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SOME NUMERIC RESULTS ON ROOT SYSTEMS

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Let Φ be an irreducible root system (sometimes we denote Φ by $\Phi(X)$ to indicate its type X). Choose a simple root system Π in Φ . Let Φ^+ (resp. Φ^-) be the corresponding positive (resp. negative) root system of Φ . By a subsystem Φ' of Φ (resp. of Φ^+), we mean that Φ' is a subset of Φ (resp. of Φ^+) which itself forms a root system (resp. a positive root system). We refer the readers to Bourbaki's book for the detailed information about root systems. Among all subsystems of Φ , the subsystems of Φ of rank 2 and of type $\neq A_1 \times A_1$ are of particular importance in the theory of Weyl groups and affine Weyl groups (see the papers by Jian-yi Shi). In the present paper, we shall compute the number of such subsystems of Φ for an irreducible root system Φ of any type. Some interesting properties of Φ are also obtained.

1. The number $h(\alpha)$. Let $\langle \cdot, \cdot \rangle$ be an inner product of the euclidean space E spanned by Φ . For any $\alpha \in \Phi$, we denote by $|\alpha|$ the length of α , by α^\vee the dual root $2\alpha/\langle \alpha, \alpha \rangle$ of α and by s_α the reflection in E which sends any vector $v \in E$ to $s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$. For $\alpha, \beta \in \Phi$, we write $\alpha < \beta$ if $\beta - \alpha$ is a sum of some positive roots.

For $\alpha \in \Phi$, we define the sets $D(\alpha) = \{\beta \in \Phi \mid \alpha + \beta \in \Phi\}$, $D^+(\alpha) = D(\alpha) \cap \Phi^+$ and $D^-(\alpha) = D(\alpha) \cap \Phi^-$. Let $d(\alpha)$ be the cardinality of the set $D^+(\alpha)$. Also, we denote by $\text{ht}(\alpha)$ the height of α , i.e. $\text{ht}(\alpha) = \sum_{\beta \in \Pi} a_\beta$ if $\alpha = \sum_{\beta \in \Pi} a_\beta \beta$ with $a_\beta \in \mathbb{Z}$.

For any $\alpha \in \Phi^+$, there exists a sequence ξ of roots $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_r$ in Φ^+ such that $\alpha_r \in \Pi$ and for every $i, 1 < i \leq r$, we have $\alpha_{i-1} > \alpha_i = s_{\delta_i}(\alpha_{i-1})$ for some $\delta_i \in \Pi$. Such a sequence ξ is called a root path from α to Π . We denote by $h(\alpha, \xi)$ the length r of ξ . We shall deduce a formula for the number $h(\alpha, \xi)$, from which we shall see that $h(\alpha, \xi)$ is actually independent on the choice of a root path ξ from α to Π but only dependent on the root α .

Note that if the root system Φ contains roots of two different lengths and if $\alpha = \sum_{\beta \in \Pi} a_\beta \beta$ is a long root of Φ with $a_\beta \in \mathbb{Z}$ then each coefficient a_β with β short is divisible by $|\alpha|^2/|\beta|^2$.

LEMMA 1.1. *Let $\alpha = \sum_{\beta \in \Pi} a_\beta \beta$, $a_\beta \in \mathbb{Z}$, be a root of Φ^+ and let ξ be a root path from α to Π . Then*

(i) If either all the roots of Φ have the same length or α is a short root of Φ with Φ containing roots of two different lengths, then $h(\alpha, \xi) = \text{ht}(\alpha)$;

(ii) If α is a long root of Φ with Φ containing roots of two different lengths, then

$$h(\alpha, \xi) = \sum_{\beta \in \Pi} \frac{|\beta|^2}{|\alpha|^2} a_\beta.$$

Proof. Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_r$ be a root path from α to Π . Then in case (i), we have $\text{ht}(\alpha_i) = \text{ht}(\alpha_{i+1}) + 1$ for any $i, 1 \leq i < r$, by the fact that $\langle \alpha_i, \delta_i^\vee \rangle = 1$, where $\delta_i \in \Pi$ satisfies the relation $\delta_i(\alpha_{i-1}) = \alpha_i$. So assertion (i) follows immediately by applying induction on $\text{ht}(\alpha) \geq 1$. Next assume that we are in case (ii). Again apply induction on $\text{ht}(\alpha) \geq 1$. If $\text{ht}(\alpha) = 1$, then $\alpha \in \Pi$ and the result is obviously true. Now assume $\text{ht}(\alpha) > 1$. Let $\xi : \alpha_1 = \alpha, \alpha_2, \dots, \alpha_r$ be a root path from α to Π . Then $\xi' : \alpha_2, \alpha_3, \dots, \alpha_r$ is a root path from α_2 to Π with $\text{ht}(\alpha_2) < \text{ht}(\alpha)$ and $\alpha_2 = s_\delta(\alpha)$ for some $\delta \in \Pi$. Note that α_2 is a long root of Φ . Write

$$\alpha_2 = \sum_{\beta \in \Pi} a'_\beta \beta, \quad a'_\beta \in \mathbb{Z}.$$

Then by inductive hypothesis, we have

$$h(\alpha_2, \xi') = \sum_{\beta \in \Pi} \frac{|\beta|^2}{|\alpha_2|^2} a'_\beta.$$

Since $\langle \alpha, \delta^\vee \rangle = |\alpha|^2/|\delta|^2$ by the assumption $s_\delta(\alpha) < \alpha$, we have

$$\alpha = \alpha_2 + \frac{|\alpha|^2}{|\delta|^2} \delta = \sum_{\substack{\beta \in \Pi \\ \beta \neq \delta}} a'_\beta \beta + \left(a'_\delta + \frac{|\alpha|^2}{|\delta|^2} \right) \delta.$$

This implies that

$$\begin{aligned} h(\alpha, \xi) &= h(\alpha_2, \xi') + 1 = \sum_{\beta \in \Pi} \frac{|\beta|^2}{|\alpha_2|^2} a'_\beta + 1 \\ &= \sum_{\substack{\beta \in \Pi \\ \beta \neq \delta}} \frac{|\beta|^2}{|\alpha_2|^2} a'_\beta + \frac{|\delta|^2}{|\alpha_2|^2} \left(a'_\delta + \frac{|\alpha|^2}{|\delta|^2} \right) = \sum_{\beta \in \Pi} \frac{|\beta|^2}{|\alpha|^2} a_\beta \end{aligned}$$

by noting $|\alpha| = |\alpha_2|$. □

We see from Lemma 1.1 that, for any $\alpha \in \Phi^+$, the length of a root path ξ from α to Π is only dependent on α but not on the choice of the path ξ . So we can denote $h(\alpha, \xi)$ simply by $h(\alpha)$.

Let Φ^\vee be the dual root system of Φ , i.e. $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. Then $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$ and $(\Phi^\vee)^+ = \{\alpha^\vee \mid \alpha \in \Phi^+\}$ are a simple root system and the corresponding positive root system of Φ^\vee , respectively. We can define the number $h^\vee(\alpha^\vee)$ for any $\alpha^\vee \in (\Phi^\vee)^+$ in the same way as that for a root of Φ . That is, $h^\vee(\alpha^\vee)$ is the length of a root path from α^\vee to Π^\vee in $(\Phi^\vee)^+$.

LEMMA 1.2. *For any $\alpha \in \Phi^+$, we have $h(\alpha) = h^\vee(\alpha^\vee)$.*

Proof. For any $\delta \in \Pi$, we have the following equivalence.

$$(1) \quad s_\delta(\alpha) < \alpha \Leftrightarrow \langle \alpha, \delta^\vee \rangle > 0 \Leftrightarrow \langle \alpha^\vee, \delta \rangle > 0 \Leftrightarrow s_{\delta^\vee}(\alpha^\vee) < \alpha^\vee.$$

Apply induction on $h(\alpha) \geq 1$. When $h(\alpha) = 1$, we have $\alpha \in \Pi$ and hence $\alpha^\vee \in \Pi^\vee$. So $h^\vee(\alpha^\vee) = 1$, and the result is true in this case. Now assume $h(\alpha) > 1$. Then there exists some $\delta \in \Pi$ with $\langle \alpha, \delta^\vee \rangle > 0$. So $h(s_\delta(\alpha)) = h(\alpha) - 1$. By inductive hypothesis, we have

$$(2) \quad h(s_\delta(\alpha)) = h^\vee((s_\delta(\alpha))^\vee) = h^\vee(s_{\delta^\vee}(\alpha^\vee)).$$

But by (1), we have

$$h^\vee(s_{\delta^\vee}(\alpha^\vee)) = h^\vee(\alpha^\vee) - 1.$$

Thus we get $h(\alpha) = h^\vee(\alpha^\vee)$. □

2. The number $d(\alpha)$. We shall deduce a formula for the number $d(\alpha)$ for any $\alpha \in \Phi^+$.

For $\alpha, \beta \in \Phi$, we call all roots of the form $\alpha + i\beta$ ($i \in \mathbb{Z}$) the β -string through α . Let $\alpha \in \Phi^+$ and $\delta \in \Pi$ satisfy the inequality $\langle \alpha, \delta^\vee \rangle > 0$. Then it is easily seen that $\alpha, \alpha - \delta, \dots, \alpha - \langle \alpha, \delta^\vee \rangle \delta$ is the δ -string through α except for the case when α is the highest short root of the root system of type G_2 .

LEMMA 2.1. *Given $\alpha \in \Phi^+$ and $\delta \in \Pi$ with $\langle \alpha, \delta^\vee \rangle > 0$. Let $\alpha' = s_\delta(\alpha)$. Then (i) $D(\alpha') = s_\delta(D(\alpha))$.*

(ii) $s_\delta(D^+(\alpha')) = D^+(\alpha) \cup \{-\delta\}$, provided that α is not the highest short root of the root system of type G_2 ;

(iii) $d(\alpha') = d(\alpha) + 1$ under the same assumption as that in (ii).

Proof. (i) $\beta \in D(\alpha') \Leftrightarrow \beta + \alpha' \in \Phi \Leftrightarrow s_\delta(s_\delta(\beta) + \alpha) \in \Phi \Leftrightarrow s_\delta(\beta) + \alpha \in \Phi \Leftrightarrow s_\delta(\beta) \in D(\alpha) \Leftrightarrow \beta \in s_\delta(D(\alpha))$.

(ii) First we shall show $s_\delta(D^+(\alpha)) \subset D^+(\alpha')$. Let $\beta \in s_\delta(D^+(\alpha))$. Then $\beta \in D(\alpha')$ by (i). If $\beta \in D^-(\alpha') \subseteq \Phi^-$, then by the fact $s_\delta(\beta) \in D^+(\alpha) \subseteq \Phi^+$, we have $\beta = -\delta$. Since $\alpha, \alpha - \delta, \dots, \alpha - \langle \alpha, \delta^\vee \rangle \delta$ is the δ -string through α by the above remark, we see that $\alpha + s_\delta(\beta) = \alpha + \delta \notin \Phi$ which contradicts the condition $s_\delta(\beta) \in D^+(\alpha)$. Thus we have $\beta \in D^+(\alpha')$ and so $s_\delta(D^+(\alpha)) \subset D^+(\alpha')$, i.e. $D^+(\alpha) \subset s_\delta(D^+(\alpha'))$.

It is obvious that $\{-\delta\} \subseteq s_\delta(D^+(\alpha'))$. Thus it remains to show the reversing inclusion. Now assume $\beta \in s_\delta(D^+(\alpha'))$. Then $s_\delta(\beta) \in D^+(\alpha')$. This implies that $s_\delta(\beta) + \alpha' \in \Phi$ and $s_\delta(\beta) \in \Phi^+$. Hence $\beta + \alpha \in \Phi$ and $s_\delta(\beta) \in \Phi^+$. But then we have either $\beta \in D^+(\alpha)$ or $\beta = -\delta$, which implies $s_\delta(D^+(\alpha')) \subseteq D^+(\alpha) \cup \{-\delta\}$.

(iii) This is an immediate consequence of (ii). □

REMARK. In the case when the type of Φ is G_2 , let $\Pi = \{\gamma, \delta\}$ with δ short. Then $D^+(2\delta + \gamma) = \{\delta, \delta + \gamma\}$, $D^+(\delta + \gamma) = \{\delta, 2\delta + \gamma\}$ and $\delta + \gamma = s_\delta(2\delta + \gamma)$. Thus the results (ii), (iii) of Lemma 2.1 do not hold in this case.

In Φ^+ , let α^l be the highest long root and let α^s be the highest short root, where we stipulate $\alpha^s = \alpha^l$ in the case when all the roots of Φ have the same length.

THEOREM 2.2. *Given $\alpha \in \Phi^+$.*

(i) *If α is short and if the type of Φ is not G_2 , then*

$$h(\alpha) + d(\alpha) = \text{ht}(\alpha^l).$$

(ii) *If α is long, then*

$$h(\alpha) + d(\alpha) = \text{ht}(\alpha^s).$$

Proof. First assume that the result has been shown to be true in the case when $\alpha = \alpha^s$ in (i) and $\alpha = \alpha^l$ in (ii). Apply reversing induction on $h(\alpha) \leq h(\alpha^s)$ in (i) and on $h(\alpha) \leq h(\alpha^l)$ in (ii). Now assume that α is either short with $h(\alpha) < h(\alpha^s)$ or long with $h(\alpha) < h(\alpha^l)$. Then there must exist some $\delta \in \Pi$ with $\langle \alpha, \delta^\vee \rangle < 0$. So $\alpha' = s_\delta(\alpha) > \alpha$ with $h(\alpha') = h(\alpha) + 1$. We see $\langle \alpha', \delta^\vee \rangle > 0$. By Lemma 2.1(iii), we

have $d(\alpha') = d(\alpha) - 1$. So by inductive hypothesis, we get

$$\begin{aligned} h(\alpha) + d(\alpha) &= (h(\alpha') - 1) + (d(\alpha') + 1) \\ &= h(\alpha') + d(\alpha') \\ &= \begin{cases} \text{ht}(\alpha') & \text{if } \alpha \text{ is short,} \\ \text{ht}(\alpha^s) & \text{if } \alpha \text{ is long,} \end{cases} \end{aligned}$$

by noting $|\alpha| = |\alpha'|$.

Thus it remains to show that assertion (i) is true for $\alpha = \alpha^s$ and that assertion (ii) is true for $\alpha = \alpha^l$.

In the case when the Dynkin diagram is simply laced, we have $h(\alpha^s) = \text{ht}(\alpha^s)$ by Lemma 1.1(i). Clearly, $d(\alpha^s) = 0$. So our result is true in this case. Now assume that Φ contains roots of two different lengths. If Φ has type B_n , then $h(\alpha^s) = n$, $d(\alpha^s) = n - 1$, $\text{ht}(\alpha^l) = 2n - 1$, $d(\alpha^l) = 0$ and $h(\alpha^l) = h^\vee((\alpha^l)^\vee) = \text{ht}((\alpha^l)^\vee) = \text{ht}(\alpha^s) = 2n - 2$ by Lemmas 1.2 and 1.1(i). If Φ has type C_n , then $h(\alpha^s) = 2n - 2$, $d(\alpha^s) = 1$, $\text{ht}(\alpha^l) = 2n - 1$ and $d(\alpha^l) = 0$. We also have

$$h(\alpha^l) = h^\vee((\alpha^l)^\vee) = \text{ht}((\alpha^l)^\vee) = \text{ht}(\alpha^s) = n$$

by Lemmas 1.2 and 1.1(i). If Φ has type F_4 , then $h(\alpha^s) = 8$, $d(\alpha^s) = 3$, $\text{ht}(\alpha^l) = 11$ and $d(\alpha^l) = 0$. By the same reason as above, we have

$$h(\alpha^l) = h^\vee((\alpha^l)^\vee) = \text{ht}((\alpha^l)^\vee) = \text{ht}(\alpha^s) = 8.$$

If Φ has type G_2 , then $d(\alpha^l) = 0$ and $h(\alpha^l) = \text{ht}(\alpha^s) = 3$. Thus in all the cases, our result is true. \square

COROLLARY 2.3. *Assume that the type of Φ is not G_2 . Then for any short root α of Φ^+ , we have the equation*

$$\text{ht}(\alpha) + d(\alpha) = h - 1,$$

where h is the Coxeter number of Φ .

Proof. We have $h(\alpha) = \text{ht}(\alpha)$ by Lemma 1.1(i). Since $\text{ht}(\alpha^l) = h - 1$, our result follows immediately from Theorem 2.2(i). \square

3. The number of certain rank 2 subsystems in Φ . Let $g(\Phi)$ be the number of subsystems of Φ of rank 2 and of type other than $A_1 \times A_1$. Then $g(\Phi)$ is also equal to the number of positive subsystems of Φ^+ of rank 2 and of type $\neq A_1 \times A_1$. In this section, we shall compute the number $g(\Phi)$ for Φ of any type.

LEMMA 3.1. *If the Dynkin diagram of Φ is simply laced, then*

$$(3) \quad g(\Phi) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d(\alpha).$$

Proof. Under our assumption, the only possible type for a subsystem of Φ^+ of rank 2 and of type $\neq A_1 \times A_1$ is A_2 . Each of such subsystems could be obtained by first taking a root $\alpha \in \Phi^+$ and then taking any root β in the set $D^+(\alpha)$ to form a subsystem $\{\alpha, \beta, \alpha + \beta\}$. Since such a subsystem is obtained twice in the above way, this implies the required formula (3) for the number $g(\Phi)$. \square

Define

$$H(\Phi) = \sum_{\alpha \in \Phi^+} \text{ht}(\alpha), \quad H^s(\Phi) = \sum_{\substack{\alpha \in \Phi^+ \\ \text{short}}} \text{ht}(\alpha) \quad \text{and}$$

$$H^l(\Phi) = \sum_{\substack{\alpha \in \Phi^+ \\ \text{long}}} \text{ht}(\alpha).$$

These numbers could be computed for any irreducible root system Φ . Define $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ for any integers $m, n, 0 \leq n \leq m$.

LEMMA 3.2.

Type of Φ	$H(\Phi)$	$H^s(\Phi)$	$H^l(\Phi)$
$A_n (n \geq 1)$	$\binom{n+2}{3}$		
$B_n (n \geq 2)$	$\frac{n(n+1)(4n-1)}{6}$	$\binom{n+1}{2}$	$4 \binom{n+1}{3}$
$C_n (n \geq 2)$	$\frac{n(n+1)(4n-1)}{6}$	$\frac{n(n-1)(4n+1)}{6}$	n^2
$D_n (n \geq 4)$	$\frac{n(n-1)(2n-1)}{3}$		
E_6	156		
E_7	399		
E_8	1240		
F_4	110	46	64
G_2	16	6	10

\square

Now we can compute the numbers $g(\Phi)$ for Φ of types A_n , $n \geq 1$, D_m , $m \geq 4$, and E_i , $i = 6, 7, 8$ as follows.

THEOREM 3.3.

Type of Φ	$g(\Phi)$
A_n ($n \geq 1$)	$\binom{n+1}{3}$
D_n ($n \geq 4$)	$4 \binom{n}{3}$
E_6	120
E_7	336
E_8	1120

Proof. By Corollary 2.3 and Lemma 3.1, we have

$$\begin{aligned} g(\Phi) &= \frac{1}{2} \sum_{\alpha \in \Phi^+} d(\alpha) = \frac{1}{2} \sum_{\alpha \in \Phi^+} (h - 1 - \text{ht}(\alpha)) \\ &= \frac{1}{2} ((h - 1)|\Phi^+| - H(\Phi)). \end{aligned}$$

Thus we have $g(\Phi(A_n)) = \frac{1}{2}(n\binom{n+1}{2} - \binom{n+2}{3}) = \binom{n+1}{3}$ for $n \geq 1$. For $n \geq 4$, we have

$$g(\Phi(D_n)) = \frac{1}{2} \left((2n - 3)n(n - 1) - \frac{n(n - 1)(2n - 1)}{3} \right) = 4 \binom{n}{3}.$$

Also, we have $g(\Phi(E_6)) = \frac{1}{2}(11 \cdot 36 - 156) = 120$,

$$g(\Phi(E_7)) = \frac{1}{2}(17 \cdot 63 - 399) = 336,$$

and $g(\Phi(E_8)) = \frac{1}{2}(29 \cdot 120 - 1240) = 1120$. □

Now assume that Φ contains roots of two different lengths and that the type of Φ is not G_2 . Then the possible types for a subsystem Φ' of Φ of rank 2 and of type $\neq A_1 \times A_1$ are A_2 and B_2 . Let $u(\Phi)$ be the cardinality of the set

$$\{ \{\alpha, \beta\} \mid \alpha, \beta \in \Phi^+ \text{ have different lengths with } \alpha + \beta \in \Phi^+ \}.$$

Then it is easily seen that the following formula for $g(\Phi)$ holds.

$$(4) \quad g(\Phi) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d(\alpha) - u(\Phi).$$

First let us consider the case when Φ has type C_n , $n \geq 2$. We see that a subsystem Φ' of Φ has type A_2 only if all the roots in Φ' are short. This implies that for each long root $\beta \in \Phi^+$, the set $D^+(\beta)$ contains no long root and hence $u(\Phi) = \sum_{\beta \in \Phi^+ \text{ long}} d(\beta)$. So by (4), we get

$$\begin{aligned} g(\Phi) &= \frac{1}{2} \sum_{\alpha \in \Phi^+} d(\alpha) - \sum_{\substack{\beta \in \Phi^+ \\ \text{long}}} d(\beta) = \frac{1}{2} \left(\sum_{\substack{\alpha \in \Phi^+ \\ \text{short}}} d(\alpha) - \sum_{\substack{\beta \in \Phi^+ \\ \text{long}}} d(\beta) \right) \\ &= \frac{1}{2} \left(\sum_{\substack{\alpha \in \Phi^+ \\ \text{short}}} (h-1 - \text{ht}(\alpha)) - \sum_{i=1}^n (i-1) \right) \end{aligned}$$

by Theorem 2.2, Corollary 2.3 and Lemma 1.2. Then by Lemma 3.2, we have

$$\begin{aligned} g(\Phi) &= \frac{1}{2} \left((2n-1)n(n-1) - \frac{n(n-1)(4n+1)}{6} - \frac{n(n-1)}{2} \right) \\ &= \frac{n(n-1)(4n-5)}{6}. \end{aligned}$$

Since the root system of type B_n is the dual of the one of type C_n , there exists a bijection from the set of subsystems of the root system of type C_n to that of type B_n by sending Φ' to Φ'^\vee . Such a bijective map preserves the ranks of subsystems and also preserves the types of them whenever their ranks are not greater than 2. This implies that we also have $g(\Phi) = \frac{n(n-1)(4n-5)}{6}$ when Φ has type B_n .

Next assume that Φ has type F_4 . By Theorem 2.2, Lemma 3.2 and Lemmas 1.1, 1.2, we get

$$\begin{aligned} \frac{1}{2} \sum_{\alpha \in \Phi^+} d(\alpha) &= \frac{1}{2} \left(\sum_{\substack{\alpha \in \Phi^+ \\ \text{short}}} (\text{ht}(\alpha^l) - \text{ht}(\alpha)) + \sum_{\substack{\beta \in \Phi^+ \\ \text{long}}} (\text{ht}(\alpha^s) - \text{ht}(\beta^\vee)) \right) \\ &= \frac{1}{2} \left(\frac{1}{2} |\Phi^+| (\text{ht}(\alpha^l) + \text{ht}(\alpha^s)) - 2H^s(\Phi) \right) \\ &= \frac{1}{2} \left(\frac{1}{2} \cdot 24 \cdot (11+8) - 92 \right) \\ &= 68. \end{aligned}$$

Also, by a direct computation, we get $u(\Phi) = 18$. So by (4), we have

$$g(\Phi) = 68 - 18 = 50.$$

Finally, it is easily seen that $g(\Phi) = 3$ when Φ has type G_2 . Summing up, we get the following table.

THEOREM 3.4.

Type of Φ	$g(\Phi)$
B_n or C_n ($n \geq 2$)	$\frac{n(n-1)(4n-5)}{6}$
F_4	50
G_2	3

□

From the above discussion, we can deduce even more precise conclusion. We note that in any irreducible root system Φ , there exist at most two different types of subsystems which have rank 2 and types $\neq A_1 \times A_1$. Let $g'(\Phi)$ be the number of subsystems of Φ of type A_2 and let $g''(\Phi)$ be the number of subsystems of Φ of type B_2 or G_2 . Then by Theorem 3.3, we have

$$g'(\Phi(B_n)) = g'(\Phi(C_n)) = g(\Phi(D_n)) = 4 \binom{n}{3} \quad \text{for } n \geq 4$$

by noting that all the long (resp. short) roots of $\Phi(B_n)$ (resp. $\Phi(C_n)$) form a root system of type D_n . Hence we also have

$$\begin{aligned} g''(\Phi(B_n)) &= g''(\Phi(C_n)) = g(\Phi(B_n)) - g'(\Phi(B_n)) \\ &= \frac{n(n-1)(4n-5)}{6} - 4 \binom{n}{3} \\ &= \binom{n}{2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g''(\Phi(F_4)) &= u(\Phi(F_4)) = 18 \quad \text{and} \\ g'(\Phi(F_4)) &= g(\Phi(F_4)) - g''(\Phi(F_4)) = 50 - 18 = 32. \end{aligned}$$

Finally, it is obvious that $g'(\Phi(G_2)) = 2$ and $g''(\Phi(G_2)) = 1$. Summing up, we have the following table.

THEOREM 3.5.

Type of Φ	$g'(\Phi)$	$g''(\Phi)$
$B_n, C_n (n \geq 2)$	$4 \binom{n}{3}$	$\binom{n}{2}$
F_4	32	18
G_2	2	1

□

Proof. By the above discussion, it remains to show the result for Φ being of types B_m or C_m , $m = 2, 3$. But this could be checked directly.

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