Pacific Journal of Mathematics

A NOTE ON MORTON'S CONJECTURE CONCERNING THE LOWEST DEGREE OF A 2-VARIABLE KNOT POLYNOMIAL

PETER RICHARD CROMWELL

Volume 160 No. 2 October 1993

Pacific Journal of Mathematics

A NOTE ON MORTON'S CONJECTURE CONCERNING THE LOWEST DEGREE OF A 2-VARIABLE KNOT POLYNOMIAL

PETER RICHARD CROMWELL

Volume 160 No. 2 October 1993

PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 2, 1993

A NOTE ON MORTON'S CONJECTURE CONCERNING THE LOWEST DEGREE OF A 2-VARIABLE KNOT POLYNOMIAL

PETER R. CROMWELL

This note is concerned with the behaviour of the 'HOMFLY' polynomial of oriented links, $P_L(v, z)$. In particular, we show that the gap between the two lowest powers of v can be made arbitrarily large. This casts doubt on whether Morton's conjecture on the least v-degree can be established in general by the kind of combinatorial approach that has been successfully applied to some special cases.

Introduction. The two-variable knot polynomial $P_L(v, z)$ of a link L, announced in **[FYHLMO]**, **[PT]**, can be written in the form

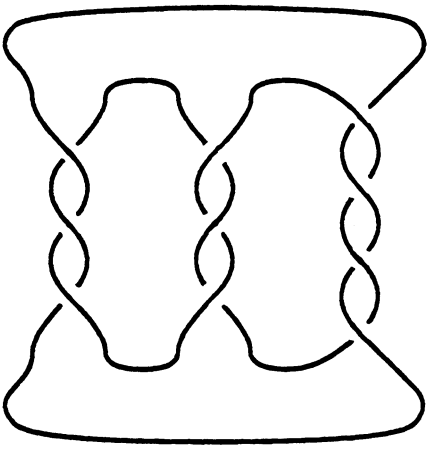
$$P_L(v, z) = \sum_{i=e}^{E} a_i(z)v^i$$

where $a_i(z)$ is a polynomial in z for each i, $a_e(z) \neq 0$, and $a_E(z) \neq 0$. Let $f(P_L)$ denote the least degree in v in the polynomial P_L. Say that $f(P_L)$ is the *first* degree of P_L. With the above formulation $f(P_L) = e$. Let $s(P_L)$ be the least $i > e$ such that $a_i(z) \neq 0$. Say that $s(P_L)$ is the *second* degree of P_L.

In **[Mo3]** H. Morton conjectured that

$$f(P_L) \leq 1 - \chi(L)$$

for all links L where $\chi(L)$ is the maximum Euler characteristic over all orientable surfaces spanning L. In **[Cr]** I showed that the conjecture is satisfied by the homogeneous links (a class containing the positive and alternating links as special cases). A computer search for counterexamples in other classes of links showed up an interesting phenomenon: sometimes polynomials were produced where $s(P_L) - f(P_L)$ was quite large and $a_e(z) = 1$. In these cases it was only the term v^e, isolated from the other non-zero terms in the polynomial, which saved the conjecture from being violated. This prompted the question of whether $s(P_L) - f(P_L)$ could be arbitrarily large. Here I provide examples to show that it can.



$(3, -3, 4)$

FIGURE 1

EXAMPLES. The simplest examples that I have found can be viewed as pretzel knots of the form $(3, -3, 2a)$ for any $a \in \mathbb{N}$ (see Figure 1). Writing the polynomial of this knot as $P(3, -3, 2a)$ we get

$$\begin{aligned} P(3, -3, 2a) &= v^2 P(3, -3, 2(a-1)) \\ &\quad + vzP \text{ (two component trivial link)} \\ &= v^2 P(3, -3, 2(a-1)) - v^2 + 1 \\ &= v^{2a}(P(3, -3, 0) - 1) + 1. \end{aligned}$$

Now $(3, -3, 0)$ is a square or reef knot—the connected sum of a trefoil and its mirror image. Its polynomial is

$$P(3, -3, 0) = (-2 - z^2)v^{-2} + (5 + 4z^2 + z^4) + (-2 - z^2)v^2.$$

Letting K denote the pretzel knot $(3, -3, 2a)$ we obtain

$$s(P_K) - f(P_K) = \begin{cases} 2, & 0 \le a \le 2, \\ 2(a-1), & 2 < a. \end{cases}$$

Thus $s(P_K) - f(P_K)$ can be made as large as we please.

Applying Seifert's algorithm to the standard diagram of the pretzel knot, K, shows that $1 - \chi(K) \le 6$. So whenever $a > 4$, we have $s(P_K) > 1 - \chi(K)$ and the constant term in P_K is the only term which validates the conjecture. The difference $s(P_K) - (1 - \chi(K))$ can also be made arbitrarily large. These examples suggest that it may be difficult to prove Morton's conjecture true in general using a combinatorial approach like that in [**Cr**].

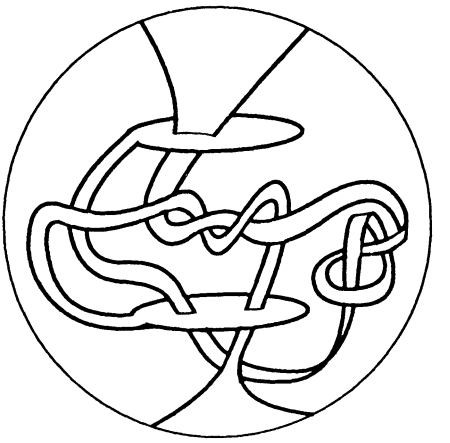


FIGURE 2

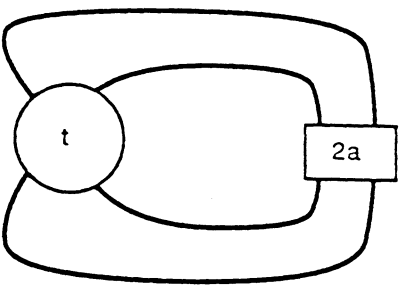


FIGURE 3

Many other examples are easily constructed. All that is required is a tangle t such that its numerator $N(t)$ is the trivial link with two components and its denominator $D(t)$ is a non-trivial knot (using Conway's notation for the closures of a tangle [**Co**]). Such tangles are easily constructed: take two discs embedded in the interior of a ball and connect each of them to the boundary of the ball by a ribbon. The ribbons may pass through the discs in ribbon singularities. An example is shown in Figure 2.

Inserting $2a$ positive half-twists into $D(t)$, as shown in Figure 3, produces the same behaviour in the polynomial as before. That is

$$P(D(t) \text{ with } 2a \text{ half-twists}) = v^{2a}(P(D(t)) - 1) + 1 .$$

Substituting $v = 1$ in this expression shows that all of the knots derived from $D(t)$ in this way have the same Conway polynomial. Thus the square knot, 8_{20}, and 10_{140} all have the same Conway polynomial since they are $(3, -3, 0)$, $(3, -3, 2)$ and $(3, -3, 4)$ respectively.

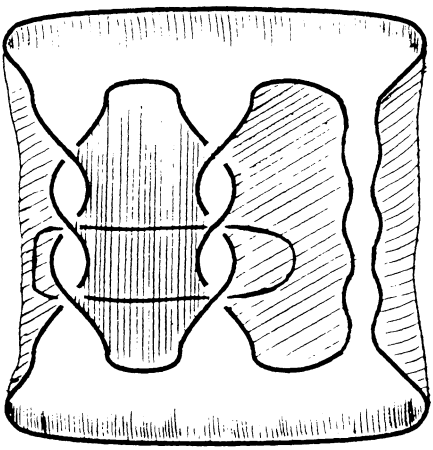


FIGURE 4

REMARKS. Morton observed that the connected sum of two trefoils is a fibred knot and that the insertion of twists described above can be achieved by $(1, n)$ Dehn surgery about an unknotted untwisted curve in the fibre surface. Such a curve is shown in Figure 4. Hence there is an infinite family of fibred knots all having the same Alexander polynomial but which can be distinguished by $P(v, z)$. This provides further counterexamples to the conjecture (made in [**Ne**]) that at most finitely many fibred knots could have the same Alexander polynomial. It was Morton who showed that the conjecture is false [**Mo1**], [**Mo2**].

The referee drew attention to a related result of Akio Kawauchi [**Ka**] who has shown that a gap in the z-degree can also be made as large as desired. More specifically, he constructed a family of knots whose polynomials have the form

$$P_L(v, z) = 1 + \sum_{i=m}^{M} b_i(v) z^i$$

(where $b_m(v)$ and $b_M(v)$ are non-zero polynomials in v). The value of m can be made arbitrarily large.

REFERENCES

[Co] J. H. Conway, *An enumeration of knots and links and some of their related properties*, Computational problems in abstract algebra, (1969), 329–358.

[Cr] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc., (2) **39** (1989), 535–552.

[FYHLMO] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. C. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc., **12** (1985), 239–246.

[Ka] A. Kawauchi, *Imitations of* (3, 1)*-dimensional manifold pairs*, Sugaku, **40** (1988), 193–204 (in Japanese); Sugaku Expositions, **2** (1989), 141–156 (in English).

[Mo1] H. R. Morton, *Infinitely many fibred knots with the same Alexander polynomial*, Topology, **17** (1978), 101–104.

[Mo2] ——, *Fibred knots with a given Alexander polynomial*, Noeuds, tresses et singularités, Plan-sur-Bex (1982) L'enseignement Math. monograph 31, 205–222.

[Mo3] ——, *Seifert circles and knot polynomials*, Math. Proc. Cambridge Philosophical Soc., **99** (1986), 107–141.

[Ne] L. P. Neuwirth, *The Status of Some Problems Related to Knot Groups*, Lecture Note in Math. 375, Springer-Verlag, (1974), 209–230.

[PT] J. H. Prztycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math., **4** (1987), 115–139.

Received July 1, 1991 and in revised form August 24, 1992. The author was supported by a S.E.R.C. post-doctoral fellowship.

UNIVERSITY COLLEGE OF NORTH WALES
DEAN STREET, BANGOR
GWYNEDD, LL57 1UT

Current address: The University
PO Box 147
Liverpool, L69 3BX UK

PACIFIC JOURNAL OF MATHEMATICS

Founded by

E. F. BECKENBACH (1906–1982) F. WOLF (1904–1989)

EDITORS

S. Y. A. CHANG
(Managing Editor)
University of California
Los Angeles, CA 90024-1555
chang@math.ucla.edu

F. MICHAEL CHRIST
University of California
Los Angeles, CA 90024-1555
christ@math.ucla.edu

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112
clemens@math.utah.edu

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093
tenright@ucsd.edu

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

R. FINN
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106
mgscharl@henri.ucsb.edu

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

V. S. VARADARAJAN
University of California
Los Angeles, CA 90024-1555
vsv@math.ucla.edu

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
UNIVERSITY OF MONTANA
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Julie Honig, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 75 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $200.00 a year (10 issues). Special rate: $100.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

This publication was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX,
the American Mathematical Society's TEX macro system.

Copyright © 1993 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 160 No. 2 October 1993

A note on Morton's conjecture concerning the lowest degree of a 2-variable knot polynomial 201
PETER RICHARD CROMWELL

The soft torus and applications to almost commuting matrices 207
RUY EXEL

A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory 219
M. FURI and MARIA PATRIZIA PERA

Positive 2-spheres in 4-manifolds of signature $(1, n)$ 245
KAZUNORI KIKUCHI

Tau functions for the Dirac operator in the Euclidean plane 259
JOHN NELSON PALMER

Moduli of linear differential equations on the Riemann sphere with fixed Galois groups 343
MICHAEL SINGER