A CONVERSE TO A THEOREM OF KOMLÓS FOR CONVEX SUBSETS OF $L_1$

Christopher John Lennard
A CONVERSE TO A THEOREM OF KOMLÓS
FOR CONVEX SUBSETS OF $L_1$

CHRIS LENNARD

A theorem of Komlós is a subsequence version of the strong law of large numbers. It states that if $(f_n)_n$ is a sequence of norm-bounded random variables in $L_1(\mu)$, where $\mu$ is a probability measure, then there exists a subsequence $(g_k)_k$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that for all further subsequences $(h_m)_m$, the sequence of successive arithmetic means of $(h_m)_m$ converges to $f$ almost everywhere.

In this paper we show that, conversely, if $C$ is a convex subset of $L_1(\mu)$ satisfying the conclusion of Komlós’ theorem, then $C$ must be $L_1$-norm bounded.

Introduction. A version of the strong law of large numbers in probability theory states that if $(f_n)_{n=1}^\infty$ is a sequence of independent, scalar-valued integrable functions (random variables), on a probability measure space $(\Omega, \Sigma, \mu)$, each having the same distribution with mean $m$, then

$$\frac{1}{n} \sum_{j=1}^{n} f_j \to m \quad \text{almost everywhere.}$$

In (1967) Komlós [Ko] showed that arbitrary sequences of integrable random variables whose absolute values have uniformly bounded expectations always have subsequences that satisfy a version of the strong law. Indeed, for all sequences $(f_n)_{n=1}^\infty$ in $L_1(\mu)$ with

$$\sup_n \int_{\Omega} |f_n| \, d\mu < \infty,$$

there exists a subsequence $(g_k)_{k=1}^\infty$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that all further subsequences $(h_m)_m$ of $(g_k)_k$ satisfy

$$\frac{1}{N} \sum_{m=1}^{N} h_m \to f \quad \text{almost everywhere.}$$

This result became the archetype for what Chatterji [C2] in the early 1970s called “the subsequence principle in probability theory”. This heuristic principle led Chatterji [C1], [C2], [C3] (see also Gaposhkin [Ga]) to find subsequence versions of the central limit theorem and
the law of the iterated logarithm, analogous to Komlós's subsequence version of the strong law.

Chatterji [C1] and Gaposhkin [Ga] extended Komlós's theorem to all $L_p$ spaces, for $0 < p < 2$. Aldous [A] and Berkes and Péter [B-P], amongst others, continued the investigation of the subsequence principle using the notion of an exchangeable sequence of random variables.

A recent extension of Komlós's theorem, due to N. J. Kalton, may be found in Godefroy [Go]. Kalton strengthens the conclusion of Komlós's theorem so that the Cesàro means converge almost everywhere and in weak $L_1$.

For other recent developments concerning Komlós's theorem and further references, we refer the reader to Balder [B1], [B2], [B3] and Trautner [T].

In this paper we show that every convex set $C$ in $L_1(\mu)$ that satisfies the conclusion of Komlós's theorem, must be $L_1$-norm bounded. To prove this we proceed by contradiction. We create a sliding hump sequence of functions on the domain $\Omega$, each a member of $C$, for which certain convex combinations have Cesàro averages with an $L_0$-limit that lies outside of $L_1(\mu)$.

Finally, we characterize those convex subsets of $L_1$ that are almost everywhere Cesàro compact in the sense of the conclusion of Komlós's theorem, using a result of Bukhvalov and Lozanovski [B-L].

I thank Joe Diestel, Nigel Kalton, Amine Khamsi and Anton Schep for helpful comments and suggestions. Thanks also to Catherine for typing the manuscript.

The author is grateful for the support of a University of Pittsburgh Internal Research Grant during part of the preparation of this paper.

1. Preliminaries and Komlós sets. $N$ denotes the set of all positive integers, while "the scalars" refers to the real or complex numbers. For a Banach space $X$, $B_X$ denotes the closed unit ball of $X$.

Throughout this paper $\Omega$ will be a non-empty set, $\Sigma$ a $\sigma$-algebra of subsets of $\Omega$, and $\mu$ will be a complete, positive, $\sigma$-finite, countably additive measure on $\Sigma$. $L_p(\mu)$ is the $F$-space or Banach space of all (equivalence classes of) measurable functions $f: \Omega \rightarrow$ the scalars for which $\|f\|_p < \infty$,

$$\|f\|_1 := \int_\Omega |f| d\mu,$$
$$\|f\|_\infty := \text{ess-sup}\{|f(\omega)| : \omega \in \Omega\},$$
and
\[ \|f\|_0 := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(E_n)} \int_{E_n} \frac{|f|}{1+|f|} \, d\mu. \]

Here \( (E_n)_{n=1}^{\infty} \) is a \( \Sigma \)-partition of \( \Omega \) into sets with \( 0 < \mu(E_n) < \infty \), for each \( n \). Such a \( \Sigma \)-partition exists as \( \mu \) is \( \sigma \)-finite. If \( \mu \) is finite we have the simpler definition,
\[ \|f\|_0 := \int_{\Omega} \frac{|f|}{1+|f|} \, d\mu. \]

The \( L_0(\mu) \)-topology restricted to \( L_1(\mu) \) will be called the topology of convergence locally in measure (clm); or the topology of convergence in measure (cm) when \( \mu \) is finite. \( \theta \) will denote the zero element in \( L_1(\mu) \).

1.1. DEFINITION. A subset \( S \) of \( L_0(\mu) \) will be called a Komlós set if for every sequence \( (f_n)_{n=1}^{\infty} \) in \( S \), there exists a subsequence \( (g_k)_{k=1}^{\infty} \) of \( (f_n)_{n=1}^{\infty} \) and \( f \in S \) such that for every subsequence \( (h_m)_{m=1}^{\infty} \) of \( (g_k)_{k=1}^{\infty} \),
\[ \frac{1}{N} \sum_{m=1}^{N} h_m \to f \quad \text{almost everywhere.} \]

Komlós showed that \( B_{L_1(\mu)} \) is a Komlós set.

Note that if \( (f_n)_{n=1}^{\infty} \) is a sequence in \( L_0(\mu) \) and \( f_n \to f \) almost everywhere, then
\[ \frac{1}{N} \sum_{n=1}^{N} f_n \to f \quad \text{almost everywhere.} \]

It follows that every clm-compact subset \( S \) of \( L_0(\mu) \) must be a Komlós set. Consequently, even when Komlós sets are contained in \( L_1(\mu) \), they need not be \( L_1 \)-norm bounded (see §2 for an example). Further, it is easy to check that Komlós sets are forced to be \( L_0 \)-closed. So, the concept of a Komlós subset of \( L_1 \) lies strictly between that of a clm-closed set and a clm-compact set in \( L_1 \).

2. Convex Komlós sets in \( L_1 \) are norm bounded.

2.1. THEOREM. Let \( (\Omega, \Sigma, \mu) \) be a finite measure space. Suppose \( C \) is a subset of \( L_1(\mu) \) that is convex and a Komlós set. Then \( C \) must be \( \| \cdot \|_1 \)-bounded.
Proof. Suppose, to get a contradiction, that \( C \) fails to be norm bounded. Then there exists a sequence \( (g_n)_{n=1}^{\infty} \) in \( C \) such that \( \|g_n\|_1 \to \infty \).

By assumption, \( C \) is a Komlós set. So, by passing to a subsequence if necessary, we may assume that there exists \( g \in C \) such that

\[
\frac{1}{N} \sum_{m=1}^{N} h_m \to g \quad \text{almost everywhere},
\]

for every subsequence \( (h_m)_m \) of \( (g_n)_n \).

Note that \( C - g \) is another convex Komlós set in \( L_1(\mu) \), \( \theta \in C - g \) and

\[
\frac{1}{N} \sum_{m=1}^{N} (h_m - g) \to \theta \quad \text{almost everywhere},
\]

for all subsequences \( (h_m)_m \) of \( (g_n)_n \). Clearly, by relabelling each \( g_n - g \) as \( g_n \) and \( C - g \) as \( C \), we have that the following is true. \( C \) is a convex Komlós set in \( L_1(\mu) \), \( (g_n)_n \) is a sequence in \( C \) with \( \|g_n\|_1 \to \infty \), \( \theta \in C \) and for every subsequence \( (h_m)_m \) of \( (g_n)_n \),

\[
\frac{1}{N} \sum_{m=1}^{N} h_m \to \theta \quad \text{almost everywhere}.
\]

We shall now use \( (g_n)_n \) to construct another sequence \( (f_n)_n \) in \( C \) such that \( f_n \to \theta \) almost everywhere and \( \|f_n\|_1 \to \infty \). Let \( u_1 := 1 \) and \( f_1 := g_{u_1} \). Since \( \|g_n\|_1 \to \infty \), there exists \( u_2 \in \mathbb{N} \) with \( u_2 > u_1 \) such that

\[
\|g_{u_2}\|_1 > \|g_{u_1}\|_1 + 2(2^2).
\]

Define \( f_2 \) by

\[
f_2 := \frac{1}{2}(g_{u_1} + g_{u_2}),
\]

\( f_2 \in C \) because \( C \) is convex. Also,

\[
\|f_2\|_1 \geq \frac{1}{2}(\|g_{u_2}\|_1 - \|g_{u_1}\|_1) > \frac{1}{2} \cdot 2(2^2) = 2^2.
\]

Next choose \( u_3 \in \mathbb{N} \) with \( u_3 > u_2 \) and

\[
\|g_{u_3}\|_1 > \|g_{u_1}\|_1 + \|g_{u_2}\|_1 + 3(2^3);
\]
and define
\[ f_3 := \frac{1}{3}(g_{u_1} + g_{u_2} + g_{u_3}). \]

Then \( f_3 \in C \) and \( ||f_3||_1 > 2^3 \).

Continuing inductively, we produce a subsequence \( (g_{u_n})_{n=1}^{\infty} \) of \( (g_n)_n \) and a sequence \( (f_n)_{n=1}^{\infty} \) in \( C \) such that \( ||f_n||_1 \to \infty \) and
\[
f_n = \frac{1}{n} \sum_{j=1}^{n} g_{u_j}, \quad \text{for all } n \in \mathbb{N}.
\]

From above, we know that \( f_n \to \theta \) almost everywhere.

We will now inductively construct a strictly increasing sequence \( (n_k)_{k=0}^{\infty} \) in \( \mathbb{N} \), a nonincreasing sequence \( (E_n)_{n=0}^{\infty} \) in \( \Sigma \) and a sequence \( (\delta_k)_{k=0}^{\infty} \) of positive real numbers with the following properties. \( E_1 = \Omega \); and for each \( k \in \mathbb{N} \) statements (1) to (5) below are true.

1. \( \delta_k < \delta_{k-1}/2 \).
2. For each \( E \in \Sigma \) with \( \mu(E) < \delta_k \), we have that \( \int_E |f_{n_k}| \, d\mu < 1 \).
3. \( ||f_{n_k} \chi_{E_k}||_1 > 2^k (2 + \mu(\Omega)) \).
4. \( ||f_{n_k} \chi_{E_k \setminus E_k}||_{\infty} < 1 \), for all \( n \geq n_k \).
5. \( \mu(E_k) < \delta_{k-1} \).

Define \( E_0 := \Omega \), \( \delta_0 := 2\mu(\Omega) \) and \( n_0 := 1 \). Next define \( E_1 := \Omega \). Since \( ||f_n||_1 \to \infty \), we can choose \( n_1 \in \mathbb{N} \) so large that \( n_1 > n_0 \),
\[
||f_{n_1} \chi_{E_1}||_1 > 2^1 (2 + \mu(\Omega)), \quad \text{and} \quad ||f_{n_1} \chi_{E_0 \setminus E_1}||_{\infty} < 1, \quad \text{for all } n \geq n_1.
\]
By the absolute continuity of the measure \( |f_{n_1}| \, d\mu \) with respect to \( \mu \), there exists \( \delta_1 \in (0, \mu(\Omega)) \) such that for every \( E \in \Sigma \) with \( \mu(E) < \delta_1 \), we have
\[
\int_E |f_{n_1}| \, d\mu < 1.
\]
Of course, \( \mu(E_1) < \delta_0 \).

Fix \( m \in \mathbb{N} \) with \( m > 1 \). Suppose that we have constructed a strictly increasing sequence \( (n_k)_{k=0}^{m-1} \) in \( \mathbb{N} \), a non-increasing sequence \( (E_k)_{k=0}^{m-1} \) in \( \Sigma \) and a sequence \( (\delta_k)_{k=0}^{m-1} \) of positive real numbers, such that statements (1) to (5) are true for each \( k \in \{1, \ldots, m-1\} \). We know that \( f_n \to \theta \) almost everywhere on \( E_{m-1} \). So we can find, with the aid of Egoroff's theorem, \( E_m \in \Sigma \) with \( E_m \subseteq E_{m-1} \), such that
\[
\mu(E_m) < \delta_{m-1} \quad \text{and} \quad ||f_n \chi_{E_{m-1} \setminus E_m}||_{\infty} \to 0.
\]
But statement (4) is true for each $k \in \{1, \ldots, m-1\}$; and hence we see that

$$\|f_n \chi_{\Omega \setminus E_{m-1}}\|_\infty < 1, \quad \text{for all } n \geq n_{m-1}.\]

Since $\|f_n\|_1 \to \infty$, it follows that

$$\sup_{n \in \mathbb{N}} \|f_n \chi_{E_m}\|_1 = \infty.$$

Choose $n_m \in \mathbb{N}$ with $n_m > n_{m-1}$, such that

$$\|f_{n_m} \chi_{E_m}\|_1 > 2^m (2 + \mu(\Omega)), \quad \text{and} \quad \|f_n \chi_{E_{m-1} \setminus E_m}\|_\infty < 1, \quad \text{for all } n \geq n_m.$$

Now, the measure $|f_{n_m}| \, d\mu$ is absolutely continuous w.r.t. $\mu$. Therefore there exists $\delta_m > 0$ satisfying $\delta_m < \delta_{m-1}/2$; and such that for every $E \in \Sigma$ with $\mu(E) < \delta_m$, we have that

$$\int_E |f_{n_m}| \, d\mu < 1.$$

Our inductive construction is complete.

For convenience, let us relabel each $f_{n_k}$ as $f_k$. We note that statements (2), (3) and (4) above still hold true, with $n_k$ replaced everywhere by $k$. We will refer to (2), (3) and (4), modified in this way, as (2)*, (3)* and (4)* respectively.

For each $k \in \mathbb{N}$, define

$$\psi_k := \sum_{j=1}^{k} \frac{1}{2^j} f_j.$$

Since $\theta \in C$, each $\psi_k \in \text{co}(C) = C$. Also define, for every $m \in \mathbb{N}$,

$$\varphi_m := \left( \frac{1}{2^m} |f_m| - \sum_{j=1}^{m-1} \frac{1}{2^j} |f_j| - 1 \right) \chi_{E_m \setminus E_{m+1}}.$$

$(\psi_k)_{k=1}^\infty$ is a sequence in $C$, which is a Komlós set in $L_1(\mu)$. So there exists a subsequence $(\psi_{k_l})_{l=1}^\infty$ of $(\psi_k)_{k=1}^\infty$ and $q \in C$ such that

$$(\diamond) \quad q_N := \frac{1}{N} \sum_{l=1}^{N} \psi_{k_l} \to q \quad \text{almost everywhere.}$$
Moreover, note that $q \in C \subseteq L_1(\mu)$; so that

(1) \hspace{1cm} \|q\|_1 < \infty.

Let $k_0 := 0$. It is simple to verify that for all $N \in \mathbb{N}$,

$$q_N = \sum_{j=1}^{N} \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t.$$

In the calculations below, when we have a pointwise inequality between two measurable functions, we mean that the inequality holds almost everywhere.

Fix $m \in \mathbb{N}$ and consider $E_m \setminus E_{m+1}$. Note that there is a unique $i \in \mathbb{N}$ such that $k_{i-1} < m \leq k_i$. Next fix $N \in \mathbb{N}$ with $N \geq i$. By property (4)* above, $|f_j| < 1$ on $E_m \setminus E_{m+1}$, for all $j \geq m + 1$.

Temporarily, let $c_m := \chi_{E_m \setminus E_{m+1}}$. Then,

$$|q_N c_m| = \left| \left( \sum_{1 \leq j \leq N, j \neq i} \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t \right) + \frac{N-i+1}{N} \sum_{t=k_{i-1}+1}^{k_i} \frac{1}{2^t} f_t \right| c_m$$

$$\geq \left( \frac{N-i+1}{N} \frac{1}{2^m} |f_m| - \sum_{1 \leq t < m} \frac{1}{2^t} |f_t| - \sum_{m < t \leq k_N} \frac{1}{2^t} |f_t| \right) c_m$$

$$\geq \frac{-(i-1)}{N} \frac{1}{2^m} |f_m| c_m + \varphi_m + c_m - \left( \sum_{m < t \leq k_N} \frac{1}{2^t} \right) c_m$$

$$\geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| c_m.$$

Thus, we have shown the following.

(2) \hspace{1cm} \text{For all } m \in \mathbb{N}, \text{ there exists } i \in \mathbb{N} \text{ such that for all } N \in \mathbb{N} \text{ with } N \geq i,

$$|q_N \chi_{E_m \setminus E_{m+1}}| \geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| \chi_{E_m \setminus E_{m+1}}.$$
Again fix $m \in \mathbb{N}$. We see that
\[
\int_{\Omega} \varphi_m \, d\mu = \frac{1}{2^m} \int_{E_m \setminus E_{m+1}} |f_m| \, d\mu
- \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| \, d\mu - \mu(E_m \setminus E_{m+1})
= \frac{1}{2^m} \|f_m\chi_{E_m}\|_1 - \frac{1}{2^m} \int_{E_{m+1}} |f_m| \, d\mu
- \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| \, d\mu - \mu(E_m \setminus E_{m+1}).
\]
$\mu(E_{m+1}) < \delta_m$, from (5); and so by (2)*,
\[
\int_{E_{m+1}} |f_m| \, d\mu < 1.
\]
Also, by (5) and (1) we have that for all $j \in \{1, \ldots, m - 1\}$,
\[
\mu(E_m \setminus E_{m+1}) \leq \mu(E_m) < \delta_{m-1} \leq \delta_j;
\]
and consequently from (2)*,
\[
\int_{E_m \setminus E_{m+1}} |f_j| \, d\mu < 1.
\]
Using (3)* above,
\[
\int_{\Omega} \varphi_m \, d\mu > \frac{1}{2^m} \|f_m\chi_{E_m}\|_1 - \frac{1}{2^m} - \sum_{j=1}^{m-1} \frac{1}{2^j} - \mu(\Omega)
> \frac{1}{2^m} (2 + \mu(\Omega)) - 1 - \mu(\Omega) = 1.
\]
In summary,
\[
\bigcirc \int_{\Omega} \varphi_m \, d\mu > 1, \quad \text{for all } m \in \mathbb{N}.
\]
We now estimate $\|q\|_1$ from below. Fix $m \in \mathbb{N}$. By (3), there exists $i \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ with $N \geq i$,
\[
|q_N(\omega)| \geq \varphi_m(\omega) - \frac{i - 1}{N} \frac{1}{2^m} |f_m(\omega)|, \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.
\]
From (\bigcirc), we therefore have that
\[
|q(\omega)| \geq \varphi_m(\omega), \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.
\]
\(E_1 = \Omega\), and \(\mu(E_m) \to 0\), by (1) and (5). Thus, \((E_m \setminus E_{m+1})_{m=1}^{\infty}\) is a \(\Sigma\)-partition of \(\Omega\). Consequently, using (\(\clubsuit\)) and (\(\heartsuit\)), we are led to the following contradiction.

\[
\infty > \|g\|_1 = \sum_{m=1}^{\infty} \int_{E_m \setminus E_{m+1}} |g(\omega)| \, d\mu(\omega) \geq \sum_{m=1}^{\infty} \int_{E_m \setminus E_{m+1}} \varphi_m(\omega) \, d\mu(\omega) \\
= \sum_{m=1}^{\infty} \int_{\Omega} \varphi_m \, d\mu \geq \sum_{m=1}^{\infty} (1)^m = \infty.
\]

The previous theorem extends to the case where \(\mu\) is a \(\sigma\)-finite measure. The proof below is simpler than our original one. It was suggested by Anton Schep.

2.2. THEOREM. Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space. Let \(C\) be a convex Komlós set in \(L_1(\mu)\). Then \(C\) must be norm bounded.

Proof. Fix \(g \in L_1(\mu)\) such that

\(g(\omega) > 0,\) for all \(\omega \in \Omega\).

Such a \(g\) exists because \(\mu\) is \(\sigma\)-finite. Define the finite measure \(\nu\) by \(d\nu := gd\mu\), and define the linear isometry \(T\) from \(L_1(\mu)\) onto \(L_1(\nu)\) by

\(Tf := fg^{-1}\), for all \(f \in L_1(\mu)\).

Since \(\mu\) and \(\nu\) have the same sets of measure zero, it is easy to see that a subset \(C\) of \(L_1(\mu)\) is a Komlós set if and only if \(T(C)\) is a Komlós set in \(L_1(\nu)\). By Theorem 2.1, \(T(C)\) is \(L_1(\nu)\)-norm bounded; and consequently \(C\) is \(L_1(\mu)\)-norm bounded. \(\Box\)

Note that every clm-compact subset of \(L_1\) is automatically a Komlós set. So the example

\[C := \{n^2 \chi_{[0,1/n]} : n \in \mathbb{N}\} \cup \{0\}\]

is a Komlós set in \(L_1[0, 1]\) that fails to be \(L_1\)-norm bounded.

We also remark that a corollary to Theorem 2.1 is that every clm-compact, convex subset of \(L_1(\mu)\) must be \(L_1\)-norm bounded. This is a result of Khamsi and Turpin [K-T], that can be generalized to the setting of a large class of tvs topologies \(\tau\) on a Banach space \(X\) (see, for example, Khamsi [Kh]).

3. A second dual characterization of Komlós convex sets in \(L_1\). In this section the symbol \(\cong\) will denote isometric isomorphism between
Banach spaces. Let \( j \) be the natural embedding of \( L_1 \) into \( L_1^{**} \). It is a fact that

\[ L_1^{**} = j(L_1) \oplus_1 S, \]

for some subspace \( S \) of \( L_1^{**} \). Indeed, \( L_1^* \cong L_\infty(\mu) \) and so \( L_1^{**} \cong L_\infty^* \), which is isometrically isomorphic to the space of all bounded, finitely additive measures on \( \Sigma \) that vanish on \( \mu \)-null sets. Hence, by the Yoshida-Hewitt decomposition theorem \([Y-H]\) and the Radon-Nikodým theorem,

\[ L_\infty^* \cong L_1 \oplus_1 pfa(\mu), \]

where \( pfa(\mu) \) denotes the space of all bounded, purely finitely additive measures on \( \Sigma \) that vanish on \( \mu \)-null sets. We identify \( pfa(\mu) \) with a subspace \( S \) of \( L_1^{**} \), and we denote by \( P \) the natural projection of \( L_1^{**} \) onto \( j(L_1) \).

Recall the following result, which we will use to establish Theorem 3.1 below.

**Theorem (Bukhvalov and Lozanovski \[B-L\] Theorem 1).** Let \( C \) be a convex subset of \( L_1(\mu) \) and let \( W \) be the weak*-closure of \( j(C) \) in \( L_1^{**} \).

(a) If \( C \) is clm-closed then \( P(W) = j(C) \).

(b) If \( C \) is \( L_1 \)-norm bounded and \( P(W) = j(C) \) then \( C \) is clm-closed.

3.1. **Theorem.** Let \( C \) be a convex subset of \( L_1(\mu) \) and \( W \) be the weak*-closure of \( j(C) \) in \( L_1^{**} \). Then the following statements are equivalent.

(a) \( C \) is a Komlós set.

(b) \( C \) is \( L_1 \)-norm bounded and clm-closed.

(c) \( C \) is \( L_1 \)-norm bounded and \( P(W) = j(C) \).

**Proof.** (a) \( \Rightarrow \) (b). By Theorem 2.2, \( C \) is \( L_1 \)-norm bounded. Moreover, Komlós sets are clm-closed, as we observed above.

(b) \( \Rightarrow \) (a). Fix \( (f_n)_{n=1}^\infty \) in \( C \). By Komlós's theorem \([Ko]\), there exists a subsequence \( (g_k)_{k=1}^\infty \) of \( (f_n)_{n=1}^\infty \) and \( f \in L_1(\mu) \), such that for all subsequences \( (h_m)_{m=1}^\infty \) of \( (g_k)_{k=1}^\infty \) we have

\[ q_N := \frac{1}{N} \sum_{m=1}^N h_m \Rightarrow f \quad \text{almost everywhere.} \]

\( C \) is convex, and hence each \( q_N \in C \). But \( C \) is clm-closed and consequently, \( f \in C \).

(b) \( \Leftrightarrow \) (c). This follows from \([B-L]\) Theorem 1. \( \square \)
REFERENCES


Received October 3, 1991.

UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA 15260
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>An application of the very weak Bernoulli condition for amenable</td>
<td>1</td>
</tr>
<tr>
<td>groups</td>
<td></td>
</tr>
<tr>
<td>Scot Robert Adams and Jeffrey Edward Steif</td>
<td></td>
</tr>
<tr>
<td>An application of homogenization theory to harmonic analysis on</td>
<td>19</td>
</tr>
<tr>
<td>solvable Lie groups of polynomial growth</td>
<td></td>
</tr>
<tr>
<td>G. Alexopoulos</td>
<td></td>
</tr>
<tr>
<td>The standard double soap bubble in $\mathbb{R}^2$ uniquely</td>
<td>47</td>
</tr>
<tr>
<td>minimizes perimeter</td>
<td></td>
</tr>
<tr>
<td>Joel Foisy, Manuel Alfaro Garcia, Jeffrey Farlowe Brock, Nickelous</td>
<td></td>
</tr>
<tr>
<td>Hodges and Jason Zimba</td>
<td></td>
</tr>
<tr>
<td>Pseudo regular elements and the auxiliary multiplication they</td>
<td>61</td>
</tr>
<tr>
<td>induce</td>
<td></td>
</tr>
<tr>
<td>Barry E. Johnson</td>
<td></td>
</tr>
<tr>
<td>A converse to a theorem of Komlós for convex subsets of $L_1$</td>
<td>75</td>
</tr>
<tr>
<td>Christopher John Lennard</td>
<td></td>
</tr>
<tr>
<td>General Kac-Moody algebras and the Kazhdan-Lusztig conjecture</td>
<td>87</td>
</tr>
<tr>
<td>Wayne L. Neidhardt</td>
<td></td>
</tr>
<tr>
<td>The flow space of a directed $G$-graph</td>
<td>127</td>
</tr>
<tr>
<td>William Lindall Paschke</td>
<td></td>
</tr>
<tr>
<td>Primitive ideals and derivations on noncommutative Banach algebras</td>
<td>139</td>
</tr>
<tr>
<td>Mark Phillip Thomas</td>
<td></td>
</tr>
<tr>
<td>Equivariant Nielsen numbers</td>
<td>153</td>
</tr>
<tr>
<td>Peter N-S Wong</td>
<td></td>
</tr>
<tr>
<td>Volumes of tubular neighbourhoods of real algebraic varieties</td>
<td>177</td>
</tr>
<tr>
<td>Richard Alexander Wongkew</td>
<td></td>
</tr>
<tr>
<td>The intrinsic group of Majid’s bicrossproduct Kac algebra</td>
<td>185</td>
</tr>
<tr>
<td>Takehiko Yamanouchi</td>
<td></td>
</tr>
</tbody>
</table>