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**JORDAN ANALOGS OF THE BURNSIDE AND JACOBSON  
DENSITY THEOREMS**

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## JORDAN ANALOGS OF THE BURNSIDE AND JACOBSON DENSITY THEOREMS

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If  $\mathcal{A}$  is an (associative) algebra of linear operators on a vector space, it is well known that 2-transitivity for  $\mathcal{A}$  implies density and, in certain situations, transitivity guarantees 2-transitivity. In this paper we consider analogs of these results for Jordan algebras of linear operators with the standard Jordan product.

**0. Introduction.** Let  $\mathcal{L}(\mathcal{V})$  be the algebra of all linear operators on a vector space  $\mathcal{V}$  over the field  $\mathbb{F}$ . A subset  $\mathcal{S}$  of  $\mathcal{L}(\mathcal{V})$  is called transitive if  $\mathcal{S}x = \mathcal{V}$  for every nonzero  $x$  in  $\mathcal{V}$ . More generally,  $\mathcal{S}$  is called  $k$ -transitive if given linearly independent vectors  $x_1, x_2, \dots, x_k$  and arbitrary vectors  $y_1, y_2, \dots, y_k$  in  $\mathcal{V}$  there exists a member  $S$  of  $\mathcal{S}$  such that  $Sx_i = y_i$ ,  $i = 1, 2, \dots, k$ . If  $\mathcal{S}$  is  $k$ -transitive for every  $k$ , then it is called (strictly) dense. It is a remarkable fact due to Jacobson [2] that if  $\mathcal{S}$  is an (associative) subalgebra of  $\mathcal{L}(\mathcal{V})$ , then 2-transitivity implies density for arbitrary  $\mathbb{F}$ . In particular, if  $\mathcal{V}$  is finite-dimensional, then  $\mathcal{L}(\mathcal{V})$  is the only 2-transitive algebra on  $\mathcal{V}$ . There are transitive algebras that are not 2-transitive even if  $\mathbb{F}$  is algebraically closed. In the presence of certain conditions (e.g., topological) transitivity implies density. The most well-known result of this kind is Burnside's theorem [3]: if  $\mathcal{V}$  is finite-dimensional and  $\mathbb{F}$  is algebraically closed, then the only transitive algebra over  $\mathcal{V}$  is  $\mathcal{L}(\mathcal{V})$ .

In this paper we consider analogs of these results for Jordan algebras of operators: linear spaces  $\mathcal{A}$  of operators such that  $A^2$  and  $ABA$  belong to  $\mathcal{A}$  for all  $A$  and  $B$  in  $\mathcal{A}$ . If the characteristic of the field  $\mathbb{F}$  is different from 2, this is equivalent to the requirement that  $\mathcal{A}$  be closed under the Jordan bracket  $\{A, B\} = AB + BA$ . Over this kind of field a Jordan algebra  $\mathcal{A}$  may be equivalently defined as a linear space closed under taking positive integral powers. For the sake of completeness we include proofs of a few elementary facts obtainable from the general theory of Jordan algebras [4].

In what follows we often find it convenient to view members of  $\mathcal{L}(\mathcal{V})$  as matrices over  $\mathbb{F}$ ; this should cause no confusion. The set

of all  $n \times n$  matrices over  $\mathbb{F}$  will be denoted by  $\mathcal{M}_n(\mathbb{F})$ . A member  $A$  of  $\mathcal{L}(\mathcal{V})$  (or  $\mathcal{M}_n(\mathbb{F})$ ) is called a projection or an idempotent element if  $A^2 = A$ .

### 1. Transitive Jordan algebras over arbitrary fields.

1.0. All the Jordan algebras  $\mathcal{A}$  considered in this section are subalgebras of the algebra of all linear operators  $\mathcal{L}(\mathcal{V})$  on a vector space  $\mathcal{V}$  over a field  $\mathbb{F}$ . In finite dimensions Jacobson's theorem says that any 2-transitive associative algebra of linear operators of  $\mathcal{V}$  is all of  $\mathcal{L}(\mathcal{V})$  [2]. The proof of this result for Jordan algebras of operators needs some preparation.

1.1. PROPOSITION. *Let  $\mathcal{A}$  be a Jordan algebra of linear operators on a vector space  $\mathcal{V}$ . Then:*

(a)  $E\mathcal{A}E$  and  $(I - E)\mathcal{A}(I - E)$  are Jordan subalgebras of  $\mathcal{A}$  for every  $E$  in  $\mathcal{A}$ .

(b) If  $\mathcal{V}$  is finite dimensional and  $\mathcal{A}$  is 2-transitive, then for every subspace  $\mathcal{W}$  of  $\mathcal{V}$  there exists a projection  $E \in \mathcal{A}$  such that  $E\mathcal{V} = \mathcal{W}$ .

(c) If  $\mathcal{V}$  is finite dimensional and  $\mathcal{A}$  is 2-transitive, then  $I \in \mathcal{A}$ .

*Proof.* (a) follows directly from the definition and from the observation that  $(I - E)\mathcal{A}(I - E) = A - EA - AE + EAE$ .

(b) Assume first that  $\mathcal{W}$  is a 1-dimensional subspace. By 2-transitivity there exists a singular  $A \in \mathcal{A}$  such that  $A\mathcal{W} = \mathcal{W}$ . Choose  $A$  to be of minimal rank having this property and write it in the form  $A = J \oplus N$ , where  $J$  is invertible and  $N$  is nilpotent. As all the powers of  $A$  are in  $\mathcal{A}$  and its rank is minimal, we have necessarily that  $N = 0$ . The minimal polynomial  $p(t) = \sum_{0 \leq i \leq m} a_i t^i$  of  $J$  has nonzero constant term  $a_0$  because  $J$  is invertible. Thus,  $I = -(\sum_{1 \leq i \leq m} a_i J^i)/a_0$  is the identity operator on the range of  $A$  and the idempotent  $E = -(\sum_{1 \leq i \leq m} a_i A^i)/a_0 = I \oplus 0$  is in  $\mathcal{A}$ . Moreover,  $\text{rank } E = \text{rank } A$  and  $E\mathcal{W} = \mathcal{W}$ . If the rank of  $E$  is strictly greater than 1, then let  $E\mathcal{Z}$  be a 1-dimensional subspace in the range of  $E$  distinct from  $E\mathcal{W} = \mathcal{W}$ . By 2-transitivity there is a  $B \in \mathcal{A}$  such that  $BE\mathcal{W} = \mathcal{W}$  and  $BE\mathcal{Z} = 0$ . But then  $EBE\mathcal{W} = \mathcal{W}$  and  $EBE\mathcal{Z} = 0$  so that the rank of  $EBE$ , which is in  $\mathcal{A}$  by part (a), is strictly smaller than the rank of  $A$  contradicting the minimality assumption.

The rest follows by induction on the dimension of  $\mathcal{W}$ . Let  $\mathcal{Z}$  be a subspace of codimension 1 in  $\mathcal{W}$  and  $E \in \mathcal{A}$  a projection such

that  $E\mathcal{V} = \mathcal{X}$ . Note that  $\mathcal{Y} = \mathcal{W} \cap \ker E$  has dimension 1 and that  $\mathcal{W} = \mathcal{X} \oplus \mathcal{Y}$ . Let  $F \in \mathcal{A}$  be a projection such that  $F\mathcal{V} = \mathcal{Y}$ ; then  $\mathcal{V} = \mathcal{W} \oplus \mathcal{U}$ , where  $\mathcal{U} = \ker E \cap \ker F$ . Let  $P$  be a projection in  $\mathcal{V}$  on  $\mathcal{W}$  along  $\mathcal{U}$ ; then  $N = E + F - P$  has square equal to zero and therefore,  $P = 2(E + F) - (E + F)^2$  is in  $\mathcal{A}$ . To get (c) take  $\mathcal{W} = \mathcal{V}$  in (b). □

Some of the proofs of the following results could be shortened slightly, at the expense of keeping the paper self-contained, by using the Pierce decomposition associated with an idempotent.

1.2. THEOREM. *Let  $\mathcal{A}$  be a Jordan algebra of linear operators on a finite dimensional vector space  $\mathcal{V}$ . Then  $\mathcal{A}$  is 2-transitive if and only if  $\mathcal{A} = \mathcal{L}(\mathcal{V})$ .*

*Proof.*  $\mathcal{L}(\mathcal{V})$  is clearly 2-transitive. The converse is proved by induction on the dimension of  $\mathcal{V}$ . The assertion obviously holds if  $\dim \mathcal{V} = 2$ . So, let  $\dim \mathcal{V} > 2$ . Let  $\mathcal{X}$  be a 1-dimensional subspace of  $\mathcal{V}$  and find by 1.1(b) a projection  $E \in \mathcal{A}$  such that  $E\mathcal{V} = \mathcal{X}$ . Next, find a 1-dimensional subspace  $\mathcal{Y} \subset \ker E$  and corresponding projection  $F \in \mathcal{A}$  such that  $F\mathcal{V} = \mathcal{Y}$ . It is clear that  $EF = 0$  and with no loss of generality we may assume that  $FE = 0$  as well, since otherwise, we could replace  $F$  by  $F - FE = (I - E)F(I - E) \in \mathcal{A}$ . The Jordan subalgebra  $\mathcal{B} = (I - E)\mathcal{A}(I - E)$ , respectively  $\mathcal{C} = (I - F)\mathcal{A}(I - F)$ , of  $\mathcal{A}$  can be viewed as a 2-transitive algebra of operators on  $\ker E$ , respectively  $\ker F$ , and by induction hypothesis  $\mathcal{B} = \mathcal{L}(\ker E)$ , respectively  $\mathcal{C} = \mathcal{L}(\ker F)$ . The subalgebra  $\mathcal{C}$  is also called the Pierce zero-space relative to  $F$ . Choose now any  $T \in \mathcal{L}(\mathcal{V})$  and let us show that  $T \in \mathcal{A}$ . By 2-transitivity we may assume with no loss of generality that  $ETF = FTE = 0$ . But, then,  $T = R + S$ , where  $R = (I - E)T(I - E) \in \mathcal{B}$  and  $S = ET + TE - ETE \in \mathcal{C}$ . □

Theorem 1.2 can be generalized as follows for Jordan algebras of finite rank operators. For a further strengthening of this result see Theorem 3.4.

1.3. THEOREM. *Let  $\mathcal{A}$  be a Jordan algebra of finite rank operators on a vector space  $\mathcal{V}$ . If  $\mathcal{A}$  is 2-transitive, then it is dense, i.e.  $n$ -transitive for all  $n \geq 1$ .*

*Proof.* This can be done by reduction to the finite dimensional case. Observe that the proof of 1.1.(b) remains valid if we replace

the condition “ $\mathcal{V}$  is finite dimensional” by weaker conditions “ $\mathcal{V}$  is finite dimensional and the elements of  $\mathcal{A}$  have finite rank”. Therefore, we can find for every finite dimensional subspace  $\mathcal{W}$  of  $\mathcal{V}$  a projection  $E$  in  $\mathcal{A}$  such that  $E\mathcal{V} = \mathcal{W}$ . By 1.1.(a)  $E\mathcal{A}E$  is a Jordan subalgebra of  $\mathcal{A}$  and it is 2-transitive as a Jordan algebra of operators on  $E\mathcal{V} = \mathcal{W}$ . Thus,  $E\mathcal{A}E = \mathcal{L}(\mathcal{W})$  by 1.2. If vectors  $\{x_1, x_2, \dots, x_k\} \subset \mathcal{V}$  are linearly independent and vectors  $\{y_1, y_2, \dots, y_k\} \subset \mathcal{V}$  are arbitrary, then apply this consideration to the span  $\mathcal{W}$  of these two sets of vectors.  $\square$

## 2. Some characterizations of proper transitive Jordan algebras.

2.0. In this section we shall assume that the characteristic of the field  $\mathbb{F}$  is different from 2. Let  $\mathcal{S}_n(\mathbb{F})$  be the (transitive) Jordan algebra of all symmetric  $n \times n$  matrices over  $\mathbb{F}$ . We give a proof that if  $\mathbb{F}$  is algebraically closed, then  $\mathcal{S}_n(\mathbb{F})$  is, up to similarity, the only proper transitive Jordan subalgebra of  $\mathcal{M}_n(\mathbb{F})$ . This, of course, does not hold if  $\mathbb{F}$  is not algebraically closed. However, for a formally real closed field the algebra  $\mathcal{S}_n(\mathbb{F})$  has no proper transitive Jordan subalgebras. These results do not seem to be easily derivable from Jacobson’s general structure theorems for Jordan matrix algebras [4]; our presentation here is self-contained and elementary.

2.1. THEOREM. *Let  $\mathbb{F}$  be any formally real closed field. Then, the only transitive Jordan algebra of symmetric  $n \times n$  matrices over  $\mathbb{F}$  is  $\mathcal{S}_n(\mathbb{F})$ .*

*Proof.* We shall use induction on  $n$ . The assertion is trivial for  $n = 1$ . So, assume  $\mathcal{A}$  is a transitive Jordan subalgebra of  $\mathcal{S}_n(\mathbb{F})$  with  $n \geq 2$ . Let  $E$  be an idempotent of minimal positive rank in  $\mathcal{A}$ . Idempotents abound in  $\mathcal{A}$  because in the spectral decomposition

$$A = \sum \lambda_i E_i, \quad E_i^2 = E_i, \quad E_i E_j = 0, \quad i \neq j,$$

for a member  $A$  of  $\mathcal{A}$ , every  $E_i$  corresponding to a nonzero  $\lambda_i$  is a polynomial in  $A$  (with constant term zero) and thus belongs to  $\mathcal{A}$ . The existence of spectral decompositions in  $\mathcal{A}$  follows from the fact that  $\mathbb{F}$  is real-closed [5].

The transitivity of  $\mathcal{A}$  implies that it has nonscalar members, so that  $E \neq I$ . Since  $E$  is symmetric there exists an invertible matrix  $T$  with  $T^{-1} = T^t$  such that

$$T^{-1}ET = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where  $k$  is the rank of  $E$ ,  $0 < k < n$ . Replacing  $\mathcal{A}$  by  $T^{-1}\mathcal{A}T$  we can assume that

$$E = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing the corresponding matrix for a typical member of  $\mathcal{A}$

$$A = \begin{pmatrix} X & Y' \\ Y & Z \end{pmatrix},$$

we observe that

$$B = EAE - (I - E)A(I - E) = \begin{pmatrix} X & 0 \\ 0 & -Z \end{pmatrix} \in \mathcal{A},$$

and thus

$$\frac{1}{2}(BE + EB) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} - B = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$$

are in  $\mathcal{A}$ . By Proposition 1.1(a) we conclude that  $E\mathcal{A}E$  and  $(I - E)\mathcal{A}(I - E)$  are Jordan subalgebras of  $\mathcal{A}$ ; they are also easily seen to be transitive on respective spaces  $\text{im } E$  and  $\text{ker } E$ . Hence, by the inductive hypothesis,  $E\mathcal{A}E = \mathcal{S}_k(\mathbb{F})$  and  $(I - E)\mathcal{A}(I - E) = \mathcal{S}_{n-k}(\mathbb{F})$ . This means that  $\mathcal{A}$  contains all symmetric matrices of the form  $\begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$ . In particular,  $k = 1$  by minimality. To complete the proof observe that the transitivity of  $\mathcal{A}$  forces it to contain a matrix with an arbitrarily assigned first column. Thus, for a given  $(n - 1) \times 1$  matrix  $N$  there is a member  $\begin{pmatrix} L & N' \\ N & M \end{pmatrix}$  in  $\mathcal{A}$  with some  $L$  and  $M$ . Since by the argument above,  $L = L^t$  and  $M = M^t$  are arbitrary in this expression, we have that  $\mathcal{A} = \mathcal{S}_n(\mathbb{F})$ .  $\square$

The following example shows that the hypothesis of real closure in the theorem is needed. Let  $\mathbb{F}$  be the field  $\mathbb{Q}$  of rational numbers and

$$\mathcal{A} = \left\{ \begin{pmatrix} a + b & b \\ b & a - b \end{pmatrix} : a, b \in \mathbb{Q} \right\}.$$

Then,  $\mathcal{A}$  is a proper Jordan subalgebra of  $\mathcal{S}_2(\mathbb{Q})$ . It is easily seen that  $\mathcal{A}$  is transitive: it is generated by  $I$  and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The minimal polynomial of  $A$  is irreducible over  $\mathbb{Q}$ , and thus if  $x$  is any nonzero vector, then the span of  $x$  and  $Ax$  is the whole underlying space.

Our next theorem is a more general result in the case of algebraically closed fields; it includes, as a corollary, the analog of the above theorem. We need the following lemmas.

2.2. LEMMA. *Let  $\mathcal{A}$  be a transitive Jordan algebra of  $n \times n$  matrices over an algebraically closed field  $\mathbb{F}$ . Then  $\mathcal{A}$  contains an idempotent of rank 1.*

*Proof.* For  $n = 1$  this is trivially true. We first prove it for  $n = 2$ . Assume there is no idempotent of rank 1 in this case. This implies, by considering spectral projections of matrices in  $\mathcal{A}$ , that every member of  $\mathcal{A}$  has singleton spectrum, i.e., it is of the form  $N + \alpha I$  with  $N$  nilpotent. Since the characteristic of  $\mathbb{F}$  is different from 2, a member of  $\mathcal{A}$  is nilpotent if and only if it has trace zero.

If  $\mathcal{A}$  consists of nilpotents alone, then  $0 = (A + B)^2 = A^2 + B^2 + AB + BA = AB + BA$  for all  $A, B \in \mathcal{A}$  implying that if  $B \neq 0$ , then its range is invariant under every  $A \in \mathcal{A}$ . Thus,  $\mathcal{A}$  is triangularizable; this contradicts the transitivity of  $\mathcal{A}$ . Thus, we can assume that  $\mathcal{A}$  has an invertible member, which implies, by taking an appropriate polynomial, that  $I \in \mathcal{A}$ . Hence,  $N$  is in  $\mathcal{A}$  for every  $N + \alpha I$  in  $\mathcal{A}$ . Let  $\mathcal{A}_0$  be the set of all nilpotent elements in the algebra. It follows that for  $A$  and  $B$  in  $\mathcal{A}_0$  and  $\alpha \in \mathbb{F}$ , the matrix  $A + \alpha B$  has trace zero and is thus nilpotent. In particular  $A + B$  is nilpotent and hence  $AB + BA = (A + B)^2 - A^2 - B^2 = 0$ . This shows that  $\mathcal{A}_0$  is a Jordan algebra. We see, as before, that  $\mathcal{A}_0$  is triangularizable and so is  $\mathcal{A} = \mathcal{A}_0 + \mathbb{F}I$ , contradicting the transitivity of  $\mathcal{A}$ .

We can now assume  $n > 2$ . If  $\mathcal{A}$  contains a nontrivial idempotent, i.e., an idempotent  $E$  with  $0 < \text{rank } E < n$ , then by (1.1)(a)  $E\mathcal{A}E$  is a Jordan subalgebra of  $\mathcal{A}$  which is forced, by the transitivity of  $\mathcal{A}$ , to be transitive as an algebra of operators acting on the range of  $E$ . We conclude, by induction on  $n$ , that  $E\mathcal{A}E$  and thus  $\mathcal{A}$  contain idempotents of rank 1. To complete the proof we must only show the existence of a nontrivial  $E$ .

Assume  $\mathcal{A}$  contains no nontrivial idempotent. Then, as in the first paragraph of the proof, every member of  $\mathcal{A}$  is seen to be of the form  $N + \alpha I$  with  $N$  nilpotent. There must be nonzero nilpotent matrices in  $\mathcal{A}$ . (Just observe that if  $N + \alpha I \in \mathcal{A}$  with  $\alpha \neq 0$ , then, considering the characteristic polynomial of this matrix, we show that  $I \in \mathcal{A}$  and hence  $N \in \mathcal{A}$ . Surely  $\mathcal{A}$  cannot consist of scalar matrices.) Let  $N$  be a nilpotent member of  $\mathcal{A}$  with minimal positive rank. Since  $N^2$  has smaller rank than  $N$ , we must have  $N^2 = 0$ . We next show that  $N$  has rank 1. Considering members of  $\mathcal{A}$  as operators on  $\mathcal{V}$  and noting that the kernel of  $N$  contains its range, let  $\mathcal{V}_1$  be the

range of  $N$ ,  $\mathcal{V}_2$  a complement of  $\mathcal{V}_1$  in the kernel of  $N$ , and  $\mathcal{V}_3$  a complement of  $\mathcal{V}_1 \oplus \mathcal{V}_2$  in  $\mathcal{V}$ . With respect to the decomposition  $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$  (where  $\mathcal{V}_2$  may of course be zero) and with an appropriate choice of basis,  $N$  will have the form

$$\begin{pmatrix} 0 & 0 & I_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $k$  is the rank of  $N$ . If the corresponding block matrix of a typical  $A$  in  $\mathcal{A}$  is  $(A_{ij})_{i,j=1}^3$ , then  $NAN \in \mathcal{A}$  and its matrix equals

$$\begin{pmatrix} 0 & 0 & A_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix  $NAN - \alpha N$  is a nilpotent member of  $\mathcal{A}$  and all its blocks except  $A_{31} - \alpha I_k$  are zero. The minimality of the rank  $k$  forces the block  $A_{31}$  of every  $A$  to be scalar. But this would contradict the transitivity of  $\mathcal{A}$  if  $k > 1$ . Thus,  $k = 1$ . (These facts about algebras consisting of scalar translations of nilpotent operators can also be deduced from more sophisticated results on Jordan algebras [6].)

Finally, we shall exhibit a single member of  $\mathcal{A}$  with an eigenvalue 1, showing that not every member of  $\mathcal{A}$  is the sum of a nilpotent and a scalar. To this end, pick  $x \in \mathcal{V}$  with  $Nx \neq 0$ . By transitivity, there is an  $A \in \mathcal{A}$  such that  $A(Nx) = x$ . Then,  $N^2 = 0$  implies  $(NA + AN)(Nx) = NANx = Nx$ . Since  $N$  has rank 1, the matrix  $NA + AN$  has rank at most 2. Since  $n > 2$ , this matrix is a singular member of  $\mathcal{A}$ . □

**2.3. LEMMA.** *If  $\mathcal{A}$  satisfies the hypotheses of Lemma 2.2, then  $\mathcal{A}$  contains idempotents  $E_i$ ,  $i = 1, 2, \dots, n$ , of rank one with  $E_i E_j = 0$  for  $i \neq j$ .*

*Proof.* We shall use induction on  $n$ . Let  $n \geq 2$  and assume the assertion true for  $n - 1$ . Let  $E_1 = E$  be an idempotent of rank one in  $\mathcal{A}$  as in Lemma 1.1. We can assume with no loss of generality that  $E = \text{diag}(1, 0, \dots, 0)$ . The Jordan algebra  $(I - E)\mathcal{A}(I - E)$  is contained in  $\mathcal{A}$  by Proposition 1.1(a). Since it also acts transitively on the range of  $I - E$ , which has dimension  $n - 1$ , we conclude from the inductive hypothesis that  $(I - E)\mathcal{A}(I - E)$  contains idempotents  $E_2, \dots, E_n$  with the desired property. The proof is completed by observing that  $E_1 E_j = E_j E_1 = 0$  for  $j \geq 2$ . □

2.4. THEOREM. *Let  $\mathbb{F}$  be an algebraically closed field, and let  $\mathcal{A}$  be a transitive Jordan algebra of  $n \times n$  matrices over  $\mathbb{F}$ . Then either  $\mathcal{A} = \mathcal{M}_n(\mathbb{F})$  or there exists an invertible matrix  $T$  such that  $T^{-1}\mathcal{A}T = \mathcal{S}_n(\mathbb{F})$ .*

*Proof.* We shall show first that  $\mathcal{A}$  contains  $\mathcal{S}_n(\mathbb{F})$  up to a similarity. By Lemma 2.3 we can assume that  $\mathcal{A}$  contains diagonal idempotents  $E_j$  of rank one:  $E_1 = \text{diag}(1, 0, \dots, 0), \dots, E_n = \text{diag}(0, \dots, 0, 1)$ . Consider the special case of  $n = 2$ . In this case the transitivity of  $\mathcal{A}$  implies that its dimension is either 3 or 4. If the dimension is 4, then  $\mathcal{A} = \mathcal{M}_2(\mathbb{F})$ ; if it is 3, then  $\mathcal{A}$  contains a nonzero matrix of the form  $\begin{pmatrix} 0 & t \\ s & 0 \end{pmatrix}$  (after adding a suitable linear combination of  $E_1$  and  $E_2$ ). Now, both  $s$  and  $t$  have to be nonzero by transitivity. Thus, we have shown that, when  $n = 2$ , the algebra  $\mathcal{A}$  must contain a matrix of the above form with  $s = 1$  and  $t \neq 0$ .

Returning now to the general case, let  $\{M_{ij}\}$  be the set of matrix units, i.e., the only nonzero entry of  $M_{ij}$  occurs at the  $(i, j)$  position and equals 1. Observe that for  $j > 1$  the Jordan subalgebra  $(E_1 + E_j)\mathcal{A}(E_1 + E_j)$  acts transitively on the 2-dimensional range of  $E_1 + E_j$ . As in the paragraph above, it must contain, together with  $E_1$  and  $E_j$ , at least one matrix of the form  $A_j = M_{1j} + t_j M_{j1}$ . Letting  $T = \text{diag}(1, \sqrt{t_2}, \dots, \sqrt{t_n})$ , we see that the Jordan algebra  $T\mathcal{A}T^{-1}$  contains the symmetric matrices

$$B_{j1} = \frac{1}{\sqrt{t_j}} T A_j T^{-1} = M_{j1} + M_{1j}$$

(and, of course,  $E_1, \dots, E_n$ ).

If 1,  $i$ , and  $j$  are distinct, then  $B_{ij} = B_{i1}B_{j1} + B_{j1}B_{i1} \in T\mathcal{A}T^{-1}$ . Observe that  $B_{ij} = M_{ij} + M_{ji}$ . We have shown that  $T\mathcal{A}T^{-1}$  contains a basis for symmetric matrices. Hence,  $T\mathcal{A}T^{-1} \supset \mathcal{S}_n(\mathbb{F})$ .

To complete the proof of the theorem it suffices to show that if  $\mathcal{A}$  contains  $\mathcal{S}_n(\mathbb{F})$  properly, then  $\mathcal{A} = \mathcal{M}_n(\mathbb{F})$ . Thus, assume  $\mathcal{A}$  contains a nonsymmetric matrix  $C = (c_{ij})$ . Some principal  $2 \times 2$  submatrix must be nonsymmetric and by passing from  $\mathcal{A}$  to  $P^{-1}\mathcal{A}P$ , where  $P$  is a permutation matrix, we can assume  $c_{12} \neq c_{21}$ . Observe that the matrix

$$M = (E_1 + E_2)C(E_1 + E_2) - c_{11}E_1 - c_{22}E_2 - c_{21}B_{21}$$

belongs to  $\mathcal{A}$  and is a nonzero scalar multiple of  $M_{12}$ . We shall show that  $M_{ij} \in \mathcal{A}$ ,  $i, j = 1, \dots, n$ . Every  $M_{ii}$  is in  $\mathcal{A}$  and we have just seen that  $M_{12}$  and hence  $M_{21} = B_{21} - M_{12}$  are in  $\mathcal{A}$ .

For  $j > 2$ ,  $M_{1j} = M_{12}B_{2j} + B_{2j}M_{12}$ , and  $M_{j1} = B_{j1} - M_{1j}$  which implies that  $M_{1j}$  and  $M_{j1}$  are in  $\mathcal{A}$ . Similarly, for  $j > i > 1$ ,  $M_{ji} = M_{1i}B_{j1} + B_{j1}M_{1i}$ , and  $M_{ij} = B_{ij} - M_{ji}$  so that  $M_{ji}$  and  $M_{ij}$  are in  $\mathcal{A}$ . □

**2.5. COROLLARY.** *Let  $\mathbb{F}$  be an algebraically closed field. If  $\mathcal{A}$  is a transitive Jordan algebra of symmetric  $n \times n$  matrices over  $\mathbb{F}$ , then  $\mathcal{A} = \mathcal{S}_n(\mathbb{F})$ .*

The example given after Theorem 2.1 can be modified to show that the algebraic closure hypothesis is essential in the preceding result. Consider  $\mathbb{Q}(i)$  instead of  $\mathbb{Q}$  and let

$$\mathcal{A} = \left\{ \begin{pmatrix} a+b & b \\ b & a-b \end{pmatrix} : a, b \in \mathbb{Q}(i) \right\}.$$

Then,  $\mathcal{A}$  is a proper Jordan subalgebra of  $\mathcal{S}_2(\mathbb{Q}(i))$ ; it is also transitive.

The following example shows that the assumption  $\text{char } \mathbb{F} \neq 2$  is essential in the results above: the 3-dimensional Jordan algebra spanned over  $\mathbb{F}_2$  by  $\{I, M_{12}, M_{21}\}$  is transitive and contains no idempotent of rank 1.

**3. Results on ideals.**

**3.0.** We continue to assume that the characteristic of the field  $\mathbb{F}$  is different from 2. In the associative algebra case some transitivity properties are inherited by ideals. This is of course trivial if  $\dim \mathcal{V}$  is finite, since then  $\mathcal{L}(\mathcal{V})$  is simple. In the Jordan case, restriction to ideals seems to be accompanied with some loss of transitivity. The following result is well known for general associative algebras [2].

**3.1. PROPOSITION.** *Let  $\mathcal{I} \neq 0$  be an ideal in an associative algebra  $\mathcal{A}$  of operators on a vector space  $\mathcal{V}$ . If  $\mathcal{A}$  is  $n$ -transitive, then so is  $\mathcal{I}$ .*

Here are our results on this question for Jordan algebras of linear operators and their Jordan ideals.

**3.2. THEOREM.** *Every Jordan ideal  $\mathcal{I} \neq 0$  of an  $(n + 1)$ -transitive Jordan algebra  $\mathcal{A}$  of operators on a vector space  $\mathcal{V}$  is  $n$ -transitive,  $n \geq 1$ .*

*Proof.* If  $\mathcal{V}$  is  $(n + 1)$ -dimensional then  $\mathcal{A} = \mathcal{L}(\mathcal{V})$  and, by [1, Theorem 1], we have  $\mathcal{I} = \mathcal{A}$ . So, assume with no loss of generality

that  $\mathcal{V}$  contains  $n + 2$  linearly independent vectors. Fix a linearly independent set of vectors  $\{x_1, \dots, x_n\} \subset \mathcal{V}$  and let  $\mathcal{X}$  and  $\mathcal{Y}$  be the span of  $\{x_1, \dots, x_n\}$  and of  $\{x_1, \dots, x_{n-1}\}$ , respectively. We will show first that

(a)  $\exists J \in \mathcal{F}$  such that  $J\mathcal{Y} = 0, Jx_n \notin \mathcal{X}$ .

Assume the contrary; then

(b)  $J \in \mathcal{F}$  and  $J\mathcal{Y} = 0$  implies  $Jx_n \in \mathcal{X}$ .

Then, for any  $A \in \mathcal{A}$  such that  $A\mathcal{Y} = 0$  it holds that  $K = JA + AJ$  is in  $\mathcal{F}$  and that  $K\mathcal{Y} = 0$ . Therefore by (b),  $Kx_n$  belongs to  $\mathcal{X}$ . Let  $\alpha$  be such that  $Kx_n = (J + \alpha)Ax_n$ . Since  $\mathcal{A}$  is  $n$ -transitive,  $A$  may be chosen so that it satisfies the required conditions and that  $Ax_n$  is an arbitrary vector in  $\mathcal{V}$ . This shows that

(c)  $J\mathcal{Y} = 0$  implies  $(J + \alpha)\mathcal{V} \subseteq \mathcal{X}$  for some  $\alpha$ .

Thus,

$$J = -\alpha + \sum_{1 \leq i \leq n} x_i \otimes f_i$$

for some linear functionals  $f_i, i = 1, \dots, n$ . But then, for an arbitrary  $A \in \mathcal{A}$  with  $A\mathcal{Y} = 0$  define  $K$  as above and use the expression for  $J$  to get

$$K = -2\alpha A + Ax_n \otimes f_n + \sum_{1 \leq i \leq n} x_i \otimes f_i A.$$

Thus  $K$  belongs to  $\mathcal{F}$  and  $K\mathcal{Y} = 0$ . Hence, by (c), it must be of the same form as  $J$ , i.e.,

$$K = -\beta + \sum_{1 \leq i \leq n} x_i \otimes g_i.$$

Now, we choose vectors  $u, v \in \mathcal{V}$  such that  $\{x_1, \dots, x_n, u, v\}$  are linearly independent, and find an  $A \in \mathcal{A}$  such that  $A\mathcal{Y} = 0$ , and that  $Ax_n = u, Au = v$ . Then

$$Ku = -2\alpha v + f_n(u)u + \sum_{1 \leq i \leq n} f_i(v)x_i = -\beta u + \sum_{1 \leq i \leq n} g_i(u)x_i,$$

which forces  $\alpha = 0$ . A similar argument with  $K$  playing the role of  $J$  shows that  $\beta = 0$ . Also, using the freedom in the choice of  $u$ , we conclude that  $f_n$  is trivial. Thus, from the fact that (b) holds for every  $J \in \mathcal{F}$  we obtain

(d)  $J \in \mathcal{F}$  and  $J\mathcal{Y} = 0$  implies  $J\mathcal{V} \subset \mathcal{Y}$ .

The conclusion (d) contradicts the assumption that  $\mathcal{F} \neq 0$  in case  $n = 1$ . In other words, we have shown that given  $x \neq 0$ , there is a  $J \in \mathcal{F}$  such that  $x$  and  $Jx$  are linearly independent. Observe that this proves the theorem for  $n = 1$ : if  $x \neq 0$  and  $y$  are given

and  $J \in \mathcal{F}$  is such that  $x$  and  $Jx$  are linearly independent, then by 2-transitivity choose an  $A \in \mathcal{A}$  with  $Ax = x$  and  $AJx = y - Jx$ . Then  $AJ + JA \in \mathcal{F}$  and  $(AJ + JA)x = y$ .

Assume now for  $n > 1$  inductively that  $\mathcal{F}$  is  $(n - 1)$ -transitive and find an  $E \in \mathcal{F}$  such that  $Ex_i = x_i, i = 1, \dots, n - 1$ . As  $E^2 \in \mathcal{F}$  and equals  $E$  on  $\mathcal{Y}$ , we have by (d) that  $(E - E^2)\mathcal{V} \subset \mathcal{Y}$ . Assume now that there exists a vector  $u \in \mathcal{V}$  such that  $u$  and  $Eu$  do not belong to  $\mathcal{Y}$ . It follows from  $(E - E^2)\mathcal{V} \subset \mathcal{Y}$  that  $E^2u$  equals the sum of  $Eu$  and a vector from  $\mathcal{Y}$  (so  $E^2u \notin \mathcal{Y}$ ). By  $n$ -transitivity of  $\mathcal{A}$  find an  $A \in \mathcal{A}$  such that  $Ax_i = x_i, i = 1, \dots, n - 1$ , and  $AEu = 0$ . This implies for  $K = EA + AE \in \mathcal{F}$  that  $Kx_i = 2x_i, i = 1, \dots, n - 1$  and  $KEu \in \mathcal{Y}$ . Hence,  $K - 2E$  annihilates  $\mathcal{Y}$  and its image is not contained in  $\mathcal{Y}$ , because  $(K - 2E)Eu$  equals the sum of  $-2E^2u$  and a vector from  $\mathcal{Y}$ , contradicting (d). The freedom in the choice of  $u$  shows that for every  $E \in \mathcal{F}$  such that  $Ex_i = x_i, i = 1, \dots, n - 1$ , we have necessarily that  $E\mathcal{V} \subset \mathcal{Y}$  and  $E$  is a projection on  $\mathcal{Y}$ . Choose now a nonzero vector  $u \in \ker E$  and find by  $(n + 1)$ -transitivity of  $\mathcal{A}$  an  $A \in \mathcal{A}$  such that  $Ax_1 = u, Ax_i = 0, i = 2, \dots, n - 1$ , and  $Au = x_1$ . Then,  $K = EA + AE - 2EAE \in \mathcal{F}$ , and  $Kx_1 = u, Kx_i = 0, i = 2, \dots, n - 1$ , and  $Ku = x_1$ . It follows for  $L = K^2 - EK^2E \in \mathcal{F}$  that  $L\mathcal{Y} = 0$  and  $Lu = u$  contradicting (d). Consequently, we have shown that (b) leads to a contradiction and we have (a). So, fix a  $J \in \mathcal{F}$  such that  $J\mathcal{Y} = 0$  and  $Jx_n \notin \mathcal{X}$ . Now, pick by  $(n + 1)$ -transitivity of  $\mathcal{A}$  an  $A \in \mathcal{A}$  such that  $A\mathcal{X} = 0$ , and  $AJx_n = u$  an arbitrary vector from  $\mathcal{V}$ . Thus, for  $K = AJ + JA \in \mathcal{F}$  we have that  $K\mathcal{Y} = 0$ , and  $Kx_n = JAx_n + AJx_n = u$ . The  $n$ -transitivity of  $\mathcal{F}$  in the theorem now follows easily by cyclicly permuting the vectors  $x_i, i = 1, \dots, n$  and taking sums of corresponding operators  $K$ .  $\square$

The following result can also be obtained from work of Osborn and Racine [7].

3.3. COROLLARY. *Every Jordan ideal of a dense Jordan algebra is dense.*

3.4. THEOREM. *Let  $\mathcal{A}$  be a 2-transitive Jordan algebra of operators on a vector space  $\mathcal{V}$ . If  $\mathcal{A}$  contains at least one operator of finite rank, then the Jordan ideal  $\mathcal{F}$  of all finite rank operators of  $\mathcal{A}$  is strictly dense, and so is  $\mathcal{A}$ .*

*Proof.* Assume with no loss of generality that  $\mathcal{V}$  is not finite dimensional. By 3.2  $\mathcal{F}$  is transitive. Thus, we may find an  $E \in \mathcal{F}$

such that  $Ex = x$  and such that it is of minimal rank with this property. Similarly as in the proof of 1.1.(b), we may find that  $E$  is a projection of rank one and that for every finite dimensional subspace  $\mathcal{W}$  of  $\mathcal{V}$  we may find a projection  $E \in \mathcal{J}$  such that  $E\mathcal{V} = \mathcal{W}$ . Now, for any  $\{x, y\}$  linearly independent and  $\{u, v\}$  arbitrary vectors of  $\mathcal{V}$ , let  $\mathcal{W}$  denote the linear span of these four vectors and let  $E \in \mathcal{J}$  be the corresponding projection. Use 2-transitivity of  $\mathcal{A}$  to find  $A \in \mathcal{A}$  such that  $Ax = u$  and  $Ay = v$ , use 1.1.(a) to see that  $B = EAE \in \mathcal{J}$ , and observe that again  $Bx = u$  and  $By = v$ . Thus,  $\mathcal{J}$  is 2-transitive and it is strictly dense by 1.3.  $\square$

The reader will no doubt have noticed that we left the following questions unanswered.

*Question 1.* Is there an  $n$ -transitive Jordan algebra  $\mathcal{A}$  with a Jordan ideal  $\mathcal{J} \neq 0$  which is not  $n$ -transitive?

*Question 2.* Is there an  $n$ -transitive Jordan algebra which is not  $(n + 1)$ -transitive for any  $n \geq 2$ ?

#### REFERENCES

- [1] I. N. Herstein, *On the Lie and Jordan rings of a simple associative ring*, Amer. J. Math., 77 (1955), 279–285.
- [2] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, RI, 1984.
- [3] ———, *Lectures in Abstract Algebra*, vol. II, Van Nostrand, (1953).
- [4] ———, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., vol. 39, Amer. Math. Soc., Providence, RI, 1968.
- [5] S. Lang, *Algebra*, Addison Wesley, (1971).
- [6] J. M. Osborn, *Jordan and associative rings with nilpotent and invertible elements*, J. Algebra, 15 (1970), 301–308.
- [7] J. M. Osborn and M. L. Racine, *Jordan rings with nonzero socle*, Trans. Amer. Math. Soc., 251 (1979), 375–387.

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