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A theorem of Komlós is a subsequence version of the strong law of large numbers. It states that if $(f_n)_n$ is a sequence of norm-bounded random variables in $L_1(\mu)$, where μ is a probability measure, then there exists a subsequence $(g_k)_k$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that for all further subsequences $(h_m)_m$, the sequence of successive arithmetic means of $(h_m)_m$ converges to f almost everywhere.

In this paper we show that, conversely, if C is a convex subset of $L_1(\mu)$ satisfying the conclusion of Komlós' theorem, then C must be L_1 -norm bounded.

Introduction. A version of the strong law of large numbers in probability theory states that if $(f_n)_{n=1}^\infty$ is a sequence of independent, scalar-valued integrable functions (random variables), on a probability measure space (Ω, Σ, μ) , each having the same distribution with mean m , then

$$\frac{1}{n} \sum_{j=1}^n f_j \xrightarrow{n} m \quad \text{almost everywhere.}$$

In (1967) Komlós [Ko] showed that arbitrary sequences of integrable random variables whose absolute values have uniformly bounded expectations always have subsequences that satisfy a version of the strong law. Indeed, for all sequences $(f_n)_{n=1}^\infty$ in $L_1(\mu)$ with

$$\sup_n \int_{\Omega} |f_n| d\mu < \infty,$$

there exists a subsequence $(g_k)_{k=1}^\infty$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that all further subsequences $(h_m)_m$ of $(g_k)_k$ satisfy

$$\frac{1}{N} \sum_{m=1}^N h_m \xrightarrow{N} f \quad \text{almost everywhere.}$$

This result became the archetype for what Chatterji [C2] in the early 1970s called “the subsequence principle in probability theory”. This heuristic principle led Chatterji [C1], [C2], [C3] (see also Gaposhkin [Ga]) to find subsequence versions of the central limit theorem and

the law of the iterated logarithm, analogous to Komlós's subsequence version of the strong law.

Chatterji [C1] and Gaposhkin [Ga] extended Komlós's theorem to all L_p spaces, for $0 < p < 2$. Aldous [A] and Berkes and Péter [B-P], amongst others, continued the investigation of the subsequence principle using the notion of an exchangeable sequence of random variables.

A recent extension of Komlós's theorem, due to N. J. Kalton, may be found in Godefroy [Go]. Kalton strengthens the conclusion of Komlós's theorem so that the Cesàro means converge almost everywhere and in weak L_1 .

For other recent developments concerning Komlós's theorem and further references, we refer the reader to Balder [B1], [B2], [B3] and Trautner [T].

In this paper we show that every convex set C in $L_1(\mu)$ that satisfies the conclusion of Komlós's theorem, must be L_1 -norm bounded. To prove this we proceed by contradiction. We create a sliding hump sequence of functions on the domain Ω , each a member of C , for which certain convex combinations have Cesàro averages with an L_0 -limit that lies outside of $L_1(\mu)$.

Finally, we characterize those convex subsets of L_1 that are almost everywhere Cesàro compact in the sense of the conclusion of Komlós's theorem, using a result of Bukhvalov and Lozanovski [B-L].

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1. Preliminaries and Komlós sets. \mathbf{N} denotes the set of all positive integers, while "the scalars" refers to the real or complex numbers. For a Banach space X , \mathbf{B}_X denotes the closed unit ball of X .

Throughout this paper Ω will be a non-empty set, Σ a σ -algebra of subsets of Ω , and μ will be a complete, positive, σ -finite, countably additive measure on Σ . $L_p(\mu)$ is the F -space or Banach space of all (equivalence classes of) measurable functions $f: \Omega \rightarrow$ the scalars for which $\|f\|_p < \infty$,

$$\|f\|_1 := \int_{\Omega} |f| d\mu,$$

$$\|f\|_{\infty} := \text{ess-sup}\{|f(\omega)|: \omega \in \Omega\},$$

and

$$\|f\|_0 := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(E_n)} \int_{E_n} \frac{|f|}{1+|f|} d\mu.$$

Here $(E_n)_{n=1}^{\infty}$ is a Σ -partition of Ω into sets with $0 < \mu(E_n) < \infty$, for each n . Such a Σ -partition exists as μ is σ -finite. If μ is finite we have the simpler definition,

$$\|f\|_0 := \int_{\Omega} \frac{|f|}{1+|f|} d\mu.$$

The $L_0(\mu)$ -topology restricted to $L_1(\mu)$ will be called the topology of convergence locally in measure (clm); or the topology of convergence in measure (cm) when μ is finite. θ will denote the zero element in $L_1(\mu)$.

1.1. DEFINITION. A subset S of $L_0(\mu)$ will be called a Komlós set if for every sequence $(f_n)_{n=1}^{\infty}$ in S , there exists a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ and $f \in S$ such that for every subsequence $(h_m)_{m=1}^{\infty}$ of $(g_k)_{k=1}^{\infty}$,

$$\frac{1}{N} \sum_{m=1}^N h_m \xrightarrow{N} f \text{ almost everywhere.}$$

Komlós showed that $\mathbf{B}_{L_1(\mu)}$ is a Komlós set.

Note that if $(f_n)_{n=1}^{\infty}$ is a sequence in $L_0(\mu)$ and $f_n \xrightarrow{n} f$ almost everywhere, then

$$\frac{1}{N} \sum_{n=1}^N f_n \xrightarrow{N} f \text{ almost everywhere.}$$

It follows that every clm-compact subset S of $L_0(\mu)$ must be a Komlós set. Consequently, even when Komlós sets are contained in $L_1(\mu)$, they need not be L_1 -norm bounded (see §2 for an example). Further, it is easy to check that Komlós sets are forced to be L_0 -closed. So, the concept of a Komlós subset of L_1 lies strictly between that of a clm-closed set and a clm-compact set in L_1 .

2. Convex Komlós sets in L_1 are norm bounded.

2.1. THEOREM. Let (Ω, Σ, μ) be a finite measure space. Suppose C is a subset of $L_1(\mu)$ that is convex and a Komlós set. Then C must be $\|\cdot\|_1$ -bounded.

Proof. Suppose, to get a contradiction, that C fails to be norm bounded. Then there exists a sequence $(g_n)_{n=1}^{\infty}$ in C such that $\|g_n\|_1 \xrightarrow{n} \infty$.

By assumption, C is a Komlós set. So, by passing to a subsequence if necessary, we may assume that there exists $g \in C$ such that

$$\frac{1}{N} \sum_{m=1}^N h_m \xrightarrow{N} g \quad \text{almost everywhere,}$$

for every subsequence $(h_m)_m$ of $(g_n)_n$.

Note that $C - g$ is another convex Komlós set in $L_1(\mu)$, $\theta \in C - g$ and

$$\frac{1}{N} \sum_{m=1}^N (h_m - g) \xrightarrow{N} \theta \quad \text{almost everywhere,}$$

for all subsequences $(h_m)_m$ of $(g_n)_n$. Clearly, by relabelling each $g_n - g$ as g_n and $C - g$ as C , we have that the following is true. C is a convex Komlós set in $L_1(\mu)$, $(g_n)_n$ is a sequence in C with $\|g_n\|_1 \xrightarrow{n} \infty$, $\theta \in C$ and for every subsequence $(h_m)_m$ of $(g_n)_n$,

$$\frac{1}{N} \sum_{m=1}^N h_m \xrightarrow{N} \theta \quad \text{almost everywhere.}$$

We shall now use $(g_n)_n$ to construct another sequence $(f_n)_n$ in C such that $f_n \xrightarrow{n} \theta$ almost everywhere and $\|f_n\|_1 \xrightarrow{n} \infty$. Let $u_1 := 1$ and $f_1 := g_{u_1}$. Since $\|g_n\|_1 \xrightarrow{n} \infty$, there exists $u_2 \in \mathbf{N}$ with $u_2 > u_1$ such that

$$\|g_{u_2}\|_1 > \|g_{u_1}\|_1 + 2(2^2).$$

Define f_2 by

$$f_2 := \frac{1}{2}(g_{u_1} + g_{u_2}),$$

$f_2 \in C$ because C is convex. Also,

$$\|f_2\|_1 \geq \frac{1}{2}(\|g_{u_2}\|_1 - \|g_{u_1}\|_1) > \frac{1}{2} \cdot 2(2^2) = 2^2.$$

Next choose $u_3 \in \mathbf{N}$ with $u_3 > u_2$ and

$$\|g_{u_3}\|_1 > \|g_{u_1}\|_1 + \|g_{u_2}\|_1 + 3(2^3);$$

and define

$$f_3 := \frac{1}{3}(g_{u_1} + g_{u_2} + g_{u_3}).$$

Then $f_3 \in C$ and $\|f_3\|_1 > 2^3$.

Continuing inductively, we produce a subsequence $(g_{u_n})_{n=1}^\infty$ of $(g_n)_n$ and a sequence $(f_n)_{n=1}^\infty$ in C such that $\|f_n\|_1 \xrightarrow{n} \infty$ and

$$f_n = \frac{1}{n} \sum_{j=1}^n g_{u_j}, \quad \text{for all } n \in \mathbf{N}.$$

From above, we know that $f_n \xrightarrow{n} \theta$ almost everywhere.

We will now inductively construct a strictly increasing sequence $(n_k)_{k=0}^\infty$ in \mathbf{N} , a nonincreasing sequence $(E_n)_{n=0}^\infty$ in Σ and a sequence $(\delta_k)_{k=0}^\infty$ of positive real numbers with the following properties. $E_1 = \Omega$; and for each $k \in \mathbf{N}$ statements (1) to (5) below are true.

- (1) $\delta_k < \delta_{k-1}/2$.
- (2) For each $E \in \Sigma$ with $\mu(E) < \delta_k$, we have that $\int_E |f_n| d\mu < 1$.
- (3) $\|f_{n_k} \chi_{E_k}\|_1 > 2^k(2 + \mu(\Omega))$.
- (4) $\|f_n \chi_{E_{k-1} \setminus E_k}\|_\infty < 1$, for all $n \geq n_k$.
- (5) $\mu(E_k) < \delta_{k-1}$.

Define $E_0 := \Omega$, $\delta_0 := 2\mu(\Omega)$ and $n_0 := 1$. Next define $E_1 := \Omega$. Since $\|f_n\|_1 \xrightarrow{n} \infty$, we can choose $n_1 \in \mathbf{N}$ so large that $n_1 > n_0$,

$$\begin{aligned} \|f_{n_1} \chi_{E_1}\|_1 &> 2^1(2 + \mu(\Omega)), \quad \text{and} \\ \|f_n \chi_{E_0 \setminus E_1}\|_\infty &< 1, \quad \text{for all } n \geq n_1. \end{aligned}$$

By the absolute continuity of the measure $|f_{n_1}| d\mu$ with respect to μ , there exists $\delta_1 \in (0, \mu(\Omega))$ such that for every $E \in \Sigma$ with $\mu(E) < \delta_1$, we have

$$\int_E |f_{n_1}| d\mu < 1.$$

Of course, $\mu(E_1) < \delta_0$.

Fix $m \in \mathbf{N}$ with $m > 1$. Suppose that we have constructed a strictly increasing sequence $(n_k)_{k=0}^{m-1}$ in \mathbf{N} , a non-increasing sequence $(E_k)_{k=0}^{m-1}$ in Σ and a sequence $(\delta_k)_{k=0}^{m-1}$ of positive real numbers, such that statements (1) to (5) are true for each $k \in \{1, \dots, m-1\}$. We know that $f_n \xrightarrow{n} \theta$ almost everywhere on E_{m-1} . So we can find, with the aid of Egoroff's theorem, $E_m \in \Sigma$ with $E_m \subseteq E_{m-1}$, such that

$$\mu(E_m) < \delta_{m-1} \quad \text{and} \quad \|f_n \chi_{E_{m-1} \setminus E_m}\|_\infty \xrightarrow{n} 0.$$

But statement (4) is true for each $k \in \{1, \dots, m-1\}$; and hence we see that

$$\|f_n \chi_{\Omega \setminus E_{m-1}}\|_\infty < 1, \quad \text{for all } n \geq n_{m-1}.$$

Since $\|f_n\|_1 \xrightarrow{n} \infty$, it follows that

$$\sup_{n \in \mathbf{N}} \|f_n \chi_{E_m}\|_1 = \infty.$$

Choose $n_m \in \mathbf{N}$ with $n_m > n_{m-1}$, such that

$$\begin{aligned} \|f_{n_m} \chi_{E_m}\|_1 &> 2^m(2 + \mu(\Omega)), \quad \text{and} \\ \|f_n \chi_{E_{m-1} \setminus E_m}\|_\infty &< 1, \quad \text{for all } n \geq n_m. \end{aligned}$$

Now, the measure $|f_{n_m}| d\mu$ is absolutely continuous w.r.t. μ . Therefore there exists $\delta_m > 0$ satisfying $\delta_m < \delta_{m-1}/2$; and such that for every $E \in \Sigma$ with $\mu(E) < \delta_m$, we have that

$$\int_E |f_{n_m}| d\mu < 1.$$

Our inductive construction is complete.

For convenience, let us relabel each f_{n_k} as f_k . We note that statements (2), (3) and (4) above still hold true, with n_k replaced everywhere by k . We will refer to (2), (3) and (4), modified in this way, as (2)*, (3)* and (4)* respectively.

For each $k \in \mathbf{N}$, define

$$\psi_k := \sum_{j=1}^k \frac{1}{2^j} f_j.$$

Since $\theta \in C$, each $\psi_k \in \text{co}(C) = C$. Also define, for every $m \in \mathbf{N}$,

$$\varphi_m := \left(\frac{1}{2^m} |f_m| - \sum_{j=1}^{m-1} \frac{1}{2^j} |f_j| - 1 \right) \chi_{E_m \setminus E_{m+1}}.$$

$(\psi_k)_{k=1}^\infty$ is a sequence in C , which is a Komlós set in $L_1(\mu)$. So there exists a subsequence $(\psi_{k_l})_{l=1}^\infty$ of $(\psi_k)_{k=1}^\infty$ and $q \in C$ such that

$$(\diamond) \quad q_N := \frac{1}{N} \sum_{l=1}^N \psi_{k_l} \xrightarrow{N} q \quad \text{almost everywhere.}$$

Moreover, note that $q \in C \subseteq L_1(\mu)$; so that

$$(\spadesuit) \quad \|q\|_1 < \infty.$$

Let $k_0 := 0$. It is simple to verify that for all $N \in \mathbf{N}$,

$$q_N = \sum_{j=1}^N \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t.$$

In the calculations below, when we have a pointwise inequality between two measurable functions, we mean that the inequality holds almost everywhere.

Fix $m \in \mathbf{N}$ and consider $E_m \setminus E_{m+1}$. Note that there is a unique $i \in \mathbf{N}$ such that $k_{i-1} < m \leq k_i$. Next fix $N \in \mathbf{N}$ with $N \geq i$. By property (4)* above, $|f_j| < 1$ on $E_m \setminus E_{m+1}$, for all $j \geq m+1$. Temporarily, let $c_m := \chi_{E_m \setminus E_{m+1}}$. Then,

$$\begin{aligned} |q_N c_m| &= \left| \left(\sum_{1 \leq j \leq N, j \neq i} \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t \right. \right. \\ &\quad \left. \left. + \frac{N-i+1}{N} \sum_{t=k_{i-1}+1}^{k_i} \frac{1}{2^t} f_t \right) c_m \right| \\ &\geq \left(\frac{N-i+1}{N} \frac{1}{2^m} |f_m| - \sum_{1 \leq t < m} \frac{1}{2^t} |f_t| - \sum_{m < t \leq k_N} \frac{1}{2^t} |f_t| \right) c_m \\ &\geq \frac{-(i-1)}{N} \frac{1}{2^m} |f_m| c_m + \varphi_m + c_m - \left(\sum_{m < t \leq k_N} \frac{1}{2^t} \right) c_m \\ &\geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| c_m. \end{aligned}$$

Thus, we have shown the following.

- (♣) For all $m \in \mathbf{N}$, there exists $i \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ with $N \geq i$,

$$|q_N \chi_{E_m \setminus E_{m+1}}| \geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| \chi_{E_m \setminus E_{m+1}}.$$

Again fix $m \in \mathbf{N}$. We see that

$$\begin{aligned} \int_{\Omega} \varphi_m d\mu &= \frac{1}{2^m} \int_{E_m \setminus E_{m+1}} |f_m| d\mu \\ &\quad - \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| d\mu - \mu(E_m \setminus E_{m+1}) \\ &= \frac{1}{2^m} \|f_m \chi_{E_m}\|_1 - \frac{1}{2^m} \int_{E_{m+1}} |f_m| d\mu \\ &\quad - \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| d\mu - \mu(E_m \setminus E_{m+1}). \end{aligned}$$

$\mu(E_{m+1}) < \delta_m$, from (5); and so by (2)*,

$$\int_{E_{m+1}} |f_m| d\mu < 1.$$

Also, by (5) and (1) we have that for all $j \in \{1, \dots, m-1\}$,

$$\mu(E_m \setminus E_{m+1}) \leq \mu(E_m) < \delta_{m-1} \leq \delta_j;$$

and consequently from (2)*,

$$\int_{E_m \setminus E_{m+1}} |f_j| d\mu < 1.$$

Using (3)* above,

$$\begin{aligned} \int_{\Omega} \varphi_m d\mu &> \frac{1}{2^m} \|f_m \chi_{E_m}\|_1 - \frac{1}{2^m} - \sum_{j=1}^{m-1} \frac{1}{2^j} - \mu(\Omega) \\ &> \frac{1}{2^m} 2^m (2 + \mu(\Omega)) - 1 - \mu(\Omega) = 1. \end{aligned}$$

In summary,

$$(\heartsuit) \quad \int_{\Omega} \varphi_m d\mu > 1, \quad \text{for all } m \in \mathbf{N}.$$

We now estimate $\|q\|_1$ from below. Fix $m \in \mathbf{N}$. By (), there exists $i \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ with $N \geq i$,

$$|q_N(\omega)| \geq \varphi_m(\omega) - \frac{i-1}{N} \frac{1}{2^m} |f_m(\omega)|, \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.$$

From (), we therefore have that

$$|q(\omega)| \geq \varphi_m(\omega), \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.$$

$E_1 = \Omega$, and $\mu(E_m) \rightarrow 0$, by (1) and (5). Thus, $(E_m \setminus E_{m+1})_{m=1}^\infty$ is a Σ -partition of Ω . Consequently, using (\spadesuit) and (\heartsuit) , we are led to the following contradiction.

$$\begin{aligned} \infty > \|q\|_1 &= \sum_{m=1}^\infty \int_{E_m \setminus E_{m+1}} |q(\omega)| d\mu(\omega) \geq \sum_{m=1}^\infty \int_{E_m \setminus E_{m+1}} \varphi_m(\omega) d\mu(\omega) \\ &= \sum_{m=1}^\infty \int_{\Omega} \varphi_m d\mu \geq \sum_{m=1}^\infty (1)^m = \infty. \end{aligned}$$

The previous theorem extends to the case where μ is a σ -finite measure. The proof below is simpler than our original one. It was suggested by Anton Schep.

2.2. THEOREM. *Let (Ω, Σ, μ) be a σ -finite measure space. Let C be a convex Komlós set in $L_1(\mu)$. Then C must be norm bounded.*

Proof. Fix $g \in L_1(\mu)$ such that

$$g(\omega) > 0, \quad \text{for all } \omega \in \Omega.$$

Such a g exists because μ is σ -finite. Define the finite measure ν by $d\nu := g d\mu$, and define the linear isometry T from $L_1(\mu)$ onto $L_1(\nu)$ by

$$Tf := fg^{-1}, \quad \text{for all } f \in L_1(\mu).$$

Since μ and ν have the same sets of measure zero, it is easy to see that a subset C of $L_1(\mu)$ is a Komlós set if and only if $T(C)$ is a Komlós set in $L_1(\nu)$. By Theorem 2.1, $T(C)$ is $L_1(\nu)$ -norm bounded; and consequently C is $L_1(\mu)$ -norm bounded. \square

Note that every clm-compact subset of L_1 is automatically a Komlós set. So the example

$$C := \{n^2 \chi_{[0, 1/n]} : n \in \mathbf{N}\} \cup \{0\}$$

is a Komlós set in $L_1[0, 1]$ that fails to be L_1 -norm bounded.

We also remark that a corollary to Theorem 2.1 is that every clm-compact, convex subset of $L_1(\mu)$ must be L_1 -norm bounded. This is a result of Khamsi and Turpin [K-T], that can be generalized to the setting of a large class of tvs topologies τ on a Banach space X (see, for example, Khamsi [Kh]).

3. A second dual characterization of Komlós convex sets in L_1 . In this section the symbol \cong will denote isometric isomorphism between

Banach spaces. Let j be the natural embedding of L_1 into L_1^{**} . It is a fact that

$$L_1^{**} = j(L_1) \oplus_1 S,$$

for some subspace S of L_1^{**} . Indeed, $L_1^* \cong L_\infty(\mu)$ and so $L_1^{**} \cong L_\infty^*$, which is isometrically isomorphic to the space of all bounded, finitely additive measures on Σ that vanish on μ -null sets. Hence, by the Yoshida-Hewitt decomposition theorem [Y-H] and the Radon-Nikodým theorem,

$$L_\infty^* \cong L_1 \oplus_1 pfa(\mu),$$

where $pfa(\mu)$ denotes the space of all bounded, purely finitely additive measures on Σ that vanish on μ -null sets. We identify $pfa(\mu)$ with a subspace S of L_1^{**} , and we denote by P the natural projection of L_1^{**} onto $j(L_1)$.

Recall the following result, which we will use to establish Theorem 3.1 below.

THEOREM (*Bukhvalov and Lozanovski [B-L] Theorem 1*). *Let C be a convex subset of $L_1(\mu)$ and let W be the weak*-closure of $j(C)$ in L_1^{**} .*

(a) *If C is clm-closed then $P(W) = j(C)$.*

(b) *If C is L_1 -norm bounded and $P(W) = j(C)$ then C is clm-closed.*

3.1. THEOREM. *Let C be a convex subset of $L_1(\mu)$ and W be the weak*-closure of $j(C)$ in L_1^{**} . Then the following statements are equivalent.*

(a) *C is a Komlós set.*

(b) *C is L_1 -norm bounded and clm-closed.*

(c) *C is L_1 -norm bounded and $P(W) = j(C)$.*

Proof. (a) \Rightarrow (b). By Theorem 2.2, C is L_1 -norm bounded. Moreover, Komlós sets are clm-closed, as we observed above.

(b) \Rightarrow (a). Fix $(f_n)_{n=1}^\infty$ in C . By Komlós's theorem [Ko], there exists a subsequence $(g_k)_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ and $f \in L_1(\mu)$, such that for all subsequences $(h_m)_{m=1}^\infty$ of $(g_k)_{k=1}^\infty$ we have

$$q_N := \frac{1}{N} \sum_{m=1}^N h_m \xrightarrow{N} f \quad \text{almost everywhere.}$$

C is convex, and hence each $q_N \in C$. But C is clm-closed and consequently, $f \in C$.

(b) \Leftrightarrow (c). This follows from [B-L] Theorem 1. □

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