A NONEXISTENCE RESULT FOR THE $n$-LAPLACIAN

TILAK BHATTACHARYA
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Let $P$ be a point in $\mathbb{R}^n$, $n \geq 2$; then the problem

$$\text{div}(|\nabla u|^{n-2}\nabla u) = e^u \quad \text{with} \quad u \in W^{1,n}_{\text{loc}} \cap L^\infty_{\text{loc}} \quad \text{has no subsolutions in} \quad \mathbb{R}^n \setminus \{P\}.$$

**Introduction.** Let $P = P(x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$, $n \geq 2$, and $\Omega = \mathbb{R}^n \setminus \{P\}$. Without any loss of generality we will take $P$ to be the origin. Consider the problem

$$\begin{cases}
L_p u = e^u & \text{in } \Omega, \\
u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega), & p > 1.
\end{cases}$$

Here $L_p u \equiv \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian with $1 < p < \infty$. By a subsolution $u$ of (1.1) we will mean that $u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, and

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \psi + \int_{\Omega} e^u \psi \leq 0, \quad \forall \psi \in C^\infty_0(\Omega) \text{ and } \psi \geq 0.$$

It is known that for $1 < p < n$, (1.1) has no subsolutions in the exterior of a compact set [AW]. However, for $p = n$ there exist radial subsolutions for large values of $|x|$. We show that (1.1) has no subsolutions in $\Omega$, thus extending the results of [AW], namely

**Theorem 1.** The following problem

$$L_n u = e^u \quad \text{in } \Omega, \quad n \geq 2,$$

has no subsolutions in $W^{1,n}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$.

The proof of Theorem 1 will be a consequence of a comparison principle and nonexistence of global radial solutions. The proof is presented in §4.

2. Preliminary results.

**Lemma 2.1.** Consider

$$C(x) = \frac{(1 + x)^{1/n}}{1 + x^{1/n}} \quad \text{in} \quad 0 \leq x \leq 1.$$ 

Then $C(x)$ is decreasing on $[0, 1]$. 

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Proof. Elementary computations show that
\[
\frac{dC}{dx} = \frac{(1+x)^{1/n}(1-x^{(1-n)/n})}{n(1+x^{1/n})^{2}(1+x)} \leq 0
\]
in \(0 \leq x \leq 1\). Furthermore, \(C(0) = 1\) and \(C(1) = 2^{1-n/n}\), and \(C(x) \to 1\) as \(x \to 0\).

We now state an elementary inequality that is easy to prove
\[
(2.1) \quad x^n - b^n \geq (x - b)^n, \quad \text{for } x \geq b \geq 0.
\]

**Lemma 2.2.** Suppose \(u(r) \in C^1\) satisfies the following differential inequality in \((a, R)\),
\[
\dot{u} \geq A \left( e^{u/n} + \frac{B - b}{R - r} \right),
\]
where \(\dot{u}\) represents differentiation with respect to \(r\), \(0 < A < 1\), \(0 < b < 1\), \(0 < a < R\) and \(B \geq \frac{n}{A} + b\). Then there is an \(\bar{r}\) in \((a, R)\) such that \(u(r) \to \infty\) as \(r \to \bar{r}\).

**Proof.** Setting \(v = e^{-u/n}\), we obtain that
\[
\dot{v} + \frac{c}{R - r} v \leq -\frac{A}{n}, \quad a < r < R,
\]
where \(c = \frac{A(B-b)}{n}\). Using the integrating factor \(\phi(r) = \left(\frac{1}{R-r}\right)^c\) and setting \(Z = v(r)\phi(r) - v(a)\phi(a)\), we obtain
\[
Z \leq \begin{cases} 
\left( -\frac{A}{n} \right) \ln \frac{R - a}{R - r}; & c = 1, \\ 
\left( -\frac{A}{n} \right) \left( \frac{1}{c - 1} \right) \left\{ \left( \frac{1}{R - r} \right)^{c-1} - \left( \frac{1}{R - a} \right)^{c-1} \right\}; & c > 1. 
\end{cases}
\]

It is clear that for each \(c \geq 1\), there is an \(\bar{r} \in (a, R)\) such that \(v(r) \to 0\) as \(r \to \bar{r}\), and hence \(u(r) \to \infty\) as \(r \to \bar{r}\). \(\Box\)

We present a comparison lemma; please refer to [AW] for its proof.

**Lemma 2.3.** In a region \((\Omega) \subseteq \mathbb{R}^n\), \(n \geq 2\), suppose \(u, v \in W^{1,p}_\text{loc}(\Omega) \cap L^{\infty}_\text{loc}(\Omega)\), and \((u - v)^+ \in W^{1,p}_0(\Omega)\). If \(g\) is a nondecreasing function and
\[
L_p u \geq g(u) \quad \text{in } D'(\Omega),
\]
\[
L_p v \leq g(v) \quad \text{in } D'(\Omega),
\]
then \(u \leq v\) a.e. in \((\Omega)\).
3. Nonexistence of radial subsolutions. Consider the following problem

\[ (n - 1)|\dot{u}|^{n-2} \left( \dot{u} + \frac{\dot{u}}{r} \right) = e^u, \quad 0 < r < \infty, \]

\[ u(R) = a, \quad \text{and} \quad \dot{u}(R) = b; \quad a, b \in R. \]

**Lemma 3.1.** For the problem in (3.1), there exists a \( C^1 \) radial solution \( u(r) \) such that at least one of the following occurs.

(i) There is an \( \bar{r} \) in \((0, R)\) such that \( u(r) \to \infty \) as \( r \to \bar{r} \).

(ii) There is an \( \bar{r} \) in \((R, \infty)\) such that \( u(r) \to \infty \) as \( r \to \bar{r} \).

Furthermore, there are values of \( b \) for which both (i) and (ii) occur.

**Proof.** We divide the proof into three parts.

**Case 1.** Take \( b = 0 \). Let \( u(r) \) be the solution defined by

\[ u(r) = a + \int_R^r \frac{1}{t} \left\{ \int_R^t s^{n-1} e^{u(s)} \, ds \right\}^{1/(n-1)} \, dt, \]

in \( r > R \). The existence and uniqueness in a small interval follows from Picard’s iteration. It can be shown by differentiating that \( u \) solves (3.1). From (3.2) it is clear that \( r\dot{u} \) is increasing and thus \( \dot{u} \geq 0 \) in \((R, r)\), and hence \( u \) is increasing. Continue \( u \) by (3.2). By differentiating (3.2) once,

\[ \dot{u}(r) = \frac{1}{r} \left\{ \int_R^r s^{n-1} e^{u(s)} \, ds \right\}^{1/(n-1)}. \]

Thus,

\[ \frac{d}{dr} \left\{ \frac{\dot{u}^{n-1}}{r} \right\} = \frac{r^n e^{u(r)} - n \int_R^r s^{n-1} e^{u(s)} \, ds}{r^{n+1}} \geq \frac{r^n e^{u(r)} - e^{u(r)} (r^n - R^n)}{r^{n+1}} \geq 0. \]

By simplifying the left side of the foregoing inequality,

\[ (n - 1)\dddot{u} \geq \frac{\dot{u}}{r}. \]

Note that \( u \) is \( C^2 \) except possibly where \( \dot{u} = 0 \). Noting that \( \dot{u} \geq 0 \),

(3.1) yields

\[ n(n - 1)(\dddot{u})^{n-1} \geq e^u, \quad R < r < \infty. \]
Multiplying both sides by \( \dot{u} \) and integrating once from \( R \) to \( r \),

\[
(\dot{u})^n \geq \frac{e^u - e^a}{n - 1}.
\]

For \( \varepsilon > 0 \), small enough, it follows from (3.2) and the fact that \( u \) is increasing that

\[
u(r) > a + \int_{R+\varepsilon}^r \frac{1}{t} \left\{ \int_R^{R+\varepsilon} s^{n-1}e^{u(s)} ds \right\}^{1/(n-1)} dt.
\]

Hence for some appropriate constant \( A > 0 \),

\[
u(r) > a + A \ln \frac{r}{R+\varepsilon}
\]

implying that \( u(r) \to \infty \) as \( r \) gets large. Thus in (3.3) we may take \( r > R_1 \), where \( R_1 \) is large enough so that \( e^u/2 \leq e^u - e^a \) for \( r > R_1 \). If \( u(r) \to \infty \) as \( r \to R_1 \), then we are done. Otherwise, continue \( u \) using (3.2) past \( r = R_1 \). Hence

\[
\dot{u} \geq Ce^{u/n}, \text{ in } r > R_1,
\]

for some \( C > 0 \). Integrating,

\[
\int_{u(R_1)}^{u(r)} e^{-u/n} du \geq C(r - R_1).
\]

It is clear that there exists an \( \bar{r} > R \), such that \( u(r) \to \infty \) as \( r \to \bar{r} \). The case \( b > 0 \) follows similarly.

**Case 2.** Without any loss of generality, take \( a = 0 \). Take \( b < 0 \). Now \( \dot{u}(r) < 0 \) near \( r = R \), so we obtain that \( \dot{u}(r) \) satisfies

\[
(3.4) \quad \dot{u}(r) = -\frac{1}{r} \left\{ |bR|^{n-1} - \int_R^r t^{n-1}e^{u(t)} dt \right\}^{1/(n-1)},
\]

in \( r > R \). We show that there is \( \bar{b} < 0 \) such that if \( \bar{b} < b < 0 \), there is an \( \hat{r} > R \) such that \( \dot{u}(r) \to 0 \) as \( r \to \hat{r} \). It follows from (3.4) that \( r\dot{u} \) is increasing and thus

\[
\frac{bR}{r} \leq \dot{u} \leq 0, \text{ for } r > R.
\]

Set \( c = bR \). Integrating, we find

\[
e^u \geq r^c,
\]

and so (3.4) yields

\[
\dot{u}(r) \geq -\frac{1}{r} \left\{ |c|^{n-1} - \int_R^r t^{n-1+c} dt \right\}^{1/(n-1)}.
\]
Therefore,
\[ \dot{u}(r) \geq \begin{cases} \left\{ -\frac{1}{r} \left( |c|^{n-1} - \frac{r^{n+c} - R^{n+c}}{n+c} \right) \right\}^{1/(n-1)} ; & \quad -n < c < 0, \\ - \frac{1}{r} \left\{ |c|^{n-1} - \ln \frac{r}{R} \right\}^{1/(n-1)} ; & \quad c = -n. \end{cases} \]

It is clear that there is an \( \hat{r} > R \) for which \( \dot{u}(r) \to 0 \) as \( r \to \hat{r} \). Now, take \( c < -n \), satisfying

\[ |c|^{n-1} - \frac{1}{|c| - n} \left( \frac{1}{R} \right)^{|c|-n} < n^{n-1}. \]

Now, (3.4) yields

\[ \dot{u}(r) \geq - \frac{1}{r} \left[ |c|^{n-1} - \frac{1}{|c| - n} \left\{ \left( \frac{1}{R} \right)^{|c|-n} - \left( \frac{1}{r} \right)^{|c|-n} \right\} \right]^{1/(n-1)}. \]

Using (3.5), there is an \( \hat{r} \) such that \( \dot{u}(r) \geq - \frac{n}{\hat{r}} \) for \( r > \hat{r} \). If \( \dot{u}(r) \to 0 \) as \( r \to \hat{r} \), then we are done. Otherwise, continue \( u \) past \( r = \hat{r} \). Repeating the arguments preceding (3.5), we see that \( \dot{u}(r) \to 0 \) as \( r \to \hat{r} \) for some \( \hat{r} > R \). Continuing \( u \) past \( r = \hat{r} \) using

\[ u(r) = u(\hat{r}) + \int_{\hat{r}}^{r} \frac{1}{t} \left\{ \int_{\hat{r}}^{t} s^{n-1} e^{u(s)} \, ds \right\}^{1/(n-1)} \, dt, \]

we may show, as in Case 1, that there is an \( \bar{r} > R \) where \( u \) blows up.

**Case 3.** We may again take \( a = 0 \). Let \( c < -n \), \( t = R - r \), and \( v(t) = u(r) \), where \( 0 < r \leq R \). Then \( \dot{v}(t) = -\dot{u}(r) \), where \( \dot{v} \) represents differentiation with respect to \( t \). Then

\[ (n - 1)|\dot{v}|^{n-2} \left( \dot{v} - \frac{\dot{v}}{R - t} \right) = e^{v}, \quad 0 \leq t \leq R, \]

\[ v(0) = 0 \quad \text{and} \quad \dot{v}(0) = -b. \]

A solution of (3.6) is given by

\[ v(t) = \int_{0}^{t} \frac{1}{R - s} \left\{ |c|^{n-1} + \int_{0}^{s} (R - w)^{n-1} e^{v(w)} \, dw \right\}^{1/(n-1)} \, ds. \]

Equation (3.6) yields that \( \frac{d}{dt}((R - t)\dot{v}) \geq 0 \), thus \( \dot{v} \geq 0 \) in \( t > 0 \). Integrating this inequality from 0 to \( t \), we obtain

\[ \dot{v}(t) \geq \frac{|c|}{(R - t)}. \]
Hence,

\[ e^{v(t)} \geq \left( \frac{1}{R - t} \right)^{|c|}. \]

Let \( 0 < \varepsilon_0 < 1 \) be such that

\[ |c| \geq n \left\{ \frac{1 + \varepsilon^{1/n}}{(1 + \varepsilon)^{1/n}} \right\} + \varepsilon \]

for every \( \varepsilon \) in \((0, \varepsilon_0)\). It follows from (3.7) that there is a \( t_1 < R \) such that

\[ \left( \frac{|c|}{R - t} \right)^n e^{-v(t)} < \varepsilon_0, \]

for \( t > t_1 \). If \( v(t) \to \infty \) as \( t \to t_1 \), then we are done; otherwise continue \( v(t) \) past \( t = t_1 \). Furthermore, we may take \( t_1 \) such that \( R - t_1 < \varepsilon_0 \). Rearranging the terms in (3.6), and multiplying by \( \dot{v}(t) \) yields

\[ (n - 1)(\dot{v})^{n-1}\dot{v} = e^v \dot{v} + \frac{n - 1}{R - t}(\dot{v})^n, \quad 0 \leq t < R. \]

Integrating both sides from \( 0 \) to \( t \), and noting that \( \dot{v} \geq \frac{|c|}{R - t} \), we find

\[ (\dot{v})^n \geq e^v - 1 + \left( \frac{|c|}{R - t} \right)^n, \quad 0 \leq t < R. \]

By the definition of \( t_1 \), it follows that

\[ (\dot{v})^n \geq e^v + \left( \frac{|c| - \varepsilon_0}{R - t} \right)^n, \quad t_1 < t < R. \]

Setting

\[ x = \left( \frac{|c| - \varepsilon_0}{R - t} \right)^n e^{-v}, \]

the above may be rewritten as

\[ (\dot{v})^n \geq e^v \{1 + x\}. \]

Hence,

\[ \dot{v} \geq e^{v/n} \{1 + x\}^{1/n}. \]

Using Lemma 2.1 and the definition of \( t_1 \),

\[ \dot{v} \geq C(\varepsilon_0)e^{v/n} \{1 + x^{1/n}\}. \]

Thus we obtain

\[ \dot{v} \geq C(\varepsilon_0) \left\{ e^{v/n} + \frac{|c| - \varepsilon_0}{R - t} \right\}, \quad t_1 < t < R. \]
By Lemma 2.2, there is a \( t_2 > t_1 \) such that \( \nu(t) \to \infty \) as \( t \to t_2 \). Hence there is an \( \bar{r} \in (0, R) \) for which \( u(r) \to \infty \) as \( r \to \bar{r} \). Thus for every \( c < -n \), we have a vertical asymptote in \((0, R)\). It is clear from (3.5) that there are values of \( b \) for which both (i) and (ii) happen. Call one such value to be \( b_R \).

For the case \( a \neq 0 \), we introduce the following change of variables. Let \( v(r) = u(r) - a \); then

\[
(n - 1)|\dot{v}|^{n-2} \left( \dot{v} + \frac{n-1}{r} \dot{v} \right) = e^a e^v.
\]

Setting \( t = re^{a/n} \), and \( w(t) = v(r) \), and differentiating with respect to \( t \), we have

\[
(n - 1)|\dot{w}|^{n-2} \left( \dot{w} + \frac{n-1}{t} \dot{w} \right) = e^w,
\]

\[
w(\bar{R}) = 0 \quad \text{and} \quad \dot{w}(\bar{R}) = e^{-a/n} \frac{b}{\bar{R}},
\]

where \( \bar{R} = e^{a/n} R \). There is a \( b_{\bar{R}} \) so that the corresponding solution which we continue to call \( w(t) \), blows up near zero and at a point past \( \bar{R} \). Then \( u(t) = a + w(e^{-a/n} t) \) is such a solution for the original problem.

4. Proof of Theorem 1. This follows easily from Lemma 2.3 and Lemma 3.1.

**Proof of Theorem** 1. Assume to the contrary. Let \( U(x) \) be such a subsolution in (1.2). Let

\[
a = \inf_{1/2 \leq |x| \leq 3/2} U(x).
\]

By Lemma 3.1, there is a radial solution \( u(r) \) such that \( u(1) = a - 1 \), and \( u(r) \) blows up at some \( r \in (0, 1) \) and \( \bar{r} \in (1, \infty) \). Let

\[
M = \sup_{r \leq |x| \leq \bar{r}} U(x),
\]

\( r \in (r, 1) \) and \( \bar{r} \in (1, \bar{r}) \) be such that \( u(r), \ u(\bar{r}) \geq M + 1 \). Using Lemma 2.3, \( u(x) \geq U(x) \) in \( r \leq |x| \leq \bar{r} \), a contradiction.

**Remark.** In Theorem 1, \( 1 < p \leq n \) is the best possible. For \( p > n \), take \( u = \ln(\beta^{|x|^p}) \), where \( 0 < A \leq (p - n) p^{p-1} \). Then

\[
L_p u = \frac{(p-n) p^{p-1}}{r^p} \geq A_{r^p}.
\]
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INDIAN STATISTICAL INSTITUTE
NEW DELHI-110 016 INDIA
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