Pacific Journal of Mathematics

# ADJOINT LINEAR SYSTEMS ON A SURFACE OF GENERAL TYPE IN POSITIVE CHARACTERISTIC

TOHRU NAKASHIMA

Volume 160 No. 1 September 1993

PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 1, 1993

# ADJOINT LINEAR SYSTEMS ON A SURFACE OF GENERAL TYPE IN POSITIVE CHARACTERISTIC

TOHRU NAKASHIMA

**Let $X$ be a minimal surface of general type defined over an algebraically closed field of positive characteristic $p$. For a given divisor $D$, we consider the spannedness properties of adjoint linear systems $|K+D|$ on $X$. Under some numerical conditions on $p$ and $D$, the failure of spannedness of $|K+D|$ implies the existence of divisors with special properties. This leads to the following result: Let $L$ be an ample line bundle and assume $p \geq 5$. Then $|m(K+L)|$ is base point free for $m \geq 2$ and very ample for $m \geq 3$. Our proof is based on a technique of Shepherd-Barron using unstable vector bundles.**

**1. Introduction.** After Reider introduced a new method ([**R**]), many results have been obtained concerning adjoint linear systems on algebraic surfaces defined over an algebraically closed field of characteristic $0$. Recently, Shepherd-Barron ([**SB**]) treated the positive characteristic case and obtained results on pluricanonical systems, improving the work of Ekedahl ([**E**]). He also showed Reider's analysis holds for surfaces of special type except the quasi-elliptic ones. His method is, as in [**R**], based on the theory of unstable vector bundles in the sense of Bogomolov.

In the present note we shall consider adjoint linear systems on a minimal surface of general type in characteristic $p$ and prove some results of Reider's type.

Let $X$ be a minimal surface of general type defined over an algebraically closed field $k$ of $\operatorname{char} k = p > 0$ and let $D$ be a nef divisor such that $D - K$ is nef and big. We shall prove the following

THEOREM 1. *Let $X$ and $D$ be as above and let $d := D^2$.*
(i) *Suppose that one of the following conditions holds:*
(1) $p \geq 2$, $d \geq 5$ *and $X$ is not uniruled,*
(2) $p = 3$ *and* $d \geq 12$,
(3) $p = 5$ *and* $d \geq 6$,
(4) $p \geq 7$ *and* $d \geq 5$.
*If $|K+D|$ has a base point, then there exists an effective divisor $\Delta$ such*

*that*

$$\text{either} \quad D \cdot \Delta = 1, \quad \Delta^2 = 0,$$
$$\text{or} \quad D \cdot \Delta = 0, \quad \Delta^2 = -1.$$

(ii) *Suppose that one of the following conditions holds*:
(1′) $p \geq 2$, $d \geq 10$ *and* $X$ *is not uniruled,*
(2′) $p = 3$ *and* $d \geq 30$,
(3′) $p = 5$ *and* $d \geq 13$,
(4′) $p = 7$ *and* $d \geq 11$,
(5′) $p \geq 11$ *and* $d \geq 10$.
*If* $|K + D|$ *is not very ample, then there exists an effective divisor* $\Delta$ *such that*

$$\text{either} \quad D \cdot \Delta = 0, \quad \Delta^2 = -1, -2,$$
$$\text{or} \quad D \cdot \Delta = 1, \quad \Delta^2 = -1, 0,$$
$$\text{or} \quad D \cdot \Delta = 2, \quad \Delta^2 = 0.$$

As a corollary of the above theorem, we obtain the following result on pluri-adjoint systems:

COROLLARY 2. *Let* $L$ *be an ample divisor on* $X$.
(i) *Suppose that one of the following conditions holds*:
(1) $p \geq 2$, $m \geq 2$ *and* $X$ *is not uniruled,*
(2) $p = 3$ *and* $m \geq 3$,
(3) $p \geq 5$ *and* $m \geq 2$.
*Then* $|m(K + L)|$ *is base point free.*
(ii) *Suppose that one of the following conditions holds*:
(1′) $p \geq 2$, $m \geq 3$ *and* $X$ *is not uniruled,*
(2′) $p = 3$ *and* $m \geq 4$,
(3′) $p \geq 5$ *and* $m \geq 3$.
*Then* $|m(K + L)|$ *is very ample.*

*Proof*. Apply the theorem to $D = (m-1)K + mL$. □

**2. Proof of the theorem.** Let $p$ be a base point of $|K + D|$ in (i) (resp. $p$, $q$ be the points not separated by $|K + D|$ in (ii)) and let $\pi: \widetilde{X} \to X$ be the blowing up at $p$ in (i) (resp. at $p$ and $q$ in (ii)). Put $l := \pi^{-1}(p)$, $m := \pi^{-1}(q)$, and $\widetilde{D} := \pi^* D - 2l$ in (i) (resp. $\widetilde{D} := \pi^* D - 2(l + m)$ in (ii)).

Since we have $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-\widetilde{D})) \neq 0$, we have a nonsplit sequence on $\widetilde{X}$:

$$0 \to \mathcal{O}_{\widetilde{X}} \to E \to \mathcal{O}_{\widetilde{X}}(\widetilde{D}) \to 0.$$

Since $E$ satisfies the inequality $c_1(E)^2 > 4c_2(E)$, by Theorem 1 in [SB], there exists a Frobenius map $F^e\colon \widetilde{X} \to \widetilde{X}$ and an exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(p^e\widetilde{D} - \Delta_1) \to \widetilde{E} \to \mathscr{I}_Z \otimes \mathscr{O}_{\widetilde{X}}(\Delta_1) \to 0.$$

Here $\widetilde{E} := (F^e)^*E$, $Z$ is a 0 cycle and $\Delta_1$ is some effective divisor such that $p^e\widetilde{D} - 2\Delta_1$ is contained in the positive cone of $\widetilde{X}$. We write $\Delta_1 = \pi^*\Delta + rl$ in (i) (resp: $\Delta_1 = \pi^*\Delta + rl + sm$ in (ii)) where $r$, $s$ are some integers and $\Delta$ is an effective divisor on $X$.

If $p^e = 1$, Reider's argument shows $\Delta$ satisfies the properties stated in the theorem (cf. [R]). We shall show that the case $p^e \geq p$ never occurs under our assumption.

Suppose $p^e \geq p$. Then there is a purely inseparable covering $\rho\colon Y \to \widetilde{X}$ of $\deg \rho = p^e$ and we have the following estimates (cf. [SB]).

LEMMA 3. *Assume $d := D^2 \geq 5$ in* (i) (*resp. $d \geq 10$ in* (ii)). *Then*

$$D \cdot \Delta \leq \left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - d}\right) p^e$$

*and*

$$\chi(\mathscr{O}_Y) \geq \left(\chi(\mathscr{O}_X) + \frac{p^e - 1}{12}[(2p^e - 1)\widetilde{D}^2 - 3\widetilde{D} \cdot K_{\widetilde{X}}]\right) p^e.$$

Since both $D$ and $D - K$ are nef and big, the Hodge index theorem yields $K \cdot D \leq d - 3$ and $K^2 \leq d - 5$ in (i) (resp. $K^2 \leq d - 6$ in (ii)). By these estimates, we have

$$\begin{aligned}
\omega_Y \cdot \rho^*\pi^*D &= 2(p^e - 1)D \cdot \Delta + p^e(K \cdot D - (p^e - 1)D^2) \\
&\leq 2(p^e - 1)\left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - d}\right) + p^e(d - 3 - (p^e - 1)d) \\
&= (d - 3 - (p^e - 1)\sqrt{d^2 - 4d})p^e.
\end{aligned}$$

Thus we obtain $\omega \cdot \rho^*\pi^*D < 0$. Therefore $Y$ is ruled and hence $X$ is uniruled. Let $q(X)$ be the irregularity of $X$. Then by Lemma 34 in [SB], we have $\chi(\mathscr{O}_Y) \leq 1 - q(X) \leq 1$. Assume we are in the case (i). Then Corollary 30 and Proposition 35 in the same paper yield

$$\begin{aligned}\chi(\mathcal{O}_Y) &\geq \left(\chi(\mathcal{O}_X) + \frac{p^e-1}{12}[(2p^e-1)\widetilde{D}^2 - 3\widetilde{D}\cdot K_{\widetilde{X}}]\right)p^e \\ &\geq \left(-\frac{K^2}{10} + \frac{p^e-1}{12}[(2p^e-1)(D^2-4) - 3(D\cdot K+2)]\right)p^e \\ &\geq \left(-\frac{d-5}{10} + \frac{p^e-1}{12}[(2p^e-1)(d-4) - 3(d-3) - 6]\right)p^e.\end{aligned}$$

Similarly in the case (ii) we obtain

$$\chi(\mathcal{O}_Y) \geq \left(-\frac{d-6}{10} + \frac{p^e-1}{12}[(2p^e-1)(d-8) - 3(d-3) - 12]\right)p^e.$$

However, under the assumption that one of (1) to (4) (resp. (1′) to (5′)) holds, we have $\chi(\mathcal{O}_Y) > 1$. This is a contradiction and hence the theorem is proved. □

## REFERENCES


[E] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Publ. Math. I.H.E.S., **67** (1988), 97–144.

[R] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math., **127** (1988), 309–316.

[SB] N. I. Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in characteristic p*, preprint.


Received March 18, 1991.

TOKYO METROPOLITAN UNIVERSITY
TOKYO 192-03, JAPAN

# CONTENTS



Vol. 160, No. 1 September 1993

# PACIFIC JOURNAL OF MATHEMATICS

Volume 160 No. 1 September 1993