = q^{-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-cr^2) \tau(-2xr) \tau(2rs).
\]

Now we calculate the action of \( W_n(t, s(t)) \) for \( a \in \mathcal{U} \). Note that in this case, \( \gamma(t) = 1 \). For \( f \in S(\mathcal{O}, \mathcal{P}^n) \), we have

\[
(W_n(t, s(t))f)(x) = \frac{\kappa(\tau)}{\kappa(\tau_a)} (W \begin{pmatrix} 1 & 0 \\ ba & 1 \end{pmatrix}, 1) W \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1) f)(x) = \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathcal{O}/\mathcal{P}^n} K_{baa}(ax, s) \tau(b^2 s^2 f)(s).
\]

Here we used the fact that

\[
a \in \mathcal{U} \Rightarrow W \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1) f \in S(\mathcal{O}, \mathcal{P}^n).
\]

This completes the proof of Lemma 2.
If \( a \in \mathcal{U} \), the action of \( W_n(t, s(t)) \) is therefore given by the kernel

\[
\frac{\kappa(\tau)}{\kappa(\tau_a)} K_{baa}(ax, s) \tau\left(\frac{b}{a} s^2\right).
\]

We now use this kernel to calculate the trace of \( W_n(t, s(t)) \) when \( a \in \mathcal{U} \). The kernel is a function defined on \( \mathcal{O} / \mathcal{P}_n \times \mathcal{O} / \mathcal{P}_n \), so we have

\[
(3) \text{ trace } W_n(t, s(t)) = \sum_{s \in \mathcal{O} / \mathcal{P}_n} \frac{\kappa(\tau)}{\kappa(\tau_a)} K_{baa}(as, s) \tau\left(\frac{b}{a} s^2\right)
\]

\[
= \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathcal{O} / \mathcal{P}_n} q^{-n} \sum_{r \in \mathcal{O} / \mathcal{P}_n} \tau(-b\alpha r^2) \tau(2rs(1-a)) \tau\left(\frac{b}{a} s^2\right)
\]

\[
= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} \sum_{r \in \mathcal{O} / \mathcal{P}_n} \tau(-b\alpha r^2) \sum_{s \in \mathcal{O} / \mathcal{P}_n} \tau\left(\frac{b}{a} s^2 + 2r(1-a)s\right).
\]

Suppose \( \nu(b) = k \). The inner sum can be written

\[
\sum_{u \in \mathcal{O} / \mathcal{P}_n-k} \sum_{v \in \mathcal{P}_n-k / \mathcal{P}_n} \tau\left(\frac{b}{a} (u + v)^2\right) \tau(2r(1-a)(u + v))
\]

\[
= \sum_{u \in \mathcal{O} / \mathcal{P}_n-k} \tau\left(\frac{b}{a} u^2\right) \tau(2r(1-a)u) \sum_{v \in \mathcal{P}_n-k / \mathcal{P}_n} \tau(2r(1-a)v)
\]

since \( \frac{b}{a} uv \in \mathcal{P}_n \) and \( \frac{b}{a} v^2 \in \mathcal{P}_n \).

Consider the sum

\[
\sum_{v \in \mathcal{P}_n-k / \mathcal{P}_n} \tau(2r(1-a)v).
\]

Since \( a \in \mathcal{U} \), we may have \( \nu(a-1) = 0 \) or \( \nu(a-1) > 0 \). Suppose first that \( \nu(a-1) = 0 \). Then \( \tau_{2r(1-a)} \) is trivial on \( \mathcal{P}_n-k \Leftrightarrow \omega(\tau_{2r(1-a)}) \leq n-k \Leftrightarrow r \in \mathcal{P}_k \). If \( r \notin \mathcal{P}_k \), we have

\[
\sum_{v \in \mathcal{P}_n-k / \mathcal{P}_n} \tau(2r(1-a)v) = 0,
\]

and (3) therefore equals

\[
(4) \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} q^k \sum_{r \in \mathcal{P}_k / \mathcal{P}_n} \tau(-b\alpha r^2) \sum_{u \in \mathcal{O} / \mathcal{P}_n-k} \tau\left(\frac{b}{a} u^2 + 2r(1-au)\right).
\]
The inner sum in (4) equals

\[ \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} \left( u^2 + \frac{2r(1-a)a}{b}u \right) \right) \]

\[ = \tau \left( -\frac{r^2(1-a)^2a}{b} \right) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} \left( u + \frac{r(1-a)a}{b} \right)^2 \right). \]

Since \( \nu(b) = k \) and \( u \in \mathcal{P}^k \), we have \( \nu \left( \frac{r(1-a)a}{b} \right) = \nu(r) - \nu(b) \geq 0 \), so \( \{ u + \frac{r(1-a)a}{b} \} = \mathcal{O}/\mathcal{P}^{n-k} \) and (5) equals

\[ \tau \left( -\frac{r^2(1-a)^2a}{b} \right) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} u^2 \right). \]

So if \( a \in \mathcal{U}, \ a - 1 \in \mathcal{U}, \) and \( \nu(b) = k \), we have

\[
(6) \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{P}^k / \mathcal{P}^n} \tau(-b\alpha ar^2)
\times \tau \left( -\frac{r^2(1-a)^2a}{b} \right) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} u^2 \right)
= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{P}^k / \mathcal{P}^n} \tau(c r^2) \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} u^2 \right),
\]

where \( c = -\frac{2a^2(a-1)}{b} \).

Now we consider the sum

\[ \sum_{v \in \mathcal{P}^{n-k} / \mathcal{P}^n} \tau(2r(1-a)v) \]

in the case when \( \nu(a - 1) > 0 \). We have \( a^2 - 1 = b^2\alpha \Rightarrow \nu(a - 1) + \nu(a + 1) = 2\nu(b) + \nu(\alpha) \). Since \( a - 1 \in \mathcal{P} \), we have \( a + 1 = (a - 1) + 2 \in \mathcal{U} \), so \( \nu(a - 1) = 2\nu(b) + \nu(\alpha) \). We therefore have \( \nu(a - 1) > \nu(b) = k \). This shows that \( \tau_{2r(1-a)} \) is trivial on \( \mathcal{P}^{n-k} \) for all \( r \in \mathcal{O}/\mathcal{P}^n \), and so (3) implies

\[
(7) \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(-b\alpha ar^2)
\times \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau \left( \frac{b}{a} u^2 + 2r(1-a)u \right).
\]
Considering (5) again, we have $\nu\left(\frac{(1-a)^2}{b}\right) > 0$, so if $a \in \mathcal{U}$, $a - 1 \in \mathcal{P}$, and $\nu(b) = k$, we have

\begin{equation}
\text{(8) trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{P}^{n-k}} \tau(-b\alpha r^2) \times \tau\left(-\frac{r^2(1-a)^2}{b}\right) \sum_{u \in \mathcal{P}^{n-k}} \tau\left(\frac{b}{a} u^2\right)
\end{equation}

\begin{equation}
= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^n q^k \sum_{r \in \mathcal{P}^{n-k}} \tau(cr^2) \sum_{u \in \mathcal{P}^{n-k}} \tau\left(\frac{b}{a} u^2\right),
\end{equation}

where $c = -\frac{2a^2(1-a)}{b}$.

We summarize (6) and (8) as follows

**Lemma 3.** Suppose $a \in \mathcal{U}$ and $\nu(b) = k$. Let $c = -\frac{2a^2(1-a)}{b}$. Then

\begin{equation}
\text{(9) trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathcal{P}^{n-k}} \tau(cr^2) \sum_{u \in \mathcal{P}^{n-k}} \tau\left(\frac{b}{a} u^2\right),
\end{equation}

where $l = k$ if $a - 1 \in \mathcal{U}$ and $l = 0$ if $a - 1 \in \mathcal{P}$.

To calculate these sums we need

**Lemma 4.** If $\omega(\tau) = n$ then $\sum_{x \in \mathcal{P}^{n-k}} \tau(x^2) = q^n \kappa(\tau)$.

**Proof.** Suppose $n$ is even. Then

\begin{align*}
\sum_{x \in \mathcal{P}^{n-k}} \tau(x^2) &= \sum_{u \in \mathcal{P}^{n-k}} \sum_{v \in \mathcal{P}^{n-k}} \tau((u + v)^2) \\
&= \sum_{u \in \mathcal{P}^{n-k}} \tau(u^2) \sum_{v \in \mathcal{P}^{n-k}} \tau(2uv).
\end{align*}

But $\nu \mapsto \tau(2uv)$ is trivial on $\mathcal{P}^{n-k} / \mathcal{P}^n \Leftrightarrow u = 0$, so the sum is just $q^n$ in this case.

If $n$ is odd, then

\begin{align*}
\sum_{x \in \mathcal{P}^{n-k}} \tau(x^2) &= \sum_{u \in \mathcal{P}^{n-k}} \tau(u^2) \sum_{v \in \mathcal{P}^{n-k}} \tau(2uv) \\
&= q^{n-1} \sum_{u \in \mathcal{P}^{n-k} / \mathcal{P}^{n-k+1}} \tau(u^2).
\end{align*}

In this case, $\nu \mapsto \tau(2uv)$ is trivial on $\mathcal{P}^{n-k+1} \Leftrightarrow u \in \mathcal{P}^{n-k+1}$, so the sum equals

$q^{\frac{n-1}{2}} \sum_{u \in \mathcal{P}^{n-1} / \mathcal{P}^{n+1}} \tau(u^2)$. 


Writing \( u = \pi \frac{n-1}{2} \), with \( r \in \mathcal{O}/\mathcal{P} \), the sum equals
\[
q^{n-1 \frac{1}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}} \tau(\pi^{n-1} r^2) = q^{\frac{n-1}{2}} q^{\frac{1}{2}} G(\tau) = q^{\frac{n}{2}} G(\tau).
\]

This completes the proof of Lemma 4.

Now we apply Lemma 4 to the sums in (9). First, \( \omega(\tau_c) = \omega(\tau) - \nu(\frac{b}{a}) = n - k \), so
\[
\sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a} u^2\right) = q^{\frac{n-k}{2}} \kappa(\tau_a).
\]

Suppose \( \nu(a - 1) = 0. \) Then
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(c \pi^{2k} u^2).
\]

Since \( \nu(c) = \nu\left(\frac{a-1}{b}\right) = -\nu(b) = -k \), \( \omega(\tau_{c\pi^{2k}}) = n - 2k - \nu(c) = n - k \), and we have
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = q^{\frac{n-k}{2}} \kappa(\tau_{c\pi^{2k}}) = q^{\frac{n-k}{2}} \kappa(\tau_c).
\]

Now suppose \( \nu(a - 1) > 0 \) and consider
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2).
\]

If \( \alpha = \varepsilon \) then \( \nu(a - 1) = 2\nu(b) = 2k \). We write
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) \sum_{v \in \mathcal{P}^{n-k} / \mathcal{P}^n} \tau(2cuv).
\]

But \( \omega(\tau_{2cu}) = n - \nu(cu) \leq n - k \Leftrightarrow \nu(cu) \geq k \), which is true for all \( u \in \mathcal{O} \), so
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = q^k \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) = q^k q^{\frac{n-k}{2}} \kappa(\tau_c),
\]
where we used Lemma 4 since \( \omega(\tau_c) = n - k \).

If \( \alpha = \pi \), then \( \nu(a - 1) = 2\nu(b) + 1 = 2k + 1 \). We write
\[
\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k-1}} \sum_{v \in \mathcal{P}^{n-k-1} / \mathcal{P}^n} \tau(c(u + v)^2)
\]
and argue as above to obtain
\[
\sum_{r \in \mathcal{F}/\mathcal{F}_n} \tau(cr^2) = q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c).
\]

Suppose that \(a \in \mathcal{U}\) and \(\nu(b) = k \geq 0\). We have now shown that if \(\nu(a - 1) = 0\), then we have
\[
(10) \quad \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_b) = \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_b).
\]

If \(\nu(a - 1) > 0\) and \(\alpha = \varepsilon\),
\[
(11) \quad \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{k} q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_b) = q^k \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_b).
\]

If \(\nu(a - 1) > 0\) and \(\alpha = \pi\),
\[
(12) \quad \text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_b) = q^{\frac{2k+1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_b).
\]

We can summarize (10), (11), and (12) as follows.

**Lemma 5.** If \(a \in \mathcal{U}\) and \(b \neq 0\), then
\[
\text{trace } W_n(t, s(t)) = q^{\nu(a-1)} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_b),
\]
where \(c = -\frac{a^2(1-a)}{b}\).

To calculate \(\text{trace } W_n(t, s(t))\) when \(a \in \mathcal{F}\) we need another decomposition. Note that since \(a \in \mathcal{F}\), we have \(\alpha = \varepsilon\) and \(b \in \mathcal{U}\).

**Lemma 6.**
\[
(t, s(t)) = \left(\begin{pmatrix} -\frac{1}{b} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & ab \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & \frac{a}{b} \\ 0 & 1 \end{pmatrix}, 1 \right).
\]

**Proof.** A calculation shows that the right side equals \((t, 1)\). Noting that \(s(t) = 1\) in this case completes the proof.
Suppose \( \nu(a) = m \geq 1 \) and \( \omega(\tau) = n \). Choose \( f \in S(\mathcal{O}, \mathcal{G}^n) \). Using Lemma 6, we see that

\[
(W_n(t, s(t))f)(x) = \left[ -\frac{1}{b\epsilon} \right]^\frac{1}{2} \frac{\kappa(\tau)}{\kappa(\tau_{-be})} (W \left( \begin{pmatrix} 1 \ a\epsilon \ 
\ 0 \ b\epsilon \\
1 \ 0 \end{pmatrix}, 1 \right)

\times W \left( \begin{pmatrix} 0 \ 1 \\
1 \ 0 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 1 \ a\epsilon \ 
\ 0 \ b\epsilon \\
1 \ 0 \end{pmatrix}, 1 \right) f \left( -\frac{1}{b\epsilon} x \right)

= \frac{\kappa(\tau)}{\kappa(\tau_{-be})} \tau \left( a\epsilon \left(-\frac{1}{b\epsilon}\right)^2 \right)

\times \left( W \left( \begin{pmatrix} 0 \ 1 \\
1 \ 0 \end{pmatrix}, 1 \right) W \left( \begin{pmatrix} 1 \ a\epsilon \ 
\ 0 \ b\epsilon \\
1 \ 0 \end{pmatrix}, 1 \right) f \left( -\frac{1}{b\epsilon} x \right)

= \frac{\kappa(\tau)^2}{\kappa(\tau_{-be})} \tau \left( \frac{a\epsilon}{b\epsilon} x^2 \right) q^{-\frac{n}{2}} \sum_{s \in \mathcal{G}/\mathcal{G}^n} (W \left( \begin{pmatrix} 1 \ a\epsilon \ 
\ 0 \ b\epsilon \\
1 \ 0 \end{pmatrix}, 1 \right) f \left( \frac{2sx}{b\epsilon} \right)

= \sum_{s \in \mathcal{G}/\mathcal{G}^n} K(x, s) f(s),

where

\[
K(x, s) = q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-be})} \tau \left( \frac{a\epsilon}{b\epsilon} x^2 \right) \tau \left( \frac{a\epsilon}{b\epsilon} s^2 \right) \tau \left( -\frac{2sx}{b\epsilon} \right).
\]

Since \( \frac{a\epsilon}{b\epsilon} \in \mathcal{O} \) and \( -\frac{2sx}{b\epsilon} \in \mathcal{O} \),

\[
\text{trace } W_n(t, s(t)) = \sum_{s \in \mathcal{G}/\mathcal{G}^n} K(s, s)

= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-be})} \sum_{s \in \mathcal{G}/\mathcal{G}^n} \tau \left( \frac{a\epsilon}{b\epsilon} s^2 \right) \tau \left( \frac{a\epsilon}{b\epsilon} s^2 \right) \tau \left( -\frac{2sx^2}{b\epsilon} \right)

= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-be})} \sum_{s \in \mathcal{G}/\mathcal{G}^n} \tau(cs^2),
\]

where \( c = \frac{2(a-1)}{be} \). Since \( \nu(c) = \nu(a-1) - \nu(b) = 0 \), we have \( \omega(\tau_c) = n \). Using Lemma 4, we have

**Lemma 7.** If \( a \in \mathcal{G} \), then

\[
\text{trace } W_n(t, s(t)) = \frac{\kappa(\tau)^2}{\kappa(\tau_{-be})} \kappa(\tau_c),
\]

where \( c = \frac{2(a-1)}{be} \).
3. Further calculation of the trace. We now refine the formulas in Lemma 5 and Lemma 7. Suppose $E = F(\sqrt{\varepsilon})$. Letting $T_n = T \cap K_n$, we have a filtration $T \supset T_1 \supset \ldots$, with $[T : T_1] = q + 1$ and $[T_i : T_{i+1}] = q$ for $i \geq 1$. Let $n = \omega(\tau)$.

**PROPOSITION 1.** Suppose $E/F$ is unramified.

(1) For $t \in T_k - T_{k+1}, \ k \geq 1$, trace $W_n(t, s(t)) = (-1)^{n-k} q^k$.

(2) For $t \notin T_1$, trace $W_n(t, s(t)) = \left(\frac{2(a-1)}{a}\right)^n$.

**Proof.** Assume first $t \in T_1$. Then $a - 1 \in \mathcal{P}$ and $b \in \mathcal{P}$. We have $\nu(a) = 0$. If $t \in T_k - T_{k+1}$, then $\nu(b) = k \geq 0$. We apply Lemma 5.

We have $\nu(c) = \nu\left(\frac{b}{a}\right) = k$. Also, $\nu(a-1) = 2\nu(b) = 2k$. If $n$ is even, we have $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $k$ is even, then $\kappa(\tau_c) = \kappa(\tau_a^b) = 1$ and so trace $= q^k$. If $n$ is even and $k$ is odd, $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_a^b) = G(\tau_a^b)$, so trace $= q^k G(\tau_c) G(\tau_a^b)$. Letting $b = u\pi^k$ and $a - 1 = v\pi^{2k}$, we have $c = -\frac{2a^2}{b}$, so trace $= q^k \left(-\frac{2uv}{u^2}\right) G(\tau)^2 = q^k \left(\frac{2uv}{u^2}\right) = q^k \left(\frac{2vu}{u^2}\right)$. But $a - 1 \in \mathcal{P} \Rightarrow a \in \mathcal{U}^2 \Rightarrow \left(\frac{a}{\mathcal{P}}\right) = 1$. Also, $a^2 = (1 + v\pi^{2k})^2 = 1 + 2v \pi^{2k} + v^2 \pi^{4k}$, and $1 + b^2 = 1 + u^2 \pi^{2k}$. But $a^2 = 1 + b^2 = 2v \pi^{2k} + v^2 \pi^{4k} \Rightarrow u^2 = 2v + v^2 \pi^{2k} \Rightarrow 2v = u^2 - v^2 \pi^{4k} = u^2 (1 - \frac{v^2 \pi^{2k}}{u^2}) \in u^2 (1 + \mathcal{P}) \subset \mathcal{U}^2$, which implies $2v$ is not a square $\Rightarrow \left(\frac{2v}{\mathcal{P}}\right) = -1$, so trace $= -q^k$.

If $n$ is odd then $\frac{\kappa(\tau_a)}{\kappa(\tau_c)} = \frac{G(\tau)}{G(\tau_a)} = \left(\frac{a}{\mathcal{P}}\right)$. If $k$ is even, then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_a^b) = G(\tau_a^b)$. Arguing as in the case of $n$ even and $k$ odd, we have trace $= q^k \left(\frac{a}{\mathcal{P}}\right) G(\tau_c) G(\tau_a^b) = q^k \left(\frac{2v}{u^2}\right) = -q^k$. If $k$ is odd, then $\kappa(\tau_c) = \kappa(\tau_a^b) = 1 \Rightarrow$ trace $= q^k \left(\frac{a}{\mathcal{P}}\right)$. But $a - 1 \in \mathcal{P} \Rightarrow \left(\frac{a}{\mathcal{P}}\right) = 1$, so trace $= q^k$. This completes the proof of (1) of Proposition 1.

Now assume $t \notin T_1$. Then $a - 1 \in \mathcal{U}$ or $b \in \mathcal{U}$. We consider various cases: (1) $a - 1 \in \mathcal{U}$, $b \in \mathcal{U}$; (2) $a - 1 \in \mathcal{U}$, $b \in \mathcal{P}$; (3) $a - 1 \in \mathcal{P}$, $b \in \mathcal{U}$. Case (3) cannot arise, since $a^2 - 1 = b^2 = \nu(a - 1) + \nu(a + 1) = 2\nu(b)$. Then $\nu(a - 1) > 0 \Rightarrow \nu(b) > 0$, which is a contradiction.

We first consider case (1). In this case, we have $\nu(a - 1) = 0$, $\nu(b) = 0$, and we may have $a \in \mathcal{U}$ or $a \in \mathcal{P}$. Suppose first $a \in \mathcal{U}$. We use Lemma 5. If $n$ is even, $\kappa(\tau) = \kappa(\tau_a) = 1$. Also, $\nu\left(\frac{b}{a}\right) = \nu(c) = 0$, so $\kappa(\tau_c) = \kappa(\tau_b) = 1$. Since $\nu(a - 1) = 0$, trace $= 1$.

If $n$ is odd, trace $= \frac{G(\tau)}{G(\tau_a)} G(\tau_c) G(\tau_a^b) = \left(\frac{a}{\mathcal{P}}\right) \left(\frac{c}{\mathcal{P}}\right) \left(\frac{b}{\mathcal{P}}\right) G(\tau)^2 = \left(\frac{2(a-1)}{\mathcal{P}}\right)$. Now suppose $a \in \mathcal{P}$. Then we must use Lemma 7. If $n$ is even,
\[ \kappa(\tau) = \kappa(\tau_{-b\epsilon}) = \kappa(\tau_c) = 1, \text{ so trace } = 1. \] If \( n \) is odd, \( \text{trace } = \frac{G(\tau)}{G(\tau_{-b\epsilon})} G(\tau_c) = \left( \frac{b\epsilon}{\tau} \right) = \left( \frac{2(a-1)}{-\delta} \right). \]

We next consider case (2). Now we have \( a-1 \in \mathcal{U} \) and \( b \in \mathcal{P} \), so \( a \in \mathcal{U} \) and we can use Lemma 5. If \( n \) is even, then \( \kappa(\tau) = \kappa(\tau_a) = 1. \) If in addition \( \nu(b) \) is even, then \( \kappa(\tau_c) = \kappa(\tau_{b\epsilon}) = 1, \) so trace = 1. If \( \nu(b) \) is odd, then \( \text{trace } = G(\tau_c)G(\tau_{b\epsilon}). \) Writing \( b = u\pi^{2k+1} \), this equals \( \left( \frac{-2(a-1)}{\delta} \right) \left( \frac{u\epsilon}{\tau} \right) G(\tau)^2 = \left( \frac{2(a-1)}{\delta} \right). \) We claim \( \left( \frac{2(a-1)}{\delta} \right) = 1. \) We have \( \nu(a-1) + \nu(a+1) = 2\nu(b) \geq 2, \) so \( a-1 \in \mathcal{U} \Rightarrow a+1 \in \mathcal{P} \Rightarrow a = -1+d, \) \( d \in \mathcal{P}. \) This shows \( a-1 = -2+d = -2(1-\frac{1}{2}d) \in -2\mathbb{U}_1 \subset -2\mathbb{U}^2, \) so \( \left( \frac{a-1}{\delta} \right) = (-\frac{2}{\delta}). \) Also, \( a = -1+d \in (-1)\mathbb{U}_1 \Rightarrow \left( \frac{a}{\delta} \right) = (-\frac{1}{\delta}). \)

Therefore, \( \left( \frac{2(a-1)}{\delta} \right) = \left( \frac{2}{\delta} \right) \left( \frac{a-1}{\delta} \right) = \left( \frac{2}{\delta} \right) \left( -\frac{1}{\delta} \right) \left( -\frac{2}{\delta} \right) = 1, \) so in this case trace = 1.

Now suppose \( n \) is odd. Then \( \kappa(\tau) = G(\tau) \) and \( \kappa(\tau_a) = G(\tau_a), \) so trace = \( \left( \frac{a}{\delta} \right) \kappa(\tau_c) \kappa(\tau_{b\epsilon}). \) If \( \nu(b) \) is even, \( b = u\pi^{2k}, \) then \( \text{trace } = \left( \frac{a}{\delta} \right) G(\tau_c)G(\tau_{b\epsilon}) = \left( \frac{a}{\delta} \right) \left( \frac{-2(a-1)}{\delta} \right) \left( \frac{u\epsilon}{\tau} \right) G(\tau)^2 = \left( \frac{2(a-1)}{\delta} \right). \) If \( \nu(b) \) is odd, \( \kappa(\tau_c) = \kappa(\tau_{b\epsilon}) = 1, \) so trace = \( \left( \frac{a}{\delta} \right) \). But we saw above that \( \left( \frac{2(a-1)}{\delta} \right) = 1, \) so trace = \( \left( \frac{a}{\delta} \right) = \left( \frac{2(a-1)}{\delta} \right). \) This finishes case (2) and thus completes the proof of Proposition 1.

Now we assume \( E/F \) is ramified, \( E = F(\sqrt{\pi}). \) We have a filtration \( T \supset T_0 \supset T_1 \supset \ldots, \) where \( T_n = \{ (\frac{a}{b\pi a}) | a \in 1+\mathcal{P}^{2n+1}, b \in \mathcal{P}^n \}. \) We have \([T : T_0] = 2\) and \([T_n : T_{n+1}] = q\) for \( n \geq 1. \) Recall that we have a bijection \( \phi : \mathcal{O} \rightarrow T_0, \) where we identify \( (\frac{a}{b\pi a}) \in T_0 \) with \( a+b\sqrt{\pi} \in N^1 \) [S]. \( \phi \) is given by

\[ \phi(x) = \frac{1+\pi x^2}{1-\pi x^2} + \sqrt{\pi} \frac{2x}{1-\pi x^2}, \]

\( x \in \mathcal{O}. \) Representatives for \( \mathcal{P}^n \) in \( \mathcal{O} \) can be taken to be \( \{a_0+a_1\pi+\ldots+a_{n-1}\pi^{n-1} | a_i = 0 \text{ or } a_i = e^j, 0 \leq j \leq q-2\}. \)

**Proposition 2.** Suppose \( E/F \) is ramified.

(1) Say \( t \in T_i - T_{i+1}, \) \( t = \phi(x), \) \( x = a_i\pi^i + \ldots + a_{n-1}\pi^{n-1}, \) with \( a_i = e^{j(t)}, 0 \leq j(t) \leq q-2. \) Then

\[ \text{trace } W_n(t, s(t)) = q^{\frac{2i+1}{2}} (-1)^{(t)} \left( \frac{2}{\mathcal{P}} \right) \left( \frac{-1}{\mathcal{P}} \right)^{n+i+1} G(\tau). \]

(2) Say \( t \in T - T_0. \) Then \( \text{trace } W_n(t, s(t)) = \left( \frac{-1}{\mathcal{P}} \right)^n. \)
Proof. We may use Lemma 5 in all cases. Assume first $t \in T_i - T_{i+1}$. Suppose that $n$ and $i$ are both even. With $x = a_i \pi^i + \cdots + a_{n-1} \pi^{n-1}$, $\nu(x) = i$. If $\phi(x) = a + b \sqrt{\pi}$, then $\nu(b) = i$, $\nu(a-1) = 2i+1$, and $\nu(c) = i+1$, where $c = -\frac{2a^2(a-1)}{b}$. Then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_b^a) = 1$. Therefore trace $W_n(t, s(t)) = q^{\frac{2i+1}{2}} G(\tau_c)$. But $G(\tau_c) = (-2^{\frac{i}{2}}) G(\tau_{\frac{n-1}{b}})$. Now, $\frac{a-1}{b} = \pi x$, so $G(\tau_{\frac{n-1}{b}}) = G(\tau_{bx}) = (\frac{a_i + a_{i+1} \pi + \cdots + a_{n-1} \pi^{n-i-1}}{a_i}) G(\tau)$. With $a_i = e^{i(t)}$, $a_i + a_{i+1} \pi + \cdots + a_{n-1} \pi^{n-i-1} \in \mathcal{U}$, so $G(\tau_{\frac{n-1}{b}}) = (\pi^{i(t)}) G(\tau) = (-1)^{j(t)} G(\tau)$. So

$$\text{trace} = q^{\frac{2i+1}{2}} \left(\frac{-2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau) = q^{\frac{2i+1}{2}} (-1)^{j(t)} \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right)^{n+i+1} G(\tau).$$

If $n$ is even and $i$ is odd, then $\kappa(\tau_c) = 1$ and $\kappa(\tau_b^a) = G(\tau_b^a)$, so trace $= q^{\frac{2i+1}{2}} G(\tau_c) G(\tau_b^a)$. We have

$$\frac{b}{a} = \frac{2x}{1 + \pi x^2} = \frac{2a_i \pi^i}{1 + \pi x^2} \left[1 + \frac{a_{i+1} \pi}{a_i} + \cdots + \frac{a_{n-1} \pi^{n-i-1}}{a_i}\right] \in \frac{2a_i \pi^i}{1 + \pi x^2} \mathcal{U},$$

so $G(\tau_b^a) = (\frac{2a_i}{\pi}) G(\tau) = (\frac{2e^{i(t)}}{\mathcal{P}}) G(\tau) = (\frac{\pi}{\mathcal{P}}) (-1)^{j(t)} G(\tau)$. Therefore, trace $= q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau)$.

If $n$ is odd and $i$ is even,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_b^a) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau).$$

If $n$ is odd and $i$ is odd,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_c) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau).$$

This completes the proof of (1).

Now suppose $t \notin T_0$. For elements of $T/T_0$ we use $\{t\} = \{-r\}$, $r \in T_0$. We therefore write $t = (-a - b)$, with $a \in 1 + \mathcal{P}, b \in \mathcal{P}$, and $c = -\frac{2a^2(a+1)}{b^2}$. If $n$ is even, then $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $\nu(b)$ is even, then $\kappa(\tau_c) = \kappa(\tau_b^a) = 1$, so trace $= 1$. If $\nu(b)$ is odd, trace $= G(\tau_{\frac{2i(a+1)}{b}}) G(\tau_b^a)$. Writing $b = u \pi^{2l+1}$, this equals $(\frac{1}{\mathcal{P}}) (-\frac{2(a+1)}{\mathcal{P}}) u = (\frac{2a(a+1)}{\mathcal{P}})$. But $\nu(a-1) + \nu(a+1) = 2\nu(b) + 1$, with $\nu(a+1) = 0$ and $\nu(b) > 0$, so $a - 1 \in \mathcal{P} \Rightarrow a + 1 \in 2 + \mathcal{P} \subset 2\mathcal{U}$. Then trace $= \left(\frac{2}{\mathcal{P}}\right) \left(\frac{-1}{\mathcal{P}}\right) (-1)^{j(t)} G(\tau).$
\[ (a + 1) = \left( \frac{2}{p} \right). \] Also, \( a \in 1 + \mathcal{P} \Rightarrow (\frac{a}{p}) = 1 \), so \( (\frac{2a(a+1)}{p}) = (\frac{a}{p}) = 1 \), and therefore \( \text{trace} = 1 \).

If \( n \) is odd, \( \text{trace} = \frac{G(\tau)}{G(\tau^{-1})} \kappa(\tau^{-\frac{2(a+1)}{b}}) \kappa(\tau^{-\frac{b}{a}}) \). If \( \nu(b) \) is even, write \( b = u\pi 2^k \). Then \( \text{trace} = (\frac{-1}{p}) (\frac{2(a+1)}{b}) G(\tau) (\frac{u}{p}) G(\tau) = (\frac{2(a+1)}{p}) (\frac{-1}{p}). \) But we still have \( a + 1 \in 2\mathbb{Z}/2 \), so \( \text{trace} = (\frac{-1}{p}). \) If \( \nu(b) \) is odd, \( \kappa(\tau^{-\frac{2(a+1)}{b}}) = \kappa(\tau^{-\frac{b}{a}}) = 1 \), so \( \text{trace} = (\frac{-1}{p}). \) For \( t \notin T_0 \), therefore, \( \text{trace} = (\frac{-1}{p})^n \). This completes the proof of Proposition 2.

4. Calculation of multiplicities. In this section we choose \( \chi \in \hat{T} \) with conductor \( c(\chi) \) less than or equal to \( n \), and we calculate \( \langle \chi, W_n \rangle \), the multiplicity of \( \chi \) in \( W_n \), \( \chi \) and \( W_n \) being considered as representations of \( T/T_n \).

Assume first that \( E/F \) is unramified. Let us say that the conductor of the trivial character of \( T \) is zero, and we let \( \theta_0 \) be the unique nontrivial character of conductor 1 such that \( \theta_0^2 = 0 \).

**Lemma 8.** For \( t \notin T_1 \), \( t = (\frac{a}{b}, e) \), we have \( (\frac{2(a-1)}{p}) = -\theta_0(t). \)

**Proof.** We identify \( t \in T \) with \( \lambda = a + b\sqrt{e} \in \mathbb{N}^1 \). Let \( |\lambda|_E \) be the valuation on \( E \). If \( |1 + \lambda|_E = 1 \), we can write \( \lambda = \frac{1 + x\sqrt{e}}{1 - x\sqrt{e}} \), \( x \in \mathcal{O} \). Then \( \lambda + \lambda^{-1} + 2 = \frac{4}{1 - ex} \), and \( 2(a - 1) = \lambda + \lambda^{-1} - 2 = \frac{4ex^2}{1 - ex} \). It is proved in [S-Sh] that if \( |1 + \lambda|_E = 1 \), then \( (\frac{\lambda + \lambda^{-1} + 2}{p}) = (\frac{1 - ex^2}{p}) = \theta_0(\lambda) \).

Therefore, \( (\frac{2(a-1)}{p}) = (\frac{a - 1}{p}) = (\frac{4ex^2(1 - ex^2)}{p}) = (\frac{-1 - ex^2}{p}) = -\theta_0(t). \) If \( |1 + \lambda|_E > 0 \), then \( -\lambda \in 1 + \mathcal{P}_E \left( \mathcal{P}_E \right. \) the prime ideal in \( E \) \) and \( \lambda = -s^2, s \in \mathbb{N}^1 \). Write \( s = c + d\sqrt{e} \). Then \( \lambda = -s^2 \Rightarrow 2(a - 1) = -4c^2 \), so \( (\frac{2(a-1)}{p}) = (\frac{-1}{p}). \) But we also have \( \lambda = -s^2 \Rightarrow \theta_0(\lambda) = \theta_0(-s^2) = \theta_0(-1), \) and it is proved in [S-Sh] that \( \theta_0(-1) = (\frac{-1}{p}). \) Therefore, \( (\frac{2(a-1)}{p}) = (\frac{-1}{p}) = -\theta_0(-1) = -\theta_0(\lambda). \) This completes the proof of Lemma 8.

**Proposition 3.** Suppose \( E/F \) is unramified and \( c(\chi) = i \).

1. If \( n \) is even and \( i \) is even, then \( \langle \chi, W_n \rangle = 1 \).
2. If \( n \) is even and \( i \) is odd, then \( \langle \chi, W_n \rangle = 0 \).
3. Say \( n \) is odd and \( i \) is even. Then \( \langle \chi, W_n \rangle = 0 \) if \( \chi \neq 1 \), and \( \langle 1, W_n \rangle = 1 \).
4. Say \( n \) is odd and \( i \) is odd. Then \( \langle \chi, W_n \rangle = 1 \) if \( \chi \neq \theta_0 \), and \( \langle \theta_0, W_n \rangle = 0 \).
Proof. Suppose \( n = \omega(\tau) \) is even and \( c(\chi) = i > 1 \). Then

\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n + \sum_{t \notin T_i} \bar{x}(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} \bar{x}(t)(-1)^{m}q^{m} \right].
\]

But \( \sum_{t \notin T_i} \bar{x}(t) = \sum_{t \in T} \bar{x}(t) - \sum_{t \in T_i} \bar{x}(t) = 0 \), so

\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n - \sum_{t \in T_m - T_{m+1}} \bar{x}(t)(-1)^{m}q^{m} \right].
\]

If \( i \) is even, this equals one, and if \( i \) is odd, it equals zero.

If \( n \) is even and \( c(\chi) = 1 \), then

\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n - q^{n-1} - (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^{m} \right].
\]

Also, if \( n \) is even, then

\[
\langle 1, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n + \sum_{t \notin T_i} 1 + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m}q^{m} \right] = 1.
\]

This proves (1) and (2) of Proposition 3.
Now suppose $n$ is odd. If $c(\chi) = i > 1$ then
\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left( q^n - \sum_{t \notin T_1} \overline{\chi}(t) \theta_0(t) + \sum_{m=1}^{i-1} \sum_{t \in T_m - T_{m+1}} \overline{\chi}(t)(-1)^{m+1}q^m + \sum_{m=i}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1}q^m \right).
\]

But $\sum_{t \notin T_1} \overline{\chi}(t) \theta_0(t) = 0$ and
\[
\sum_{m=1}^{i-2} \sum_{t \in T_m - T_{m+1}} \overline{\chi}(t)(-1)^{m+1}q^m = 0,
\]
so
\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n + (-1)^i q^{i-1}q^{n-i} + (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^{m+1} \right].
\]
If $i$ is even, this equals zero and if $i$ is odd, it equals one.

If $c(\chi) = 1$ or $\chi = 1$, then
\[
\langle \chi, W_n \rangle = \frac{1}{(q + 1)q^{n-1}} \left[ q^n - \sum_{t \notin T_1} \overline{\chi}(t) \theta_0(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1}q^m \right]
\]
\[
= \frac{1}{(q + 1)q^{n-1}} \left[ q^n - \sum_{t \in T} \overline{\chi}(t) \theta_0(t) + \sum_{t \in T_1} \overline{\chi}(t) \theta_0(t) \right]
\]
\[
= \frac{q^n}{(q + 1)q^{n-1}} - \langle \chi, \theta_0 \rangle + \frac{q^{n-1}}{(q + 1)q^{n-1}}
\]
\[
= 1 - \langle \chi, \theta_0 \rangle.
\]

This completes the proof of Proposition 3.

Now we assume $E/F$ is ramified. Let $\theta_0$ be the unique nontrivial character of $T/T_0$.

**Proposition 4.** Let $E/F$ be ramified. Then
1. $\langle 1, W_n \rangle = 1$ if $n$ is even or $\left( \frac{-1}{p} \right) = 1$, and equals 0 otherwise.
2. $\langle \theta_0, W_n \rangle = 1 - \langle 1, W_n \rangle$. 


Proof. We have
\[
\langle 1, W_n \rangle = \frac{1}{2q^n} \left[ q^n + \sum_{t \not\in T_0} \left( -\frac{1}{\mathcal{P}} \right)^n \right]
\]
\[
+ \sum_{i=0}^{n-1} \sum_{t \in T_i - T_{i+1}} q^{\frac{i}{2} \cdot \frac{i+1}{2}} (-1)^j \left( \frac{2}{\mathcal{P}} \right) \left( -\frac{1}{\mathcal{P}} \right)^{n+i+1} G(\tau),
\]
where \( j \) was defined in Proposition 2. Consider \( \sum_{t \in T_i - T_{i+1}} (-1)^j \).
Since \( a_i = \varepsilon^j \), and \( h \neq i \Rightarrow a_h \) can assume the values \( 0, 1, \varepsilon, \ldots, \varepsilon^{q-2} \), this sum is zero, so \( \langle 1, W_n \rangle = \frac{1}{2q^n} [q^n + (-\frac{1}{\mathcal{P}})^n q^n] \), which gives the result.

Similarly, \( \langle \theta_0, W_n \rangle = \frac{1}{2q^n} [q^n + (-\frac{1}{\mathcal{P}})^n \sum_{t \not\in T_0} \theta_0(t)] \). But \( \sum_{t \not\in T_0} \theta_0(t) = \sum_{t \in T_0} \theta_0(t) - \sum_{t \not\in T_0} \theta_0(t) = -q^n \), so \( \langle \theta_0, W_n \rangle = \frac{1}{2} [1 - (-\frac{1}{\mathcal{P}})^n] \). This completes the proof of Proposition 4.

**Proposition 5.** Assume \( c(\chi) = m > 0 \). Then \( \langle \chi, W_n \rangle \) equals 0 or 1, and exactly half of the characters \( \chi \) of conductor \( m \) satisfy \( \langle \chi, W_n \rangle = 1 \).

Proof. We have
\[
\langle \chi, W_n \rangle = \frac{1}{2q^n} \left[ q^n + \sum_{t \not\in T_0} \bar{\chi}(t) \left( -\frac{1}{\mathcal{P}} \right)^n \right]
\]
\[
+ \sum_{i=0}^{n-1} \sum_{t \in T_i - T_{i+1}} \bar{\chi}(t) q^{\frac{i}{2} \cdot \frac{i+1}{2}} \left( -\frac{1}{\mathcal{P}} \right)^{n+i+1} (-1)^{j(t)} G(\tau),
\]
where \( j(t) \) is as in Proposition 2. Since \( \chi \) is nontrivial on \( T_0 \), \( \sum_{t \not\in T_0} \bar{\chi}(t) = 0 \), so
\[
\langle \chi, W_n \rangle = \frac{1}{2q^n} \left[ q^n + \left( \frac{2}{\mathcal{P}} \right) \left( -\frac{1}{\mathcal{P}} \right)^{n+1} G(\tau) \right]
\]
\[
\times \left[ \sum_{i=0}^{m-2} \left( -\frac{1}{\mathcal{P}} \right)^i q^{\frac{i}{2} \cdot \frac{i+1}{2}} \sum_{t \in T_i - T_{i+1}} \bar{\chi}(t) (-1)^{j(t)} \right]
\]
\[
+ \left( -\frac{1}{\mathcal{P}} \right)^{m-1} q^{\frac{2m-1}{2}} \sum_{t \in T_{m-1} - T_m} \bar{\chi}(t) (-1)^{j(t)}
\]
\[
+ \sum_{i=m}^{n-1} \left( -\frac{1}{\mathcal{P}} \right)^i q^{\frac{2i+1}{2}} \sum_{t \in T_i - T_{i+1}} (-1)^{j(t)} \right].
\]
As before, \( \sum_{t \in T_{i-1}} (-1)^{j(t)} = 0 \) for \( m \leq i \leq n - 1 \). Now consider \( \sum_{t \in T_{i-1}} \overline{\chi}(t)(-1)^{j(t)} \) for \( 0 \leq i \leq m - 2 \). Write this sum as

\[
\sum_{S_1} \sum_{S_2} \overline{\chi}(\phi(a_1 \pi^i + \cdots + a_{n-1} \pi^{n-1}))(1)^{j(t)},
\]

where \( S_1 = \{a_i, a_{i+1}, \ldots, a_{m-2} | a_i \neq 0 \} \), \( S_2 = \{a_{m-1}, \ldots, a_{n-1} \} \), and \( \phi \) is the map on \( \mathcal{O} \) to \( T_0 \) which was recalled above. If \( x \in \mathcal{P}^n \), then \( \phi(x) \in T_n \). If \( x, y \in \mathcal{O} \),

\[
\frac{\phi(x) \phi(y)}{\phi(x+y)} = \frac{a - b \sqrt{\pi}}{a + b \sqrt{\pi}} = c + d \sqrt{\pi},
\]

where \( a = 1 - \pi(x^2 + xy + y^2) \), \( b = \pi xy(x + y) \), \( c = \frac{a^2 + b^2 \pi}{a^2 - b^2 \pi} \), and \( d = -\frac{2ab}{a^2 - b^2 \pi} \). Let \( x = a_1 \pi^i + \cdots + a_{m-2} \pi^{m-2} \) and \( y = a_{m-1} \pi^{m-1} + \cdots + a_{n-1} \pi^{n-1} \). Then \( \nu(x) = i \) and \( y \) either equals 0 or satisfies \( \nu(y) \geq m - 1 \). We need only consider the case \( y \neq 0 \). Then \( \nu(x + y) \geq i \), so \( \nu(c) \geq 2m + 1 \) and \( \nu(d) \geq m \). Therefore, \( c + d \sqrt{\pi} \in T_m \). Since \( \chi \equiv 1 \) on \( T_m \), we have \( \chi(\phi(x)) \chi(\phi(y)) = \chi(\phi(x + y)) \). This shows that

\[
\sum_{t \in T_{i-1}} \overline{\chi}(t)(-1)^{j(t)} = \sum_{S_1} \overline{\chi}(\phi(x))(1)^{j(t)} \sum_{S_2} \overline{\chi}(\phi(y)).
\]

But

\[
\sum_{S_2} \overline{\chi}(\phi(y)) = \sum_{t \in T_{m-1}} \overline{\chi}(t) = 0
\]

since \( \chi \not\equiv 1 \) on \( T_{m-1} \). Therefore,

\[
\sum_{t \in T_{i-1}} \overline{\chi}(t)(-1)^{j(t)} = 0
\]

for \( 0 \leq i \leq m - 2 \).

Next, consider

\[
\sum_{t \in T_m - T_m} \overline{\chi}(t)(-1)^{j(t)}.
\]

Here, \( t = \phi(a_{m-1} \pi^{m-1} + \cdots + a_{n-1} \pi^{n-1}) \), with \( a_{m-1} = e^{j(t)} \), \( 0 \leq j(t) \leq q - 2 \). Let \( x = a_{m-1} \pi^{m-1} \), \( y = a_m \pi^m + \cdots + a_{n-1} \pi^{n-1} \). As before,

\[
\frac{\phi(x) \phi(y)}{\phi(x+y)} \in T_m,
\]
which makes

\[
(13) \sum_{t \in T_{m-1} - T_{m+1}} \overline{\chi}(t)(-1)^{j(t)} = \sum_{S_2} \overline{\chi}(\phi(x))\overline{\chi}(\phi(y))(-1)^{j(t)}
\]
\[
= q^{n-m} \sum_{a_{m-1} \neq 0} \overline{\chi}(\phi(a_{m-1}\pi^{m-1}))(-1)^{j(t)},
\]

since \(\phi(y) \in T_m\) and \(\chi \equiv 1\) on \(T_m\).

We have a map

\[
\mathbb{P}^{m-1}/\mathbb{P}^m \xrightarrow{\phi} T_{m-1}/T_m \xrightarrow{\overline{\chi}} \mathbb{C}.
\]

For \(x, y \in \mathbb{P}^{m-1}\),

\[
\frac{\phi(x)\phi(y)}{\phi(x+y)} \in T_m,
\]

so \(\overline{\chi}\phi\) is an additive homomorphism on \(\mathbb{P}^{m-1}/\mathbb{P}^m\) to \(\mathbb{C}\). Letting \(\psi = \overline{\chi}\phi\), (13) becomes

\[
q^{n-m} \sum_{j=0}^{q-2} \psi(e^{j\pi^{m-1}})(-1)^j = q^{n-m} \sum_{x \in \mathcal{O}/\mathbb{P}} \psi(\pi^{m-1}x^2) = q^{n-m}q^{\frac{1}{2}}G(\psi).
\]

(Note that \(\psi_{\pi^{m-1}}\) is a character of \(\mathcal{O}/\mathbb{P}\).) We can now write

\[
\langle \chi, W_n \rangle = \frac{1}{2qn} \left[ q^n + \left( \frac{2}{\mathbb{P}} \right) \left( \frac{-1}{\mathbb{P}} \right)^{n+m} q^nG(\tau)G(\psi) \right],
\]

which equals 0 or 1. Notice that \(\psi_{\pi^{m-1}} = \tau_{\pi^{n-1}}e^{iu}\) for some \(0 \leq i \leq q-2, u \in 1+\mathbb{P}\). Then \(G(\tau)G(\psi) = \left( \frac{-e^i}{\mathbb{P}} \right) = \left( \frac{-1}{\mathbb{P}} \right)(-1)^i\), which takes on each value \(\pm 1\) for half the \(q-1\) possible values of \(i\). This completes the proof of Proposition 5.

If \(E/F\) is ramified, suppose that we replace \(\tau\) by \(\tau_u, u \in \mathcal{U}\). Then the characters of a given conductor appearing in \(W^n_\tau\) will be the same as those appearing in \(W^n_{\tau_u}\) if \(\left( \frac{u}{\mathbb{P}} \right) = 1\). If \(\left( \frac{u}{\mathbb{P}} \right) = -1\), then the two sets of characters of a given conductor \(m > 0\) appearing respectively in \(W^n_\tau\) and \(W^n_{\tau_u}\) are disjoint. By varying \(\tau\), we thus obtain all characters of conductor \(m > 0\) in the restriction to \(T\) of some \(W^\tau\).

5. Decomposition of \(W^\tau|_T\). In this section we use the results of the preceding section to determine the decomposition of \(W^\tau|_T\).
LEMMA 9. For $2k > -n$, let $H_k = S(\mathbb{P}^{-k}, \mathbb{P}^{n+k})$. Then $H_k$ is an invariant subspace for $W^\tau$ which is equivalent to $W^\tau_{n+2k}$, where $\alpha = \pi^{-2k}$.

Proof. Recall that if $\beta \in F$ and $\alpha = \beta^2$, then $W^\tau = R^{-1}W^\tau R$, where $(Rf)(x) = |\beta|^\frac{k}{2}f(\beta x)$. Let $\beta = \pi^{-k}$. Then $\omega(\tau\alpha) = n + 2k$.

Suppose $g \in K$. Then $f \in H_k \Rightarrow Rf \in S(\mathbb{O}, \mathbb{P}^{n+2k}) \Rightarrow W^\tau(g)Rf \in S(\mathbb{O}, \mathbb{P}^{n+2k}) \Rightarrow R^{-1}W^\tau(g)Rf \in H_k$. Thus $H_k$ is invariant under $W^\tau$. Also, $W^\tau(g)f = f$ if $f \in H_k$ and $g \in K_{n+2k}$. We thus have a representation of $K/K_{n+2k}$ on $H_k$ which is a subrepresentation of $W^\tau$ and which is equivalent to $W^\tau_{n+2k}$. This completes the proof of Lemma 8.

Suppose $W^\tau(t)f = \chi(t)f$ for all $t \in T$. If $f \in S(\mathbb{P}^r, \mathbb{P}^s)$, choose $k$ so that $-k \leq r$ and $n + k \geq s$. Then $S(\mathbb{P}^r, \mathbb{P}^s) \subset S(\mathbb{P}^{-k}, \mathbb{P}^{n+k}) = H_k$. Then the action of $W^\tau$ on $H_k$ is equivalent to $W^\tau_{n+2k}$, $\alpha = \pi^{-2k}$, by Lemma 9. This implies $\chi$ appears in $W^\tau_{n+2k}$. We apply Proposition 3 to each of the representations $W^\tau_{n+2k}$, $k \geq 0$, to obtain

PROPOSITION 6. Suppose $E/F$ is unramified, $\omega(\tau) = n$, and $c(\chi) = i$.

1. If $n$ is even and $i$ is even, then $\langle \chi, W^\tau|_T \rangle = 1$.
2. If $n$ is even and $i$ is odd, then $\langle \chi, W^\tau|_T \rangle = 0$.
3. If $n$ is odd and $i$ is even, then $\langle \chi, W^\tau|_T \rangle = 0$ if $\chi \neq 1$, and $\langle 1, W^\tau|_T \rangle = 1$.
4. If $n$ is odd and $i$ is odd, then $\langle \chi, W^\tau|_T \rangle = 1$ if $\chi \neq \theta_0$, and $\langle \theta_0, W^\tau|_T \rangle = 0$.

We argue in a similar fashion if $E/F$ is ramified. Applying Propositions 4 and 5, we obtain

PROPOSITION 7. Suppose $E/F$ is ramified and $\omega(\tau) = n$.

1. $\langle 1, W^\tau|_T \rangle = 1$ if $n$ is even or $(\frac{-1}{F}) = 1$, and equals 0 otherwise.
2. $\langle \theta_0, W^\tau|_T \rangle = 1 - \langle 1, W^\tau|_T \rangle$.
3. If $c(\chi) = m > 0$, then

$$\langle \chi, W^\tau|_T \rangle = 1 \Leftrightarrow G(\tau)G(\psi) = \left(\frac{2}{\mathbb{P}}\right)\left(-1\right)^{n+m},$$

where $\psi = \bar{\chi}\phi$. Otherwise, $\langle \chi, W^\tau|_T \rangle = 0$. 

(4) Exactly half the characters $\chi$ of a given conductor satisfy $\langle \chi, W^r|_T \rangle = 1$.

REFERENCES


Received August 14, 1991.

U. S. NAVAL ACADEMY
ANNAPOLIS, MD 21402
On the extension of Lipschitz functions from boundaries of subvarieties to strongly pseudoconvex domains  201
K. ADACHI and HIROSHI KAJIMOTO

On a nonlinear equation related to the geometry of the diffeomorphism group  223
DAVID DAI-WAI BAO, JACQUES LAFONTAINE and TUDOR S. RATIU

Fixed points of boundary-preserving maps of surfaces  243
ROBERT F. BROWN and BRIAN SANDERSON

On orthomorphisms between von Neumann preduals and a problem of Araki  265
L. J. Bunce and JOHN DAVID MAITLAND WRIGHT

Primitive subalgebras of complex Lie algebras. I. Primitive subalgebras of the classical complex Lie algebras  273
I. V. CHEKALOV

$L^n$ solutions of the stationary and nonstationary Navier-Stokes equations in $R^n$  293
ZHI MIN CHEN

Some applications of Bell’s theorem to weakly pseudoconvex domains  305
XIAO JUN HUANG

On isotropic submanifolds and evolution of quasicaustics  317
STANISŁAW JANECZKO

Currents, metrics and Moishezon manifolds  335
SHANYU JI

Stationary surfaces in Minkowski spaces. I. A representation formula  353
JIANGFAN LI

The dual pair $(U(1), U(1))$ over a $p$-adic field  365
COURTNEY HUGHES MOEN

Any knot complement covers at most one knot complement  387
SHICHENG WANG and YING QING WU