SOLUTIONS OF THE STATIONARY AND NONSTATIONARY
NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

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It is shown that a nonstationary exterior Navier-Stokes flow tends to a small stationary flow in $L^2$ like $t^{-3/4}$ as $t \to \infty$.

0. Introduction. In this paper we are concerned with the stationary Navier-Stokes equations

\begin{equation}
(w \cdot D)w - \Delta w + D\bar{p} = f, \quad D \cdot w = 0 \quad \text{in } G,
\end{equation}

\begin{equation}
w = 0 \quad \text{on } \partial G \quad (D = \text{grad}),
\end{equation}

and the nonstationary Navier-Stokes equations

\begin{equation}
v_t + (v \cdot D)v - \Delta v + D\bar{p} = f \quad \text{in } G \times (0, \infty),
\end{equation}

\begin{equation}
D \cdot v = 0 \quad \text{in } G \times (0, \infty),
\end{equation}

\begin{equation}
v = 0 \quad \text{on } \partial G \times (0, \infty),
\end{equation}

\begin{equation}
v|_{t=0} = a + w \quad \text{in } G \quad (v_t = \partial v/\partial t).
\end{equation}

Here and in what follows $G$ denotes a smooth exterior domain of $R^3$, $f = f(x)$ is a prescribed vector field, and $\bar{p}$ (resp. $\bar{p}$) represents unknown stationary (resp. nonstationary) scalar pressure which can be determined by the stationary solution $w$ via (0.1) (resp. nonstationary solution $v$ via (0.2)).

As is well known, it was shown by Finn [8, 9] that (0.1) admits a small solution

\begin{equation}
w \in L^\infty(G; R^3), \quad Dw \in L^3(G; R^3),
\end{equation}

\begin{equation}C_0 = \sup_{x \in G} |x| |w(x)| < \infty.
\end{equation}

If $C_0 < 1/2$ the Finn’s solution $w$ may be formed as a limit of a nonstationary solution $v$ as $t \to \infty$ in local or global $L^2$-norms (cf. Heywood [15, 14], Galdi and Rionero [11], Miyakawa and Sohr [23]) and in other norms (cf. Heywood [16], Masuda [20]). Moreover it has recently proved (cf. Borchers and Miyakawa [4]) that every weak solution of (0.2) tends the Finn’s solution in $L^2(G; R^3)$.
like \( t^{-(3/p-3/2)/2} \) with \( 6/5 < p < 2 \), provided \( C_0 < 1/2 \) and \( a \in L^2(G; R^3) \cap L^p(G; R^3) \).

In this paper we are only interested in the case \( w \in L^3(G; R^3) \), \( Dw \in L^3(R^3; G) \), or \( Dw \in L^r(G; R^9) \cap L^p(G; R^3) \) with \( 1 < r < 3/2 < p < 2 \). Under certain smallness assumptions on \( w \) we show now that every weak solution of (0.2) tends to the stationary solution \( w \) in \( L^2(G; R^3) \) like the sharp decay rate \( t^{-3/4} \).

1. Notation and main result. In this paper we use the following spaces.

\( L^p = \) the Lebesgue spaces \( L^p(G; R^3) \), with \( \| \cdot \|_p \) the associated norm,
\( C^\infty = \) the set of compactly supported solenoidal in \( C^\infty(G; R^3) \),
\( W^{k,p} = \) the Sobolev space \( W^{k,p}(G; R^3) \),
\( J^p = \) the completion of \( C^\infty \) in \( L^p \),
\( W^{1,p} = \) the completion of \( C^\infty \) in \( W^{1,p} \),
\( \widehat{W}^{1,p} = \) the completion of \( C^\infty \) under the norm \( \| D \cdot \|_p \),
\( \widehat{W}^2,p = \) the space \( \{ u \in \widehat{W}^{1,3p/(3-p)}; D^2 u \in L^p(G; R^{27}) \} \) for \( 1 < p < 3 \),
\( W^{-1,2} = \) the dual of \( W^{1,2} \),
\( \widehat{W}^{-1,p} = \) the dual of \( \widehat{W}^{1,p/(p-1)} \), with \( \| \cdot \|_{-1,p} \) the associated norm.

Moreover for \( 1 < r < \infty \) and \( n \geq 1 \), we denote by \( r' \) the real \( r/(r-1) \), by \( (\cdot, \cdot) \) the inner product in \( L^2(G; R^n) \), by \( P \) the bounded projection from \( L^r \) onto \( J^r \) (cf. [22]), by \( A \) the Stokes operators \( -P\Delta \) with the domain \( W^{1,r} \cap W^{2,r} \), by \( \overline{A} \) the Laplacian \( -\Delta \) with the domain \( W^{2,r}(R^3; R^3) \), and by \( C \) a positive constant which may vary from line to line, but is always independent of the quantities \( t, T, u, v, w, f, u_k \), and \( a \).

Now we make preparations for stating our main result. The existence of the stationary solutions \( w \) is guaranteed by the following.

**Lemma 1.1.** Let \( 1 < r \leq 3/2 < p < 2 \), and \( f \in C^\infty \). Then there is a small \( h > 0 \) such that (0.1) admits a unique solution within the class

\[ \{ w \in \widehat{W}^{1,r} \cap \widehat{W}^{1,p}; \| Dw \|_{3/2} \leq h \}, \]

provided that \( \| f \|_{-1,3/2} \leq h^2 \). Moreover

\[ \| Dw \|_r + \| Dw \|_p \leq C(\| f \|_{-1,r} + \| f \|_{-1,p}). \]
From (0.1) and (0.2) we see that $u = v - w$ and $\dot{p} = \bar{p} - \bar{p}$ solve the problem

$$
(1.1) \quad u_t + (u \cdot D)u - \Delta u + (u \cdot D)w + (w \cdot D)u + D\dot{p} = 0,
$$

$$
D \cdot u = 0 \quad \text{in } G \times (0, \infty),
$$

$$
u = 0 \quad \text{on } \partial G \times (0, \infty),
$$

$$
u|_{t=0} = a \quad \text{in } G.
$$

Weak solutions are given in the following sense.

**Definition 1.1.** Let $a \in J^2$, and $w \in \dot{W}^{1,3/2}_\sigma$ solve (0.1). A weakly continuous function $u: [0, \infty) \to J^2$ is said to be a weak solution of (1.1) if $u(0) = a$, $u \in L^\infty(0, \infty; J^2) \cap L^2(0, \infty; \dot{W}^{1,2}_\sigma)$,

$$
(1.2) \quad \|u(t)\|_2^2 + \int_s^t \|Du(z)\|_2^2 \, dz \leq \|u(s)\|_2^2,
$$

$$
(1.3) \quad (u(t), g(t)) + \int_s^t ((Du, Dg) + ((u \cdot D)w, g)
$$

$$
+ ((w \cdot D)u, g) - (u, g_z)) \, dz
$$

$$
= (u(s), g(s)) - \int_s^t ((u \cdot D)u, g) \, dz
$$

for all $t > s \geq 0$ and all $g \in C([0, \infty); W^{1,2}_\sigma) \cap C^1([0, \infty); J^2)$, where $g_z = \partial g / \partial z$.

The existence of weak solutions to (1.1) is guaranteed by the following.

**Lemma 1.2.** Let $a \in J^2$, and $w \in \dot{W}^{1,3/2}_\sigma$ such that $\|Dw\|_{3/2} < 1/8$. Then (1.1) admits a weak solution.

We are now in a position to state our main result.

**Theorem 1.1.** Let $1 < r < 3/2 < p < 2$, $a \in J^2 \cap L^1$, and let $w \in W^{1,r}_\sigma \cap W^{1,p}_\sigma$ such that $w$ solves (0.1) and $\|Dw\|_r + \|Dw\|_p$ is sufficiently small. Then every weak solution of (1.1) possesses the sharp decay property

$$
\|u(t)\|_2 = O(t^{-3/4}).
$$

Section 2 is concerned with the proof of Lemmas 1.1 and 1.2. In [23], it has been obtained an existence result on weak solutions of (1.1) with $w$ the Finn's solution such that $C_0 < 1/2$. However,
the argument of [23] heavily depends on the property (0.3). In §3, with the use of the approach developed from [7], we shall show sharp decay estimates of solutions to the linearized equations of (1.1). If $w$ only satisfies (0.3) and $C_0 < 1/2$, such estimates seem unavailable. Theorem 1.1 will be proved in §4 by making use of the estimates carried out in §3 and studying the time average $t^{-1} \int_0^t \|u(s)\|_2 \, ds$. A similar technique has been used in [23, 4]. However, we have not used the spectral decomposition of the Stokes operator $A$ in $L^2$ as usually used in earlier work concerning the $L^2$ decay problem. Moreover our proof seems much simpler.

It should be noted that the $L^2$ decay problem of (1.1) with $w = 0$ stems from Leray [19], and has affirmatively been solved (cf. [24, 3, 2] and the references therein). If $1 < p < 2$ and $u$ is a weak solution of (1.1) with $w = 0$, it has been proved that $\|u(t)\|_2 = O(t^{-(3/p - 3/2)/2})$ provided $u(0) \in J^2 \cap L^p$ (cf. [2]), and $\|u(t)\|_2 = O(t^{-3/4})$ provided $u(0) \in J^2 \cap L^1$ and $\|e^{-tA}a\|_2 \leq Ct^{-3/4}\|a\|_1$ (cf. [3]).

2. Proof of Lemmas 1.1, 1.2. To begin with we give the estimate (cf. [2, Theorem 3.6] or [12, 18] for a similar consideration)

\begin{equation}
\|Du\|_p \leq C \sup \{|(Du, Dv)|; v \in C_\sigma^\infty, \|Dv\|_{p'} = 1\}
\end{equation}

for $1 < p < n$, $u \in \hat{W}_\sigma^{1,p}$,

and the Sobolev inequality (cf. [13])

\begin{equation}
\|u\|_{3p/(3-p)} \leq 2p(3-p)^{-1}3^{-1/2}\|Du\|_p
\end{equation}

for $1 < p < n$, $u \in \hat{W}_\sigma^{1,p}$.

Proof of Lemma 1.1. Let $r$ and $p$ be given in Lemma 1.1. We rewrite (0.1) in the abstract form $Aw + P(w \cdot D)w = f$, $w \in \hat{W}_\sigma^{1,r} \cap \hat{W}_\sigma^{1,p}$. Since the proof of [5, (3.1)] implies that $A$ can be extended as a bounded and invertible operator from $\hat{W}_\sigma^{2,q}$ onto $J^q$ with $1 < q < 3/2$, we can set

$H: \hat{W}_\sigma^{1,r} \cap \hat{W}_\sigma^{1,p} \to \hat{W}_\sigma^{2,3p/(6-p)}$ such that $Hw = A^{-1}(f - P(w \cdot D)w)$.

Let $w \in \hat{W}_\sigma^{1,r} \cap \hat{W}_\sigma^{1,p}$, $r < s < p$, and $v \in C_\sigma^\infty$ with $\|Dv\|_{s'} = 1$. Integrating by parts and using the divergence condition $D \cdot w = 0$, we have

$(DHw, Dv) = (f, v) - ((w \cdot D)w, v)$
\begin{align*}
= (f, v) + ((w \cdot D)v, w) \\
\leq (f, v) + \|w\|_3 \|w\|_{3s/(3-s)} \|Dv\|_{s'}
\end{align*}
that is, by (2.1)–(2.2),
\[ \|DHw\|_s \leq C(\|f\|_{-1,s} + \|Dw\|_s \|Dw\|_{3/2}). \]

Similarly, for \( w, w^* \in W^{1,r}_\sigma \cap W^{1,p}_\sigma \) we have
\[ \|DHw - DHw^*\|_s \leq C(\|Dw\|_{3/2} + \|Dw^*\|_{3/2}) \|Dw - Dw^*\|_s. \]

Consequently, the desired assertion follows immediately from the contraction mapping principle. The proof is complete.

In [23], Miyakawa and Sohr proved that (1.1) admits a weak solution in case \( w \) is the Finn’s solution and \( C_0 < 1/2 \). However, as for our case, the argument of [23] does not work somewhere. Now we give our proof in a slightly different way. Similar to [23], we also study approximate solutions of (1.1) by applying a technique developed from [6].

**Proof of Lemma 1.2.** Let \( k > 1 \). We set \( J_k = k(k + A)^{-1} \) and \( I_k = k(k + A)^{-1}E \), where \( E \) denotes the extension operator such that \( Eu = u \) in \( G \) and \( Eu = 0 \) outside \( G \). With the use of the notation above, we have
\[
\begin{align*}
(2.3) \quad &\|J_k u\|_p \leq C(k)\|u\|_r, \quad \|I_k u\|_p \leq C(k)\|u\|_r, \\
&\text{for } 1 < r < p \leq \infty, \ u \in J^r, \\
(2.4) \quad &\|I_k u\|_r \leq \|u\|_r, \quad \|J_k u\|_r \leq C\|u\|_r, \quad \text{for } 1 < r < \infty, \ u \in J^r,
\end{align*}
\]
where \( C \) is independent of \( k \). (2.3) is a consequence of the Sobolev embedding theorem and \( L^r \)-estimates. The first inequality in (2.4) follows from the proof of [1, Lemma 10.1], and the second one from [2, Theorem 1.2].

Now we proceed to the evolution equation
\[
(2.5) \quad (d/dt)u_k + Au_k = F_k(u_k), \quad u_k(0) = J_k a \quad \text{in } J^2,
\]
where \( F_k(u) = F_k(u, u) \) with
\[
F_k(u, v) = -P(J_k u \cdot D)v - P(J_k w \cdot D)u - P(I_k u \cdot D)I_k w.
\]

For \( u, v \in W^{1,2}_\sigma \), we have
\[
(2.6) \quad \|F_k(u, v)\|_2 + \|P(J_k u \cdot D)v\|_2 \leq \|J_k u\|_\infty \|Dv\|_2 + \|J_k w\|_\infty \|Du\|_2 + \|I_k u\|_6 \|DI_k w\|_3 + \|J_k v\|_\infty \|Du\|_2 \leq C(k)(\|u\|_6 \|Dv\|_2 + \|w\|_3 \|Du\|_2 + \|u\|_6 \|I_k DEw\|_3 + \|v\|_6 \|Du\|_2), \quad \text{by } (2.3),
\]
\[
\leq C(k)\|Du\|_2(\|Dv\|_2 + \|Dw\|_{3/2}), \quad \text{by } (2.2).
\]
On the other hand, given \( k \) and \( T > 0 \), we suppose that \( u_k \) solve (2.5) over \([0, T)\), and \( u_k \in L^2(0, T; W^{1,2}_\sigma \cap W^{2,2}) \cap W^{1,2}(0, T; J^2) \). Then multiplying (2.5) by \( 2u_k \) and \( 2Au_k \), respectively, we have
\[
\frac{d}{dt}\|u_k\|_2^2 + 2\|Du_k\|_2^2 = 2(F_k(u_k), u_k),
\]
\[
\frac{d}{dt}\|Du_k\|_2^2 + 2\|Au_k\|_2^2 = 2(F_k(u_k), Au_k).
\]

The estimation of the right-hand side terms of the preceding identities can be achieved as follows.

\[
2(F_k(u_k), u_k) = 2((I_k u_k \cdot D)u_k, I_k w),
\]
since \( D \cdot J_k u_k = D \cdot J_k w = D \cdot I_k u = 0 \),
\[
\leq 2\|I_k u_k\|_6 \|Du_k\|_2 \|I_k w\|_3,
\]
\[
\leq (12/3^{-1/2}) \|w\|_3 \|Du_k\|_2^2, \quad \text{by (2.4) and (2.2),}
\]
\[
\leq 8 \|Dw\|_{3/2} \|Du_k\|_2^2, \quad \text{by (2.2),}
\]
\[
\leq \|Du_k\|_2^2, \quad \text{by setting } \|Dw\|_{3/2} < 1/8,
\]

\[
2(F_k(u_k), Au_k)
\]
\[
\leq 2\|Au_k\|_2 (\|J_k u_k\|_\infty \|Du_k\|_2 + \|J_k w\|_\infty \|Du_k\|_2
\]
\[
+ \|I_k u\|_\infty \|I_k Dw\|_2)
\]
\[
\leq C(k) \|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_{3/2} + \|Dw\|_{3/2})
\]
\[
\leq C(k) \|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_{3/2})
\]
\[
\leq 2\|Au_k\|_2^2 + C(k) \|Du_k\|_2^2 (\|u_k\|_2^2 + \|Dw\|_{3/2}^2).
\]

Consequently, we have
\[
(2.7) \quad \|u_k(t)\|_2^2 + \int_s^t \|Du_k(z)\|_2^2 \, dz \leq \|u_k(s)\|_2^2, \quad 0 \leq s < t < T,
\]
\[
(2.8) \quad \|Du_k(t)\|_2^2
\]
\[
\leq \|DJ_k a\|_2^2 + C(k) \int_0^t \|Du_k(s)\|_2^2 (\|u_k(s)\|_2^2 + \|Dw\|_{3/2}^2) \, ds
\]
\[
\leq \|DJ_k a\|_2^2 + C(k) \|J_k a\|_2^2 (\|J_k a\|_2^2 + \|Dw\|_{3/2}^2), \quad \text{by (2.7)}
\]

Thus, following the same way as in the proof of [23, Proposition 3.4] by making use of (2.6)–(2.8), we conclude that (2.5) admits a unique global solution \( u_k \) satisfying (2.6), and \( u_k \in L^2(0, T; W^{1,2}_\sigma \cap W^{2,2}) \cap W^{1,2}(0, T; J^2) \) for all \( T > 0 \).
To obtain a weak solution of (1.1), we need to study compactness of the sequence $u_k$. Let $v \in W^{1,2}_\sigma$. Applying (2.2) and (2.4) repeatedly, we have, from (2.5),

\[
((d/dt)u_k, v) \leq ||D u_k||_2 ||D v||_2 + ||D I u_k||_6 ||D I_k w||_3/2 ||v||_6
\]

\[
\leq ||D u_k||_2 ||D v||_2 + C ||v||_6 (||u_k||_3 ||D u_k||_2 + ||w||_3 ||D u_k||_2
\]

\[
+ ||u_k||_6 ||D E w||_3/2)
\]

\[
\leq C ||D v||_2 (||D u_k||_2 + ||u_k||_2^{1/2} ||D u_k||_2^{3/2} + ||D u_k||_2 ||D w||_3/2)
\]

\[
\leq C ||D v||_2 (1 + ||a||_2^{1/2} + ||D w||_3/2)(||D u_k||_2 + ||D u_k||_2^{3/2}),
\]

by (2.7) and (2.4),

with $C$ independent of $k$. This together with (2.7) implies that the sequence $u_k$ is bounded in

$L^\infty(0, \infty; J^2) \cap L^2(0, \infty; \widetilde{W}^{1,2}_\sigma) \cap W^{1,4/3}(0, T; W^{-1,2})$

for all $0 < T < \infty$. From [26, Theorem 2.1 in Chapter III] it follows readily that there are a function $u$ and a subsequence of $u_k$, denoted again $u_k$, satisfying

\[
u_k \overset{w^*}{\rightharpoonup} u \text{ in } L^\infty(0, \infty; J^2),
\]

\[
u_k \overset{w}{\rightharpoonup} u \text{ in } L^2(0, \infty; \widetilde{W}^{1,2}_\sigma),
\]

\[u_k \rightharpoonup u \text{ strongly in } L^2_{\text{loc}}(G \times (0, \infty)).
\]

As in [21], we can check that the limit $u$ is a weak solution of (1.1). The proof is complete.

3. Decay estimates. In this section, we let $t > 0$, $1 < r < 3/2 < p < 2$, and $w$ be a solution of (0.1) such that $w \in \widetilde{W}^{1,r}_\sigma \cap \widetilde{W}^{1,p}_\sigma$, and set

\[L u = Au + P(u \cdot D)w + P(w \cdot D)u,
\]

\[B^* u = -p(w \cdot D)u + P \sum_{i=1}^n u_i Dw_i,
\]

\[L^* u = Au + B^* u.
\]

Thus, we see that

\[(Lu, v) = (u, L^* v) \text{ for } u, v \in W^{1,2}_\sigma \cap W^{2,2}_\sigma,
\]
and the linearized equation of (1.1) can be stated in the form
\[(d/dt)v + Lv = 0, \quad v(0) = u.\]
Denote by \(e^{-tL}u\) the solution of the preceding equation. It is the purpose of this section to prove the following.

**Proposition 3.1.** Suppose that \(\|Dw\|_r + \|Dw\|_p\) is sufficiently small. Then there holds

\[\|e^{-tL}Pu\|_2 \leq Ct^{-3/4}\|u\|_1\]
for \(u \in L^1 \cap L^{6/5}\).

The preceding proposition is based on the following decay estimates.

\[\|e^{-tA}u\|_\infty \leq Ct^{-1/4}\|u\|_6 \quad \text{for } u \in J^6,\]
\[\|e^{-tA}u\|_s \leq Ct^{-(3q-3s)/2}\|u\|_q \quad \text{for } 1 < q \leq s < \infty, \quad u \in J^q,\]
\[\|De^{-tA}u\|_s \leq Ct^{-(1+3q-3s)/2}\|u\|_q \quad \text{for } 1 < q \leq s \leq 3, \quad u \in J^q.\]

The estimates (3.3) and (3.4) were recently obtained by Iwashita (cf. [17, Theorems 1.2, 1.3]). (3.2) will be proved in the Appendix by using the argument of [17].

With the use of (3.2)–(3.4), we can now prove the following.

**Lemma 3.1.** Let \(u \in C^\infty_0\). Then there hold

\[\|e^{-tA}u\|_\infty \leq Ct^{-3/4}\|u\|_2,\]
\[\|e^{-tA}B^*u\|_\infty + \|De^{-tA}B^*u\|_3 \leq Ct^{-3/2p}(t + 1)^{-(3/r-3/p)/2}\|u\|_\infty + \|Du\|_3)\]
\[\|Dw\|_r + \|Dw\|_p).\]

**Proof.** From (3.2), (3.3), (2.2) and the semigroup property of \(e^{-tA}\) we get (3.5) and
\[\|e^{-tA}B^*u\|_\infty \leq Ct^{-3/2b}\|B^*u\|_b \leq Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3)\]
for \(b = r, \ p\). Moreover (3.4) and (2.2) yield
\[\|De^{-tA}B^*u\|_3 \leq Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3) \quad \text{for } b = r, \ p.\]
Collecting terms, we get readily (3.6) and complete the proof.

**Proof of Proposition 3.1.** Setting \(v(t) = e^{-tL}u\) with \(u \in C^\infty_0\), we have obviously that \(v \in C([0, \infty); L^\infty \cap W^1_\sigma)\) and
\[v(t) = e^{-tA}u + \int_0^t e^{-(t-s)A}B^*v(s)\, ds.\]
This gives, by (3.4)–(3.6),
\[ \|v(t)\|_\infty + \|Dv(t)\|_3 \]
\[ \leq Ct^{-3/4}\|u\|_2 + C \int_0^t (t-s)^{-3/2p} (t-s+1)^{-3(r-3/p)/2} \]
\[ \times (\|v\|_\infty + \|Dv\|_3) \, ds (\|Dw\|_r + \|Dw\|_p). \]

Setting \( \|v\|_t = \sup_{0 \leq s \leq t} s^{3/4}(\|v(s)\|_\infty + \|Dv(s)\|_3) \), we have
\[ \|v(t)\|_\infty + \|Dv(t)\|_3 \]
\[ \leq Ct^{-3/4}\|u\|_2 + C \int_0^t (t-s)^{-3/2p} (t-s+1)^{-3(r-3/p)/2} s^{-3/4} \, ds \]
\[ \leq Ct^{-3/4}\|u\|_2 + Ct^{-3/4}(\|Dw\|_r + \|Dw\|_p)\|v\|_t \]
\[ \times \int_0^t s^{-3/2p}(s+1)^{-3(r-3/p)/2} \, ds \]
\[ + Ct^{1/4-3/2p}(t+1)^{-3(r-3/p)/2}(\|Dw\|_r + \|Dw\|_p)\|v\|_t, \]
where we have used the condition \( r < 3/2 < p \). Hence, if we presuppose that
\[ (3.7) \quad C(\|Dw\|_r + \|Dw\|_p) < 1/2 \]
with the constant \( C \) given in the last term above, we obtain
\[ (3.8) \quad \|e^{-tL^*}u\|_\infty \leq Ct^{-3/4}\|u\|_2. \]

Now we take \( u \in L^1 \cap L^{6/5} \) and \( v \in L^2 \). By (3.8) we have
\[ (e^{-tL}Pu, v) = (u, e^{-tL^*}PV) \leq \|u\|_1 \|e^{-tL^*}PV\|_\infty \leq Ct^{-3/4}\|u\|_1\|v\|_2 \]
and therefore the validity of (3.1). The proof is complete.

4. Proof of Theorem 1.1. In this section we always suppose that the stationary solution \( w \in \widetilde{W}^{1,r}_\sigma \cap \widetilde{W}^{1,p}_\sigma \) with \( 1 < r < 3/2 < p < 2 \) such that (3.7) holds. Let \( u \) be a weak solution of (1.1). Then (1.2) implies
\[ (4.1) \quad \|u(t)\|_2 \leq t^{-1} \int_0^t \|u(s)\|_2 \, ds. \]

On the other hand, taking \( v \in C^\infty_\sigma \) and applying (1.3) with \( g(z) = e^{-(t-z)L^*}v \), we have
\[ (u(t), v) + \int_0^t (Lu(s), e^{-(t-s)L^*}v) \, ds - \int_0^t (u(s), L^*e^{-(t-s)L^*}v) \, ds \]
\[ = (a, e^{-tL^*}v) - \int_0^t ((u \cdot D)u, e^{-(t-s)L^*}v) \, ds, \]
that is,

\[(u(t), v) = (e^{-tL}a, v) - \int_0^t (e^{-s}L P(u \cdot D)u(s), v) \, ds\]

\[\leq \|e^{-tL}a\|_2 \|v\|_2 + \int_0^t \|e^{-s}L P(u \cdot D)u(s)\|_2 \, ds \|v\|_2\]

\[\leq C \|v\|_2 \left( t^{-3/4} \|a\|_1 + \int_0^t (t-s)^{-3/4} \|u(s)\|_2 \|Du(s)\|_2 \, ds \right),\]

where we have used (3.1). We then get

\[\|u(s)\|_2 \leq Cs^{-3/4} \|a\|_1 + C \int_0^s (s-z)^{-3/4} \|u(z)\|_2 \|Du(z)\|_2 \, dz.\]

Integrating the above inequality from 0 to \(t\), we have

\[\int_0^t \|u(s)\|_2 \, ds \leq Ct^{1/4} \|a\|_1 + C \int_0^t d z \int_z^t (s-z)^{-3/4} \|u(z)\|_2 \|Du(z)\|_2 \, ds\]

\[\leq Ct^{1/4} \|a\|_1 + Ct^{1/4} \int_0^t \|u(s)\|_2 \|Du(s)\|_2 \, ds\]

\[\leq Ct^{1/4} \|a\|_1 + Ct^{1/4} \|a\|_2 \left( \int_0^t \|u(s)\|_2^2 \, ds \right)^{1/2}, \text{ by (1.2).}\]

Combining this with (4.1), we have

\[\|u(t)\|_2 \leq Ct^{-3/4} \|a\|_1 + Ct^{-3/4} \|a\|_2 \left( \int_0^t \|u(s)\|_2^2 \, ds \right)^{1/2},\]

that is,

(4.2) \[\|u(t)\|_2 \leq C_1 t^{-3/4} \left( 1 + \left( \int_0^t \|u(s)\|_2^2 \, ds \right)^{1/2} \right),\]

where and in what follows \(C_1 = C_1(\|a\|_1, \|a\|_2)\) may vary from line to line.

Now we apply (4.2) and (1.2) to complete our proof via a bootstrap iteration argument.

Note that

(4.3) \[\|u(t)\|_2 \leq C_1, \text{ by (1.2),}\]

and

(4.4) \[\|u(t)\|_2 \leq C_1 t^{-3/4} (1 + t^{1/2}), \text{ by (4.2) and (4.3).}\]

Combining (4.4) with (4.3), we have

(4.5) \[\|u(t)\|_2 \leq C_1 t^{-1/4}.\]
Moreover, taking (4.2) and (4.5) into account, we have
\[ \|u(t)\|_2 \leq C_1 t^{-3/4}(1 + t^{1/4}). \]

This together with (4.3) implies
\[ (4.6) \quad \|u(t)\|_2 \leq C_1 (t + 1)^{-1/2}. \]

Similarly, (4.2) and (4.6) yield
\[ \|u(t)\|_2 \leq C_1 t^{-3/4}(1 + \ln(t + 1)), \]
and so, by (4.3),
\[ (4.7) \quad \|u(t)\|_2 \leq C_1 (t + 1)^{-2/3}. \]

Finally, by (4.2) and (4.7), we arrive at the desired estimate
\[ \|u(t)\|_2 \leq C_1 t^{-3/4} \]
and complete the proof.

Remark 4.1. It should be noted that the validity of the assumption of Lemma 1.2 follows from the inequality \( \|Dw\|_{3/2} \leq \|Dw\|_{r} + \|Dw\|_{p} \) and (2.7).

Appendix: Proof of (3.2). Let \( Q \) be a domain of \( R^3 \). By \( \| \cdot \|_{k,p,Q} \) and \( \| \cdot \|_{p,Q} \) we denote respectively the norms of the Sobolev space \( W^{k,p}(Q;R^3) \) and the Lebesgue space \( L^p(Q;R^3) \). Of course, \( \| \cdot \|_{k,p} = \| \cdot \|_{k,p,G} \) and \( \| \cdot \|_{p} = \| \cdot \|_{p,G} \). \( \overline{P} \) is the bounded projection from \( L^p(R^3;R^3) \) onto \( J^p(R^3;R^3) \), where \( J^p(R^3;R^3) \) denotes the completion of the set of compactly supported solenoidal in \( C^\infty(R^3;R^3) \). Let \( h \) be a constant such that \( |x| < h - 1 \) for \( x \in \partial G \), and let \( g \in C^\infty(R^3;R) \) be a fixed function such that \( g = 1 \) for \( |x| > h \) and \( g = 0 \) for \( |x| < h - 1 \). Moreover we set \( G_h = \{ x \in G; |x| < h \} \).

In arriving at (3.2), we need the following lemmas.

Lemma A.1. Let \( 1 < p \leq q < \infty \), \( t > 0 \), \( v \in L^p(R^3;R^3) \cap L^q(R^3;R^3) \), \( n \geq 1 \), and \( u \in J^6 \). Then we have
\[ (A.1) \quad \|e^{-tA}v\|_{\infty,R^3} \leq C t^{-3/2q}(t + 1)^{-(3/p-3/q)/2}(\|v\|_{p,R^3} + \|v\|_{q,R^3}), \]
\[ (A.3) \quad \|e^{-tA}u\|_{2n,6} \leq C(t^{-n} + 1)\|u\|_6. \]

(A.1) is deduced immediately by an elementary calculation. (A.2) is a consequence of \( L^p \)-estimates (cf. [25]) and the Sobolev embedding theorem. One can also refer to [17] for details.
LEMMA A.2 ([17, Lemmas 5.3, 5.4] and (A.2)). Let \( t > 0, \ v \in J^6, \) and \( P^* \) be a certain pressure such that \( p^* = Ae^{-(t+1)A}v + \Delta e^{-(t+1)A}v. \) Then

\[
\|e^{-(t+1)A}v\|_{2,6,\mathcal{G}_h} + \|A e^{-(t+1)A}v\|_{2,6,\mathcal{G}_h} + \|p^*(t)\|_{3,6,\mathcal{G}_h} \leq Ct^{-1/4}\|v\|_6.
\]

LEMMA A.3 ([17, (5.18)] and (A.2)). Let \( v \in J^6, \) and \( t > 0. \) Then there is a function \( v^* \) such that

\[
D \cdot v^* = D \cdot (ge^{-(t+1)A}v),
\]

\[
\text{supp} v^*(t) \subset \{x \in R^3; h - 1 < |x| < h\},
\]

\[
\|v^*(t)\|_{2,6} + \|\left(\partial / \partial t\right)v^*(t)\|_6 \leq C(t + 1)^{-1/4}\|v\|_6.
\]

LEMMA A.4. Let \( t > 0, \ v \) and \( v^* \) be given in Lemma A.3. Then we have

\[
\|ge^{-(t+1)A}v - v^*(t)\|_\infty \leq C(t + 1)^{-1/4}\|v\|_6.
\]

Proof. Set \( u(t) = ge^{-(t+1)A}v - v^*(t), \ u_0 = u(0), \) and

\[
F(t) = p^*(t)Dg - 2(Dg \cdot D)e^{-(t+1)A}v - (\Delta g)e^{-(t+1)A}v
+ \Delta v^*(t) - (\partial / \partial t)v^*(t),
\]

where \( p^* \) is given in Lemma A.2. By Lemmas A.2, A.3 we have that the support of \( F(t) \) is contained in \( \{x \in R^3; h - 1 < |x| < h\}, \) and

(A.3) \[
(t + 1)^{1/4}\|F(t)\|_6 + \|u_0\|_{1,6,\mathcal{G}_h} \leq C\|v\|_6,
\]

\[
u_t - \Delta u + D(gp^*) = F, \quad D \cdot u = 0 \text{ in } R^3 \times (0, \infty).
\]

We thus rewrite \( u \) in the integral form

(A.4) \[
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}PF(s) \, ds.
\]

From (A.1), (A.3), and Sobolev's embedding theorem it follows that

\[
\|e^{-tA}u_0\|_{\infty, R^3} \leq C(t + 1)^{-1/4}(\|u_0\|_{\infty} + \|u_0\|_6) \leq Ct^{-1/4}\|v\|_6,
\]
and
\[
\left\| \int_0^t e^{-(t-s)\hat{A}} PF(s) \, ds \right\|_{\infty, R^3} \\
\leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}(\| F(s) \|_3, G_h + \| F(s) \|_6, G_h) \, ds \\
\leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4} \| F(s) \|_6 \, ds \\
\leq C \| v \|_6 \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}(s+1)^{-1/4} \, ds \\
\leq C(t+1)^{-1/4} \| v \|_6.
\]

Taking (A.4) into account, we have the desired estimate and complete the proof.

**Proof of (3.2).** Let \( v \in J^6 \). By Lemmas A.1, A.2, A.3, Sobolev inequality, and Gagliardo-Nirenberg inequality (cf. [10]), we have
\[
\| e^{-(t+1)A}v \|_\infty \leq \| ge^{-(t+1)A}v \|_\infty + \| e^{-(t+1)A}v \|_{1,6}, G_h \\
\leq \| ge^{-(t+1)A}v - v^*(t) \|_\infty + C \| v^*(t) \|_{1,6} \\
+ C \| e^{-(t+1)A}v \|_{1,6}, G_h \\
\leq C(t+1)^{-1/4} \| v \|_6 \quad \text{for } t > 0,
\]
\[
\| e^{-tA}v \|_\infty \leq C \| e^{-tA}v \|_{3/4,6}^{3/4} \| e^{-tA}v \|_{1/4,6}^{1/4} \\
\leq C(t^{-1} + 1)^{1/4} \| v \|_6 \leq Ct^{-1/4} \| v \|_6
\]
for \( 1 > t > 0 \). The proof is complete.

The author would like to thank T. Miyakawa for sending [2, 3, 4]. He would also like to thank the referee for his valuable suggestions.

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Received October 7, 1991 and in revised form March 13, 1992.

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