LACUNARY STATISTICAL CONVERGENCE

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The sequence $x$ is statistically convergent to $L$ provided that for each $\varepsilon > 0$,

$$\lim n^{-1}\{\text{the number of } k \leq n: |x_k - L| \geq \varepsilon\} = 0.$$  

In this paper we study a related concept of convergence in which the set $\{k: k \leq n\}$ is replaced by $\{k: r_{r-1} < k \leq r_r\}$, for some lacunary sequence $\{r_i\}$. The resulting summability method is compared to statistical convergence and other summability methods, and questions of uniqueness of the limit value are considered.

1. Introduction. A complex number sequence $x$ is said to be statistically convergent to the number $L$ if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S\text{-lim } x = L$ or $x_k \to L(S)$. We shall also use $S$ to denote the set of all statistically convergent sequences. The idea of statistical convergence was introduced by Fast [4] and studied by several authors [2], [3], [5], [6], [11]. There is a natural relationship [2] between statistical convergence and strong Cesàro summability:

$$|\sigma_1| := \left\{ x: \text{for some } L, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{r_i\}$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta$ will be denoted by $I_r := (k_{r-1}, k_r]$, and the ratio $k_r/k_{r-1}$ will be abbreviated by $q_r$. There is a strong connection [7] between $|\sigma_1|$ and the sequence space $N_\theta$, which is defined by

$$N_\theta := \left\{ x: \text{for some } L, \lim_{r \to \infty} \left( \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \right) = 0 \right\}.$$  

The purpose of this paper is to introduce and study a concept of convergence that is related to statistical convergence (1) in the same way that $N_\theta$ is related to $|\sigma_1|$.
DEFINITION. Let $\theta$ be a lacunary sequence; the number sequence $x$ is $S_\theta$-convergent to $L$ provided that for every $\varepsilon > 0$,

$$
\lim_r \frac{1}{h_r} |\{k \in I_r: |x_k - L| \geq \varepsilon\}| = 0.
$$

In this case we write $S_\theta$-lim $x = L$ or $x_k \to L(S_\theta)$, and we define $S_\theta := \{x: \text{for some } L, S_\theta$-lim $x = L\}$.

The limits in (1) and (2) can be expressed using matrix transformations of the characteristic function $\chi_K$ of the set $K = K(x, L, \varepsilon) := \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}$.

The limit in (1) is $\lim_n (C_1 \chi_K)_n = 0$, where $C_1$ is the Cesàro mean; the limit in (2) is $\lim_n (C_\theta \chi_K)_n = 0$, where $C_\theta$ is the matrix given by

$$
C_\theta[n, k] := \begin{cases} 
\frac{1}{h_r}, & \text{if } k \in I_r, \\
0, & \text{if } k \notin I_r.
\end{cases}
$$

In this form $S_\theta$-convergence is seen to be a part of "A-density convergence" as defined in [8] and [3].

In the next section we establish inclusion relations between $N_\theta$ and $S_\theta$ and also between $S_\theta$ and $S$. In §3 we show that the $S_\theta$-limit of a given sequence $x$ is not necessarily unique for different $\theta$'s, but different $S_\theta$-limits cannot occur if $x \in S$. In the final section we get a relationship between $S_\theta$-convergence and strong almost convergence, a concept introduced by Maddox [10] and (independently) by Freedman et al. [7].

2. Inclusion theorems. In this section we first give some inclusion relations between $N_\theta$- and $S_\theta$-convergence and show that they are equivalent for bounded sequences. We also study the inclusions $S \subseteq S_\theta$ and $S_\theta \subseteq S$ under certain restrictions on $\theta = \{k_r\}$.

THEOREM 1. Let $\theta = \{k_r\}$ be a lacunary sequence; then

(i) (a) $x_k \to L(N_\theta)$ implies $x_k \to L(S_\theta)$, and

(b) $N_\theta$ is a proper subset of $S_\theta$;

(ii) $x \in l_\infty$ and $x_k \to L(S_\theta)$ imply $x_k \to L(N_\theta)$;

(iii) $S_\theta \cap l_\infty = N_\theta \cap l_\infty$,

where $l_\infty$ denotes the set of bounded sequences.

Before proving this theorem we remark that this result is included by Theorem 8 in [3], where Connor bases the proof on the concept of ideals in $l_\infty$; we give a direct proof.
Proof. (a) If $\epsilon > 0$ and $x_k \to L(N_\theta)$ we can write

$$\sum_{k \in I_r} |x_k - L| \geq \sum_{k \in I_r, |x_k - L| \geq \epsilon} |x_k - L| \geq \epsilon \{k \in I_r: |x_k - L| \geq \epsilon\},$$

which yields the result.

(b) In order to establish that the inclusion $N_\theta \subseteq S_\theta$ in (i) is proper, let $\theta$ be given and define $x_k$ to be $1, 2, \ldots, [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in $I_r$, and $x_k = 0$ otherwise. Note that $x$ is not bounded. We have, for every $\epsilon > 0$,

$$\frac{1}{h_r} \sum_{k \in I_r} |x_k - 0| = \frac{[\sqrt{h_r}](\sqrt{h_r} + 1)}{2} \to 0 \text{ as } r \to \infty,$$

i.e., $x_k \to 0(S_\theta)$. On the other hand,

$$\frac{1}{h_r} \sum_{k \in I_r} |x_k - 0| = \frac{[\sqrt{h_r}](\sqrt{h_r} + 1)}{2} - \frac{1}{2} \neq 0,$$

hence $x_k \not\to 0(N_\theta)$.

(ii) Suppose that $x_k \to L(S_\theta)$ and $x \in l_\infty$, say $|x_k - L| \leq M$ for all $k$. Given $\epsilon > 0$, we get

$$\frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = \frac{1}{h_r} \sum_{k \in I_r, |x_k - L| \geq \epsilon} |x_k - L| + \frac{1}{h_r} \sum_{k \in I_r, |x_k - L| < \epsilon} |x_k - L|$$

$$\leq \frac{M}{h_r} \{k \in I_r: |x_k - L| \geq \epsilon\} + \epsilon,$$

from which the result follows.

We remark that the example given in (i) shows that the boundedness condition cannot be omitted from the hypothesis of Theorem 1 (ii).

(iii) This is an immediate consequence of (i) and (ii).

Since any $N_\theta$-summable sequence is $C_\theta$-summable, we conclude from Theorem 1 (ii) that any bounded $S_\theta$-summable sequence is also $C_\theta$-summable.

Lemma 2. For any lacunary sequence $\theta$, $S$-$\lim x = L$ implies $S_\theta$-$\lim x = L$ if and only if $\lim inf, q_r > 1$. If $\lim inf, q_r = 1$, then there exists a bounded $S_\theta$-summable sequence that is not $S$-summable (to any limit).
Proof. Suppose first that \( \lim \inf_r q_r > 1 \); then there exists a \( \delta > 0 \) such that \( q_r \geq 1 + \delta \) for sufficiently large \( r \), which implies that

\[
\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}.
\]

If \( x_k \rightarrow L(S) \), then for every \( \varepsilon > 0 \) and for sufficiently large \( r \), we have

\[
\frac{1}{k_r} |\{k \leq k_r : |x_k - L| \geq \varepsilon\}| \geq \frac{1}{k_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|
\]

\[
\geq \frac{\delta}{1 + \delta} \cdot \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|;
\]

this proves the sufficiency.

Conversely, suppose that \( \lim \inf_r q_r = 1 \). Proceeding as in [7; p. 510] we can select a subsequence \( \{k_{r(j)}\} \) of the lacunary sequence \( \theta \) such that

\[
\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \quad \text{where} \ r(j) \geq r(j-1) + 2.
\]

Now define a bounded sequence \( x \) by \( x_i = 1 \) if \( i \in I_{r(j)} \) for some \( j = 1, 2, \ldots \) and \( x_i = 0 \) otherwise. It is shown in [7; p. 510] that \( x \notin N_\theta \) but \( x \in |\sigma_1| \). The above Theorem 1 (ii) implies that \( x \notin S_\theta \), but it follows from Theorem 2.1 of [2] that \( x \in S \). Hence \( S \not\subseteq S_\theta \), and the proof is complete.

**Lemma 3.** For any lacunary sequence \( \theta \), \( S\)-\( \lim x = L \) implies \( S_\theta\)-\( \lim x = L \) if and only if \( \lim \sup_r q_r < \infty \). If \( \lim \sup_r q_r = \infty \), then there exists a bounded \( S \)-summable sequence that is not \( S_\theta \)-summable (to any limit).

*Proof.* If \( \lim \sup_r q_r < \infty \), then there is an \( H > 0 \) such that \( q_r < H \) for all \( r \). Suppose that \( x_k \rightarrow L(S_\theta) \), and let \( N_r := |\{k \in I_r : |x_k - L| \geq \varepsilon\}|. \) By (2), given \( \varepsilon > 0 \), there is an \( r_0 \in \mathbb{N} \) such that

\[
\frac{N_r}{h_r} < \varepsilon \quad \text{for all} \ r > r_0.
\]

Now let \( M := \max\{N_r : 1 \leq r \leq r_0\} \) and let \( n \) be any integer satisfying
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\[ k_{r-1} < n \leq k_r; \] then we can write

\[ \frac{1}{n} |\{ k \leq n : |x_k - L| \geq \varepsilon \}| \leq \frac{1}{k_{r-1}} |\{ k \leq k_r : |x_k - L| \geq \varepsilon \}| \]

\[ = \frac{1}{k_{r-1}} \{ N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r \} \]

\[ \leq \frac{M}{k_{r-1}} r_0 + \frac{1}{k_{r-1}} \left\{ \frac{N_{r_0+1}}{h_{r_0+1}} + \cdots + \frac{N_r}{h_r} \right\} \]

\[ \leq \frac{r_0 \cdot M}{k_{r-1}} + \frac{1}{k_{r-1}} \left( \sup_{r > r_0} \frac{N_r}{h_r} \right) \{ h_{r_0+1} + \cdots + h_r \} \]

\[ \leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot \frac{k_r - k_{r_0}}{k_{r-1}}, \quad \text{by (3)}, \]

\[ \leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot q_r \leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon H, \]

and the sufficiency follows immediately.

Conversely, suppose that \( \limsup_r q_r = \infty \). Following the idea in [7; p. 511] we can select a subsequence \( \{k_r(j)\} \) of the lacunary sequence \( \theta = \{k_r\} \) such that \( q_{r(j)} > j \), and define a bounded sequence by \( x_i = 1 \) if \( k_{r(j)-1} < i \leq 2k_{r(j)-1} \) for some \( j = 1, 2, \ldots \), and \( x_i = 0 \) otherwise. It is shown in [7; p. 5.11] that \( x \in N_\theta \) but \( x \notin \sigma_1 \).

By Theorem 1 (i) we conclude that \( x \in S_\theta \), but Theorem 2.1 of [2] implies that \( x \notin S \). Hence, \( S_\theta \notin S \).

Combining Lemma 2 and Lemma 3 we get

**Theorem 4.** Let \( \theta \) be a lacunary sequence; then \( S = S_\theta \) if and only if

\[ 1 < \liminf_r q_r \leq \limsup_r q_r < \infty; \]

then \( S\text{-lim} x = L \) implies \( S_\theta\text{-lim} x = L \).

For an example of a lacunary sequence satisfying the conditions of Theorem 4, we can take \( k_r = 2^r \) for \( r > 0 \), whence \( S_{\{2^r\}} = S \). We remark that the examples given in Lemmas 2 and 3 illustrate the difference between \( S\)-convergence and \( S_\theta\)-convergence.

We conclude this section with the following observation. Buck [1, Theorem 3.2] proved that if a real sequence is \( C_1\)-summable to its finite limit inferior, then the sequence "converges to that point for almost all \( n \)" (i.e., it is statistically convergent to its limit inferior [2]). Note that this result remains true if we replace limit inferior by
limit superior. For each subset $K$ of $\mathbb{N}$, define

$$D(K) := \lim_{r} (C_{\theta} \chi_{K} )_{r} = \lim_{r} \frac{|K \cap I_{r}|}{h_{r}} ;$$

then $D$ is a density [8; p. 296], and it is not hard to get a result for $S_{\theta}$-convergence that is analogous to Buck’s. To be precise, the following result is such an analogue.

**Proposition 5.** If the real number sequence $x$ is $C_{\theta}$-summable to either its finite limit inferior or finite limit superior, then $x$ is $S_{\theta}$-convergent to that value.

3. **Uniqueness of $S_{\theta}$-limit and lacunary refinements.** It is easy to see that, for any fixed $\theta$, the $S_{\theta}$-limit is unique. It is possible, however, for a sequence—even a bounded one—to have different $S_{\theta}$-limits for different $\theta$’s. This can be seen by applying Theorem 1 (i) to the sequence $x$ given in [7, proof of Theorem 2.1] for which $N_{\theta_{1}}$-lim $x = 0$ and $N_{\theta_{2}}$-lim $x = 1$. The next theorem shows that this situation cannot occur if $x \in S$; in other words, every $S_{\theta}$ method is consistent with the $S$-method.

**Theorem 6.** If $x \in S \cap S_{\theta}$, then $S_{\theta}$-lim $x = S$-lim $x$.

**Proof.** Suppose $S$-lim $x = L$ and $S_{\theta}$-lim $x = L'$, and $L \neq L'$. For $\varepsilon < \frac{1}{2}|L - L'|$ we get

$$\lim_{n} \frac{1}{n} | \{ k \leq n : |x_{k} - L'| \geq \varepsilon \} | = 1.$$

Consider the $k_{m}$th term of the statistical limit expression $n^{-1}|\{ k \leq n : |x_{k} - L'| \geq \varepsilon \}|:

$$\frac{1}{k_{m}} \left| \left\{ k \in \bigcup_{r=1}^{m} I_{r} : |x_{k} - L'| \geq \varepsilon \right\} \right|$$

$$= \frac{1}{k_{m}} \sum_{r=1}^{m} \left| \{ k \in I_{r} : |x_{k} - L'| \geq \varepsilon \} \right| = \frac{1}{\sum_{r=1}^{m} h_{r}} \sum_{r=1}^{m} h_{r} t_{r},$$

where $t_{r} = h_{r}^{-1} | \{ k \in I_{r} : |x_{k} - L'| \geq \varepsilon \} | \rightarrow 0$ because $x_{k} \rightarrow L'(S_{\theta})$. Since $\theta$ is a lacunary sequence, (4) is a regular weighted mean transform of $t$, and therefore it, too, tends to zero as $m \rightarrow \infty$. Also, since this is a subsequence of $\{ n^{-1}|\{ k \leq n : |x_{k} - L'| \geq \varepsilon \} \}_{n=1}^{\infty}$, we infer that

$$\frac{1}{n} | \{ k \leq n : |x_{k} - L'| \geq \varepsilon \} | \rightarrow 1,$$
and this contradiction shows that we cannot have \( L \neq L' \).

We now consider the inclusion of \( S_{\theta'} \) by \( S_{\theta} \), where \( \theta' \) is a lacunary refinement of \( \theta \). Recall [7] that the lacunary sequence \( \theta' = \{k'_r\} \) is called a lacunary refinement of the lacunary sequence \( \theta = \{k_r\} \) if \( \{k_r\} \subseteq \{k'_r\} \).

**Theorem 7.** If \( \theta' \) is a lacunary refinement of \( \theta \) and \( x_k \to L(S_{\theta'}) \), then \( x_k \to L(S_{\theta}) \).

**Proof.** Suppose each \( I_r \) of \( \theta \) contains the points \( \{k_r', \nu(r)\}_{i=1}^{\nu(r)} \) of \( \theta' \) so that

\[
\begin{align*}
    k_{r-1} < k_{r,1}' < k_{r,2}' < \cdots < k_{r,\nu(r)}' = k_r, \quad \text{where } I_{r,i}' = (k_{r,i-1}', k_{r,i}').
\end{align*}
\]

Note that for all \( r \), \( \nu(r) \geq 1 \) because \( \{k_r\} \subseteq \{k'_r\} \). Let \( \{I_{r,i}'\}_{j=1}^{\infty} \) be the sequence of abutting intervals \( \{I_{r,i}'\} \) ordered by increasing right end points. Since \( x_k \to L(S_{\theta'}) \), we get, for each \( \varepsilon > 0 \),

\[
\lim_j \sum_{I_{r,j}' \subseteq I_r} \frac{1}{h_r} \left| \left\{ k \in I_{r,j}' : |x_k - L| \geq \varepsilon \right\} \right| = 0.
\]

As before we write, \( h_r = k_r - k_{r-1} \), \( h_{r,i}' = k_{r,i}' - k_{r,i-1}' \), and \( h_{r,1}' = k_{r,1}' - k_{r-1} \). For each \( \varepsilon > 0 \) we have

\[
\frac{1}{h_r} \left| \left\{ k \in I_r : |x_k - L| \geq \varepsilon \right\} \right| \\
= \frac{1}{h_r} \sum_{I_{r,j}' \subseteq I_r} h_{r,j}' \frac{1}{h_{r,j}'} \left| \left\{ k \in I_{r,j}' : |x_k - L| \geq \varepsilon \right\} \right| \\
= \frac{1}{h_r} \sum_{I_{r,j}' \subseteq I_r} h_{r,j}' (C_{\theta'} \chi_K)_j ,
\]

where \( \chi_K \) is the characteristic function of the set \( K := \{ k \in \mathbb{N} : |x_k - L| \geq \varepsilon \} \). By (5), \( C_{\theta'} \chi_K \) is a null sequence, and (6) is a regular weighted mean transform of \( C_{\theta'} \chi_K \). Hence, the transform (6) also tends to zero as \( r \to \infty \).

We conclude this section by observing that Theorem 7 establishes inclusion between two lacunary methods only when one sequence is a lacunary refinement of the other. The example cited at the beginning of this section shows that \( S_{\theta} \) can be inconsistent with \( S_{\theta'} \). A general description of inclusion between two arbitrary lacunary methods is left as an open problem.
4. Strong almost convergence and $S_\theta$-convergence. The idea of almost convergence was introduced by Lorentz [9]: the sequence $x$ is said to be almost convergent to $L$ if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=m+1}^{m+n} (x_i - L) = 0, \quad \text{uniformly in } m.
\]
Maddox [10] and (independently) Freedman et al. [7] introduced the notion of strong almost convergence: the sequence $x$ is said to be strongly almost convergent to $L$ if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=m+1}^{m+n} |x_i - L| = 0, \quad \text{uniformly in } m.
\]
Let $c$, $AC$ and $[AC]$, respectively, denote the sets of all convergent, almost convergent, and strongly almost convergent sequences. It is known [10] that
\[
(7) \quad c \subsetneq [AC] \subsetneq AC \subsetneq l_\infty.
\]

**Theorem 8.** If $\mathcal{L}$ denotes the set of all lacunary sequences, then
\[
[AC] = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} S_\theta \right).
\]

**Proof.** By [7, Theorem 3.1], the relations (7) and Theorem 1 (iii), we have
\[
l_\infty \supset [AC] = \bigcap_{\theta \in \mathcal{L}} N_\theta = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} N_\theta \right) \bigcap (l_\infty \cap N_\theta)
= \bigcap_{\theta \in \mathcal{L}} (l_\infty \cap S_\theta) = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} S_\theta \right).
\]

Finally we remark that in contrast to [7, Theorem 3.1] where it was proved that $[AC] = \bigcap N_\theta$, the factor $l_\infty$ cannot be omitted from Theorem 8. For, $\bigcap S_\theta \not\subset l_\infty$ and $\bigcap N_\theta = [AC]$ is a proper subset of $\bigcap S_\theta$. To see this consider the sequence $x$ defined by $x_k = m$, if $k = m^2$ for $m = 1, 2, \ldots$, and $x_k = 0$ otherwise. Observe that $x$ is not bounded, so it is not strongly almost convergent. On the other hand, for any lacunary sequence $\theta$, we have
\[
\frac{1}{h_r} |\{k \in I_r : x_k \neq 0\}| \leq \frac{\sqrt{h_r}}{h_r} \to 0, \quad \text{as } r \to \infty;
\]
hence, $x_k \to O(S_\theta)$. 

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References


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