SOLUTIONS OF THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

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It is shown that a nonstationary exterior Navier-Stokes flow tends to a small stationary flow in $L^2$ like $t^{-3/4}$ as $t \to \infty$.

0. Introduction. In this paper we are concerned with the stationary Navier-Stokes equations

\begin{align}
(w \cdot D)w - \Delta w + D\bar{p} &= f, \quad D \cdot w = 0 \quad \text{in } G, \\
w &= 0 \quad \text{on } \partial G \quad (D = \text{grad}),
\end{align}

and the nonstationary Navier-Stokes equations

\begin{align*}
v_t + (v \cdot D)v - \Delta v + D\bar{\bar{p}} &= f \quad \text{in } G \times (0, \infty), \\
D \cdot v &= 0 \quad \text{in } G \times (0, \infty), \\
v &= 0 \quad \text{on } \partial G \times (0, \infty), \\
v|_{t=0} &= a + w \quad \text{in } G \quad (v_t = \partial v / \partial t).
\end{align*}

Here and in what follows $G$ denotes a smooth exterior domain of $R^3$, $f = f(x)$ is a prescribed vector field, and $\bar{p}$ (resp. $\bar{\bar{p}}$) represents unknown stationary (resp. nonstationary) scalar pressure which can be determined by the stationary solution $w$ via (0.1) (resp. nonstationary solution $v$ via (0.2)).

As is well known, it was shown by Finn [8, 9] that (0.1) admits a small solution

\begin{align}
w \in L^\infty(G; R^3), \quad Dw \in L^3(G; R^9), \\
C_0 = \sup_{x \in G} |x| |w(x)| < \infty.
\end{align}

If $C_0 < 1/2$ the Finn’s solution $w$ may be formed as a limit of a nonstationary solution $v$ as $t \to \infty$ in local or global $L^2$-norms (cf. Heywood [15, 14], Galdi and Rionero [11], Miyakawa and Sohr [23]) and in other norms (cf. Heywood [16], Masuda [20]). Moreover it has recently proved (cf. Borchers and Miyakawa [4]) that every weak solution of (0.2) tends the Finn’s solution in $L^2(G; R^3)$.
like $t^{-(3/p-3/2)/2}$ with $6/5 < p < 2$, provided $C_0 < 1/2$ and $a \in L^2(G; R^3) \cap L^p(G; R^3)$.

In this paper we are only interested in the case $w \in L^3(G; R^3)$, $Dw \in L^{3/2}(G; R^9)$, or $Dw \in L^r(G; R^9) \cap L^p(G; R^9)$ with $1 < r < 3/2 < p < 2$. Under certain smallness assumptions on $w$ we show now that every weak solution of (0.2) tends to the stationary solution $w$ in $L^2(G; R^3)$ like the sharp decay rate $t^{-3/4}$.

1. Notation and main result. In this paper we use the following spaces.

$L^p$ = the Lebesgue spaces $L^p(G; R^3)$, with $\| \cdot \|_p$ the associated norm,

$C^\infty_\sigma$ = the set of compactly supported solenoidal in $C^\infty(G; R^3)$,

$W^{k,p}$ = the Sobolev space $W^{k,p}(G; R^3)$,

$P^p$ = the completion of $C^\infty_\sigma$ in $L^p$,

$W^{1,p}_\sigma$ = the completion of $C^\infty_\sigma$ in $W^{1,p}$,

$\widehat{W}^{1,p}_\sigma$ = the completion of $C^\infty_\sigma$ under the norm $\| D \cdot \|_p$

$\widehat{W}^{2,p}_\sigma$ = the space $\{ u \in \widehat{W}^{1,3p/(3-p)}_\sigma ; D^2 u \in L^p(G; R^{27}) \}$
for $1 < p < 3$,

$W^{-1,2}$ = the dual of $W^{1,2}_\sigma$,

$\widehat{W}^{-1,p}_\sigma$ = the dual of $\widehat{W}^{1,p/(p-1)}_\sigma$, with $\| \cdot \|_{-1,p}$ the associated norm.

Moreover for $1 < r < \infty$ and $n \geq 1$, we denote by $r'$ the real $r/(r-1)$, by $(\cdot, \cdot)$ the inner product in $L^2(G; R^n)$, by $P$ the bounded projection from $L^r$ onto $J^r$ (cf. [22]), by $A$ the Stokes operators $-P\Delta$ with the domain $W^{1,r}_\sigma \cap W^{2,r}$, by $\overline{A}$ the Laplacian $-\Delta$ with the domain $W^{2,r}(R^3; R^3)$, and by $C$ a positive constant which may vary from line to line, but is always independent of the quantities $t$, $T$, $u$, $v$, $w$, $f$, $u_k$, and $a$.

Now we make preparations for stating our main result. The existence of the stationary solutions $w$ is guaranteed by the following.

**Lemma 1.1.** Let $1 < r \leq 3/2 < p < 2$, and $f \in C^\infty_\sigma$. Then there is a small $h > 0$ such that (0.1) admits a unique solution within the class

$\{ w \in \widehat{W}^{1,r}_\sigma \cap \widehat{W}^{1,p}_\sigma ; \| Dw \|_{3/2} \leq h \}$,

provided that $\| f \|_{-1,3/2} \leq h^2$. Moreover

$\| Dw \|_r + \| Dw \|_p \leq C(\| f \|_{-1,r} + \| f \|_{-1,p})$. 
From (0.1) and (0.2) we see that \( u = v - w \) and \( \rho = \rho - \rho \) solve the problem

\[
\begin{align*}
(1.1) \quad u_t + (u \cdot D)u - \Delta u + (u \cdot D)w + (w \cdot D)u + D\rho &= 0, \\
D \cdot u &= 0 \quad \text{in } G \times (0, \infty), \\
(\rho - \rho) &= 0 \quad \text{on } \partial G \times (0, \infty), \\
|u|_{t=0} &= a \quad \text{in } G.
\end{align*}
\]

Weak solutions are given in the following sense.

**Definition 1.1.** Let \( a \in J^2 \), and \( w \in \widetilde{W}^{1,3/2} \) solve (0.1). A weakly continuous function \( u: [0, \infty) \to J^2 \) is said to be a weak solution of (1.1) if \( u(0) = a, \ u \in L^\infty(0, \infty; J^2) \cap L^2(0, \infty; \widetilde{W}^{1,2}) \),

\[
(1.2) \quad \|u(t)\|^2_2 + \int_s^t \|Du(z)\|^2_2 \, dz \leq \|u(s)\|^2_2,
\]

\[
(1.3) \quad (u(t), g(t)) + \int_s^t ((Du, Dg) + ((u \cdot D)w, g) \\
+ ((w \cdot D)u, g) - (u, g_z)) \, dz \\
= (u(s), g(s)) - \int_s^t ((u \cdot D)u, g) \, dz
\]

for all \( t > s \geq 0 \) and all \( g \in C([0, \infty); W^{1,2}_\sigma) \cap C^1([0, \infty); J^2) \), where \( g_z = \partial g/\partial z \).

The existence of weak solutions to (1.1) is guaranteed by the following.

**Lemma 1.2.** Let \( a \in J^2 \), and \( w \in \widetilde{W}^{1,3/2} \) such that \( \|Dw\|_{3/2} < 1/8 \). Then (1.1) admits a weak solution.

We are now in a position to state our main result.

**Theorem 1.1.** Let \( 1 < r < 3/2 < p < 2 \), \( a \in J^2 \cap L^1 \), and let \( w \in W^{1,r}_\sigma \cap W^{1,p}_\sigma \) such that \( w \) solves (0.1) and \( \|Dw\|_r + \|Dw\|_p \) is sufficiently small. Then every weak solution of (1.1) possesses the sharp decay property

\[
\|u(t)\|_2 = O(t^{-3/4}).
\]

Section 2 is concerned with the proof of Lemmas 1.1 and 1.2. In [23], it has been obtained an existence result on weak solutions of (1.1) with \( w \) the Finn's solution such that \( C_0 < 1/2 \). However,
the argument of [23] heavily depends on the property (0.3). In §3, with the use of the approach developed from [7], we shall show sharp decay estimates of solutions to the linearized equations of (1.1). If \( w \) only satisfies (0.3) and \( C_0 < 1/2 \), such estimates seem unavailable. Theorem 1.1 will be proved in §4 by making use of the estimates carried out in §3 and studying the time average \( t^{-1} \int_0^t \| u(s) \|_2 \, ds \). A similar technique has been used in [23, 4]. However, we have not used the spectral decomposition of the Stokes operator \( A \) in \( L^2 \) as usually used in earlier work concerning the \( L^2 \) decay problem. Moreover our proof seems much simpler.

It should be noted that the \( L^2 \) decay problem of (1.1) with \( w = 0 \) stems from Leray [19], and has affirmatively been solved (cf. [24, 3, 2] and the references therein). If \( 1 < p < 2 \) and \( u \) is a weak solution of (1.1) with \( w = 0 \), it has been proved that \( \| u(t) \|_2 = O(t^{-3/p - 3/2}) \) provided \( u(0) \in J^2 \cap L^p \) (cf. [2]), and \( \| u(t) \|_2 = O(t^{-3/4}) \) provided \( u(0) \in J^2 \cap L^1 \) and \( \| e^{-tA}a \|_2 \leq Ct^{-3/4} \| a \|_1 \) (cf. [3]).

2. Proof of Lemmas 1.1, 1.2. To begin with we give the estimate (cf. [2, Theorem 3.6] or [12, 18] for a similar consideration)

\[
\| Du \|_p \leq C \sup \{ |(Du, Dv)| \ ; \ v \in C^\infty, \| Dv \|_{p'} = 1 \}
\]

for \( 1 < p < n, \ u \in \hat{W}^{1,p}_\sigma \),

and the Sobolev inequality (cf. [13])

\[
\| u \|_{3p/(3-p)} \leq 2p(3-p)^{-1}3^{-1/2}\| Du \|_p
\]

for \( 1 < p < n, \ u \in \hat{W}^{1,p}_\sigma \).

**Proof of Lemma 1.1.** Let \( r \) and \( p \) be given in Lemma 1.1. We rewrite (0.1) in the abstract form \( Aw + P(w \cdot D)w = f, \ w \in \hat{W}^{1,r}_\sigma \). Since the proof of [5, (3.1)] implies that \( A \) can be extended as a bounded and invertible operator from \( \hat{W}^{2,q}_\sigma \) onto \( J^q \) with \( 1 < q < 3/2 \), we can set

\[
H: \hat{W}^{1,r}_\sigma \cap \hat{W}^{1,p}_\sigma \to \hat{W}^{2,3p/(6-p)}_\sigma \text{ such that } Hw = A^{-1}(f - P(w \cdot D)w).
\]

Let \( w \in \hat{W}^{1,r}_\sigma \cap \hat{W}^{1,p}_\sigma, \ r < s < p, \) and \( v \in C^\infty, \| Dv \|_{s'} = 1 \). Integrating by parts and using the divergence condition \( D \cdot w = 0 \), we have

\[
(DHw, Dv) = (f, v) - ((w \cdot D)w, v)
\]
\[
= (f, v) + ((w \cdot D)v, w)
\]
\[
\leq (f, v) + \| w \|_{3s/(3-s)}\| Dv \|_{s'},
\]
that is, by (2.1)-(2.2),
\[ \|DHw\|_s \leq C(\|f\|_{-1,s} + \|Dw\|_s\|Dw\|_{3/2}). \]
Similarly, for \( w, w^* \in W^{1,r}_\sigma \cap W^{1,p}_\sigma \) we have
\[ \|DHw - DHw^*\|_s \leq C(\|Dw\|_{3/2} + \|Dw^*\|_{3/2})\|Dw - Dw^*\|_s. \]
Consequently, the desired assertion follows immediately from the contraction mapping principle. The proof is complete.

In [23], Miyakawa and Sohr proved that (1.1) admits a weak solution in case \( w \) is the Finn's solution and \( C_0 < 1/2 \). However, as for our case, the argument of [23] does not work somewhere. Now we give our proof in a slightly different way. Similar to [23], we also study approximate solutions of (1.1) by applying a technique developed from [6].

**Proof of Lemma 1.2.** Let \( k > 1 \). We set \( J_k = k(k + A)^{-1} \) and \( I_k = k(k + A)^{-1}E \), where \( E \) denotes the extension operator such that \( Eu = u \) in \( G \) and \( Eu = 0 \) outside \( G \). With the use of the notation above, we have
\[
(2.3) \quad \|J_k u\|_p \leq C(k)\|u\|_r, \quad \|I_k u\|_p \leq C(k)\|u\|_r \\
\text{for } 1 < r < p \leq \infty, \ u \in J',
\]
\[
(2.4) \quad \|I_k u\|_r \leq \|u\|_r, \quad \|J_k u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < \infty, \ u \in J',
\]
where \( C \) is independent of \( k \). (2.3) is a consequence of the Sobolev embedding theorem and \( L^r \)-estimates. The first inequality in (2.4) follows from the proof of [1, Lemma 10.1], and the second one from [2, Theorem 1.2].

Now we proceed to the evolution equation
\[
(2.5) \quad (d/dt)u_k + Au_k = F_k(u_k), \quad u_k(0) = J_ka \quad \text{in } J^2,
\]
where \( F_k(u) = F_k(u, u) \) with
\[
F_k(u, v) = -P(J_k u \cdot D)v - P(J_k w \cdot D)u - P(I_k u \cdot D)I_k w.
\]
For \( u, v \in W^{1,2}_\sigma \), we have
\[
(2.6) \quad \|F_k(u, v)\|_2 + \|P(J_k v \cdot D)u\|_2 \\
\leq \|J_k u\|_\infty\|Dv\|_2 + \|J_k w\|_\infty\|Du\|_2 \\
+ \|I_k u\|_6\|DI_k w\|_3 + \|J_k v\|_\infty\|Du\|_2 \\
\leq C(k)(\|u\|_6\|Dv\|_2 + \|w\|_3\|Du\|_2) \\
+ \|u\|_6\|I_k Dw\|_3 + \|v\|_6\|Du\|_2), \quad \text{by (2.3)},
\]
\[
\leq C(k)\|Du\|_2(\|Dv\|_2 + \|Dw\|_{3/2}), \quad \text{by (2.2)}.\]
On the other hand, given $k$ and $T > 0$, we suppose that $u_k$ solve (2.5) over $[0, T)$, and $u_k \in L^2(0, T; W^{1,2}_0 \cap W^{2,2}) \cap W^{1,2}(0, T; J^2)$. Then multiplying (2.5) by $2u_k$ and $2Au_k$, respectively, we have

$$(d/dt)||u_k||_2^2 + 2||Du_k||_2^2 = 2(F_k(u_k), u_k),$$

$$(d/dt)||Du_k||_2^2 + 2||Au_k||_2^2 = 2(F_k(u_k), Au_k).$$

The estimation of the right-hand side terms of the preceding identities can be achieved as follows.

$$2(F_k(u_k), u_k) = 2((I_k u_k \cdot D)u_k, I_k w),$$

since $D \cdot J_k u_k = D \cdot J_k w = D \cdot I_k u = 0$,

$$\leq 2\|I_k u_k\|_6 \|Du_k\|_2 \|I_k w\|_3$$

$$\leq (12/3^{1/2})\|w\|_3 \|Du_k\|_2^2, \quad \text{by (2.4) and (2.2)},$$

$$\leq 8\|Dw\|_3/2 \|Du_k\|_2^2, \quad \text{by (2.2)},$$

$$\leq \|Du_k\|_2^2, \quad \text{by setting } \|Dw\|_3/2 < 1/8,$$

$$2(F_k(u_k), Au_k) \leq 2\|Au_k\|_2 (\|J_k u_k\|_\infty \|Du_k\|_2 + \|J_k w\|_\infty \|Du_k\|_2$$

$$+ \|I_k u\|_\infty \|I_k DEw\|_2)$$

$$\leq C(k)\|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_3/2 + \|DEw\|_3/2),$$

by (2.3) and (2.2),

$$\leq C(k)\|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_3/2)$$

$$\leq 2\|Au_k\|_2^2 + C(k)\|Du_k\|_2^2 (\|u_k\|_2^2 + \|Dw\|_3/2).$$

Consequently, we have

$$\|u_k(t)\|_2^2 + \int_s^t \|Du_k(z)\|_2^2 \, dz \leq \|u_k(s)\|_2^2, \quad 0 \leq s < t < T,$$

(2.7)

$$\|Du_k(t)\|_2^2$$

$$\leq \|D\cdot J_k a\|_2^2 + C(k)\int_0^t \|Du_k(s)\|_2^2(\|u_k(s)\|_2^2 + \|Dw\|_3/2) \, ds$$

$$\leq \|D\cdot J_k a\|_2^2 + C(k)\|J_k a\|_2^2(\|J_k a\|_2^2 + \|Dw\|_3/2),$$

by (2.7).

Thus, following the same way as in the proof of [23, Proposition 3.4] by making use of (2.6)–(2.8), we conclude that (2.5) admits a unique global solution $u_k$ satisfying (2.6), and $u_k \in L^2(0, T; W^{1,2}_0 \cap W^{2,2}) \cap W^{1,2}(0, T; J^2)$ for all $T > 0$. 
To obtain a weak solution of (1.1), we need to study compactness of the sequence \( u_k \). Let \( v \in W^{1,2}_\sigma \). Applying (2.2) and (2.4) repeatedly, we have, from (2.5),

\[
((d/dt)u_k, v) 
\leq \|Du_k\|_2 \|Dv\|_2 + \|I_k u_k\|_3 \|Du_k\|_2 \|v\|_6 + \|J_k w\|_3 \|Du_k\|_2 \|v\|_6 \\
+ \|u_k\|_6 \|\mathcal{D} w\|_{3/2} \|v\|_6 
\leq \|Du_k\|_2 \|Dv\|_2 + C \|v\|_6 (\|u_k\|_3 \|Du_k\|_2 + \|w\|_3 \|Du_k\|_2 \\
+ \|u_k\|_6 \|\mathcal{D} w\|_{3/2}) 
\leq C \|Dv\|_2 (\|Du_k\|_2 + \|u_k\|_2^{1/2} \|Du_k\|_2^{3/2} + \|Du_k\|_2 \|Dw\|_{3/2}) 
\leq C \|Dv\|_2 (1 + \|a\|_2^{1/2} + \|Dw\|_{3/2})(\|Du_k\|_2 + \|Du_k\|_2^{3/2}) , 
\]

by (2.7) and (2.4),

with \( C \) independent of \( k \). This together with (2.7) implies that the sequence \( u_k \) is bounded in

\[ L^\infty(0, \infty; \sigma^2) \cap L^2(0, \infty; W^{1,2}_\sigma) \cap W^{1,4/3}(0, T; W^{-1,2}) \]

for all \( 0 < T < \infty \). From [26, Theorem 2.1 in Chapter III] it follows readily that there are a function \( u \) and a subsequence of \( u_k \), denoted again \( u_k \), satisfying

\[
\begin{align*}
& u_k \rightharpoonup u \text{ in } L^\infty(0, \infty; \sigma^2), \\
& u_k \to u \text{ in } L^2(0, \infty; W^{1,2}_\sigma), \\
& u_k \to u \text{ strongly in } L^2_{\text{loc}}(G \times (0, \infty)).
\end{align*}
\]

As in [21], we can check that the limit \( u \) is a weak solution of (1.1). The proof is complete.

3. Decay estimates. In this section, we let \( t > 0, 1 < r < 3/2 < p < 2 \), and \( w \) be a solution of (0.1) such that \( w \in \hat{W}^{1,r}_\sigma \cap \hat{W}^{1,p}_\sigma \), and set

\[
Lu = Au + P(u \cdot D)w + P(w \cdot D)u , \\
B^* u = -p(w \cdot D)u + P \sum_{i=1}^n u^i Dw^i , \\
L^* u = Au + B^* u .
\]

Thus, we see that

\[
(Lu, v) = (u, L^* v) \quad \text{for } u, v \in W^{1,2}_\sigma \cap W^{2,2} ,
\]
and the linearized equation of (1.1) can be stated in the form
\[(d/dt)v + Lv = 0, \quad v(0) = u.\]
Denote by \(e^{-tL}u\) the solution of the preceding equation. It is the purpose of this section to prove the following.

**PROPOSITION 3.1.** Suppose that \(\|Dw\|_r + \|Dw\|_p\) is sufficiently small. Then there holds
\[(3.1) \quad \|e^{-tL}Pu\|_2 \leq Ct^{-3/4}\|u\|_1\]
for \(u \in L^1 \cap L^{6/5} \). The preceding proposition is based on the following decay estimates.

(3.2) \(\|e^{-tA}u\|_\infty \leq Ct^{-1/4}\|u\|_6\) for \(u \in J^6\),

(3.3) \(\|e^{-tA}u\|_s \leq Ct^{(3q-3s)/2}\|u\|_q\) for \(1 < q \leq s < \infty, u \in J^q\),

(3.4) \(\|De^{-tA}u\|_s \leq Ct^{(1+3q-3s)/2}\|u\|_q\) for \(1 < q \leq s \leq 3, u \in J^q\). The estimates (3.3) and (3.4) were recently obtained by Iwashita (cf. [17, Theorems 1.2, 1.3]). (3.2) will be proved in the Appendix by using the argument of [17].

With the use of (3.2)–(3.4), we can now prove the following.

**LEMMA 3.1.** Let \(u \in C_0^\infty \). Then there hold
\[(3.5) \quad \|e^{-tA}u\|_\infty \leq Ct^{-3/4}\|u\|_2,\]

(3.6) \(\|e^{-tA}B^*u\|_\infty + \|De^{-tA}B^*u\|_3 \leq Ct^{-3/2p}(t + 1)^{(3/r-3/p)/2}(\|u\|_\infty + \|Du\|_3)(\|Dw\|_r + \|Dw\|_p).\]

**Proof.** From (3.2), (3.3), (2.2) and the semigroup property of \(e^{-tA}\) we get (3.5) and
\[
\|e^{-tA}B^*u\|_\infty \leq Ct^{-3/2b}\|B^*u\|_b \\
\leq Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3)
\]
for \(b = r, p\). Moreover (3.4) and (2.2) yield
\[
\|De^{-tA}B^*u\|_3 \leq Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3) \quad \text{for } b = r, p.
\]
Collecting terms, we get readily (3.6) and complete the proof.

**Proof of Proposition 3.1.** Setting \(v(t) = e^{-tL}u\) with \(u \in C_0^\infty \), we have obviously that \(v \in C([0, \infty); L^\infty \cap W^{1,3}_0)\) and
\[
v(t) = e^{-tA}u + \int_0^t e^{-(t-s)A}B^*v(s)\,ds.
\]
This gives, by (3.4)-(3.6),
\[ \|v(t)\|_\infty + \|Dv(t)\|_3 \]
\[ \leq C t^{-3/4} \|u\|_2 + C \int_0^t (t-s)^{-3/2}(t-s+1)^{-3(r-3/p)/2} \]
\[ \times (\|v\|_\infty + \|Dv\|_3) \, ds (\|Dw\|_r + \|Dw\|_p). \]

Setting \( \|v\|_r = \sup_{0 < s < t} s^{3/4}(\|v(s)\|_\infty + \|Dv(s)\|_3) \), we have
\[ \|v(t)\|_\infty + \|Dv(t)\|_3 \]
\[ \leq C t^{-3/4} \|u\|_2 + C (\|Dw\|_r + \|Dw\|_p) \|v\|_r \]
\[ \times \int_0^t (t-s)^{-3/2}(t-s+1)^{-3(r-3/p)/2} \, ds \]
\[ \leq C t^{-3/4} \|u\|_2 + C t^{-3/4}(\|Dw\|_r + \|Dw\|_p) \|v\|_r \]
\[ \times \int_0^t s^{-3/2}(s+1)^{-3(r-3/p)/2} \, ds \]
\[ + C t^{1/4-3/2} (t+1)^{-3(r-3/p)/2} (\|Dw\|_r + \|Dw\|_p) \|v\|_r. \]
where we have used the condition \( r < 3/2 < p \). Hence, if we presuppose that
\[ (3.7) \quad C(\|Dw\|_r + \|Dw\|_p) < 1/2 \]
with the constant \( C \) given in the last term above, we obtain
\[ (3.8) \quad \|e^{-tL^*} u\|_\infty \leq C t^{-3/4} \|u\|_2. \]

Now we take \( u \in L^1 \cap L^{6/5} \) and \( v \in L^2 \). By (3.8) we have
\[ (e^{-tL} Pu, v) = (u, e^{-tL^*} P_v) \leq \|u\|_1 \|e^{-tL^*} P_v\|_\infty \leq C t^{-3/4} \|u\|_1 \|v\|_2 \]
and therefore the validity of (3.1). The proof is complete.

4. Proof of Theorem 1.1. In this section we always suppose that the stationary solution \( w \in \vec{W}^{1,r}_{\alpha} \cap \vec{W}^{1,p}_{\alpha} \) with \( 1 < r < 3/2 < p < 2 \) such that (3.7) holds. Let \( u \) be a weak solution of (1.1). Then (1.2) implies
\[ (4.1) \quad \|u(t)\|_2 \leq t^{-1} \int_0^t \|u(s)\|_2 \, ds. \]

On the other hand, taking \( v \in C^\infty_0 \) and applying (1.3) with \( g(z) = e^{-(t-z)L^*} v \), we have
\[ (u(t), v) + \int_0^t (Lu(s), e^{-(t-s)L^*} v) \, ds - \int_0^t (u(s), L^* e^{-(t-s)L^*} v) \, ds \]
\[ = (a, e^{-tL^*} v) - \int_0^t ((u \cdot D)u, e^{-(t-s)L^*} v) \, ds, \]
that is,

\[(u(t), v) = (e^{-tL}a, v) - \int_0^t (e^{-(t-s)L}P(u \cdot D)u(s), v) \, ds\]

\[\leq \|e^{-tL}a\|_2\|v\|_2 + \int_0^t \|e^{-(t-s)L}P(u \cdot D)u(s)\|_2 \, ds\|v\|_2\]

\[\leq C\|v\|_2 \left(t^{-3/4}\|a\|_1 + \int_0^t (t-s)^{-3/4}\|u(s)\|_2\|Du(s)\|_2 \, ds\right),\]

where we have used (3.1). We then get

\[\|u(s)\|_2 \leq Cs^{-3/4}\|a\|_1 + C \int_0^s (s-z)^{-3/4}\|u(z)\|_2\|Du(z)\|_2 \, dz.\]

Integrating the above inequality from 0 to \(t\), we have

\[\int_0^t \|u(s)\|_2 \, ds \leq Ct^{1/4}\|a\|_1 + C \int_0^t \, ds \int_0^t (s-z)^{-3/4}\|u(z)\|_2\|Du(z)\|_2 \, ds\]

\[\leq Ct^{1/4}\|a\|_1 + Ct^{1/4} \int_0^t \|u(s)\|_2\|Du(s)\|_2 \, ds\]

\[\leq Ct^{1/4}\|a\|_1 + Ct^{1/4}\|a\|_2 \left(\int_0^t \|u(s)\|_2^2 \, ds\right)^{1/2}, \text{ by (1.2).}\]

Combining this with (4.1), we have

\[\|u(t)\|_2 \leq Ct^{-3/4}\|a\|_1 + Ct^{-3/4}\|a\|_2 \left(\int_0^t \|u(s)\|_2^2 \, ds\right)^{1/2},\]

that is,

\[\|u(t)\|_2 \leq C_1 t^{-3/4} \left(1 + \left(\int_0^t \|u(s)\|_2^2 \, ds\right)^{1/2}\right),\]

where and in what follows \(C_1 = C_1(\|a\|_1, \|a\|_2)\) may vary from line to line.

Now we apply (4.2) and (1.2) to complete our proof via a boot strap iteration argument.

Note that

\[\|u(t)\|_2 \leq C_1, \text{ by (1.2),}\]

and

\[\|u(t)\|_2 \leq C_1 t^{-3/4}(1 + t^{1/2}), \text{ by (4.2) and (4.3).}\]

Combining (4.4) with (4.3), we have

\[\|u(t)\|_2 \leq C_1 t^{-1/4}.\]
Moreover, taking (4.2) and (4.5) into account, we have
\[ \|u(t)\|_2 \leq C_1 t^{-3/4}(1 + t^{1/4}). \]
This together with (4.3) implies
\[ (4.6) \quad \|u(t)\|_2 \leq C_1 (t + 1)^{-1/2}. \]
Similarly, (4.2) and (4.6) yield
\[ \|u(t)\|_2 \leq C_1 t^{-3/4}(1 + \ln(t + 1)), \]
and so, by (4.3),
\[ (4.7) \quad \|u(t)\|_2 \leq C_1 (t + 1)^{-2/3}. \]
Finally, by (4.2) and (4.7), we arrive at the desired estimate
\[ \|u(t)\|_2 \leq C_1 t^{-3/4} \]
and complete the proof.

**Remark 4.1.** It should be noted that the validity of the assumption of Lemma 1.2 follows from the inequality \( \|Dw\|_{3/2} \leq \|Dw\|_r + \|Dw\|_p \) and (2.7).

**Appendix: Proof of (3.2).** Let \( Q \) be a domain of \( R^3 \). By \( \| \cdot \|_{k,p,Q} \) and \( \| \cdot \|_{p,Q} \) we denote respectively the norms of the Sobolev space \( W^{k,p}(Q; R^3) \) and the Lebesgue space \( L^p(Q; R^3) \). Of course, \( \| \cdot \|_{k,p} = \| \cdot \|_{k,p,G} \) and \( \| \cdot \|_p = \| \cdot \|_{p,G} \). \( \bar{P} \) is the bounded projection from \( L^p(R^3; R^3) \) onto \( J^p(R^3; R^3) \), where \( J^p(R^3; R^3) \) denotes the completion of the set of compactly supported solenoidal in \( C^\infty(R^3; R^3) \). Let \( h \) be a constant such that \( |x| < h - 1 \) for \( x \in \partial G \), and let \( g \in C^\infty(R^3; R) \) be a fixed function such that \( g = 1 \) for \( |x| > h \) and \( g = 0 \) for \( |x| < h - 1 \). Moreover we set \( G_h = \{ x \in G; |x| < h \} \).

In arriving at (3.2), we need the following lemmas.

**Lemma A.1.** Let \( 1 < p \leq q < \infty \), \( t > 0 \), \( v \in L^p(R^3; R^3) \cap L^q(R^3; R^3) \), \( n > 1 \), and \( u \in J^6 \). Then we have
\[ (A.1) \quad \|e^{-tA}v\|_{\infty,R^3} \leq Ct^{-3/2q}(t + 1)^{-(3/p - 3/q)/2} (\|v\|_{p,R^3} + \|v\|_{q,R^3}), \]
\[ (A.3) \quad \|e^{-tA}u\|_{2n,6} \leq C(t^{-n} + 1)\|u\|_6. \]

(A.1) is deduced immediately by an elementary calculation. (A.2) is a consequence of \( L^p \)-estimates (cf. [25]) and the Sobolev embedding theorem. One can also refer to [17] for details.
**Lemma A.2** ([17, Lemmas 5.3, 5.4] and (A.2)). Let $t > 0$, $v \in J^6$, and $P^*$ be a certain pressure such that $p^* = Ae^{-(t+1)A}v + \Delta e^{-(t+1)A}v$. Then
\[
\|e^{-(t+1)A}v\|_{2, 6, G_h} + \|Ae^{-(t+1)A}v\|_{2, 6, G_h} + \|p^*(t)\|_{3, 6, G_h} \leq Ct^{-1/4}\|v\|_6.
\]

**Lemma A.3** ([17, (5.18)] and (A.2)). Let $v \in J^6$, and $t > 0$. Then there is a function $v^*$ such that
\[
\begin{align*}
D \cdot v^* &= D \cdot (ge^{-(t+1)A}v), \\
\text{supp } v^*(t) &\subset \{x \in \mathbb{R}^3; h - 1 < |x| < h\}, \\
\|v^*(t)\|_{2, 6} + \|(\partial / \partial t)v^*(t)\|_6 &\leq C(t + 1)^{-1/4}\|v\|_6.
\end{align*}
\]

**Lemma A.4.** Let $t > 0$, $v$ and $v^*$ be given in Lemma A.3. Then we have
\[
\|ge^{-(t+1)A}v - v^*(t)\|_\infty \leq C(t + 1)^{-1/4}\|v\|_6.
\]

**Proof.** Set $u(t) = ge^{-(t+1)A}v - v^*(t)$, $u_0 = u(0)$, and
\[
\begin{align*}
F(t) &= p^*(t)Dg - 2(Dg \cdot D)e^{-(t+1)A}v - (\Delta g)e^{-(t+1)A}v \\
&\quad + \Delta v^*(t) - (\partial / \partial t)v^*(t),
\end{align*}
\]
where $p^*$ is given in Lemma A.2. By Lemmas A.2, A.3 we have that the support of $F(t)$ is contained in $\{x \in \mathbb{R}^3; h - 1 < |x| < h\}$, and
\[
(t + 1)^{1/4}\|F(t)\|_6 + \|u_0\|_{1, 6} \leq C\|v\|_6, \\
u_t - \Delta u + D(gp^*) = F, \quad D \cdot u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty).
\]

We thus rewrite $u$ in the integral form
\[
\begin{align*}
u(t) &= e^{-tA}u_0 + \int_0^t e^{-(t-s)A}FP(s) \, ds.
\end{align*}
\]

From (A.1), (A.3), and Sobolev's embedding theorem it follows that
\[
\|e^{-tA}u_0\|_{\infty, \mathbb{R}^3} \leq C(t + 1)^{-1/4}(\|u_0\|_\infty + \|u_0\|_6) \leq Ct^{-1/4}\|v\|_6,
\]
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and

\[ \left\| \int_0^t e^{-(t-s)A}F(s)\,ds \right\|_{\infty, \mathbb{R}^3} \leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}\|F(s)\|_{3, g_h} + \|F(s)\|_{6/5, g_h}\,ds \]

\[ \leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}\|F(s)\|_6\,ds \]

\[ \leq C\|v\|_6 \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}(s+1)^{-1/4}\,ds \]

\[ \leq C(t+1)^{-1/4}\|v\|_6. \]

Taking (A.4) into account, we have the desired estimate and complete the proof.

**Proof of (3.2).** Let \( v \in J^6 \). By Lemmas A.1, A.2, A.3, Sobolev inequality, and Gagliardo-Nirenberg inequality (cf. [10]), we have

\[ \|e^{-(t+1)A}v\|_\infty \leq \|ge^{-(t+1)A}v\|_\infty + \|e^{-(t+1)A}v\|_{\infty, g_h} \]

\[ \leq \|ge^{-(t+1)A}v - v^*(t)\|_\infty + C\|v^*(t)\|_{1, 6} + C\|e^{-(t+1)A}v\|_{1, 6, g_h} \]

\[ \leq C(t+1)^{-1/4}\|v\|_6 \quad \text{for } t > 0, \]

\[ \|e^{-tA}v\|_\infty \leq C\|e^{-tA}v\|^{3/4}_{6} \|e^{-tA}v\|^{1/4}_{2, 6} \]

\[ \leq C(t^{-1} + 1)^{1/4}\|v\|_6 \leq Ct^{-1/4}\|v\|_6 \]

for \( 1 > t > 0 \). The proof is complete.

The author would like to thank T. Miyakawa for sending [2, 3, 4]. He would also like to thank the referee for his valuable suggestions.

**References**


Received October 7, 1991 and in revised form March 13, 1992.
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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

This publication was typeset using \textsc{Ams-T\TeX},
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