KNOTTING TRIVIAL KNOTS AND RESULTING KNOT TYPES

Kimihiko Motegi
Let \((V, K)\) be a pattern (i.e. \(V\) is a standardly embedded solid torus in oriented \(S^3\) and \(K\) is a knot in \(V\)) and \(f\) an orientation preserving embedding from \(V\) into \(S^3\) such that \(f(V)\) is knotted.

In this paper answers to the following questions will be given depending upon whether the winding number of \(K_2\) in \(V\) is zero or not.

1. **Introduction.** Let \(K\) be a knot in \(S^3\), which is contained in a standardly embedded solid torus \(V\) \((\subset S^3)\). Assume that \(K\) is not contained in a 3-ball in \(V\). Let \(f\) be an orientation preserving embedding from \(V\) into \(S^3\) such that \(f(C)\) is knotted in \(S^3\), here \(C\) denotes a core of \(V\). Then we get a new knot \(f(K)\) in \(S^3\) called a satellite knot with a companion knot \(f(C)\). The knot \(K\) is called a preimage knot and we call the pair \((V, K)\) a pattern (see Figure 1 on the next page).

Throughout this paper for an embedding \(f\) from \(V\) into \(S^3\), we assume that it is orientation preserving and \(f(C)\) is knotted in \(S^3\).

We concern ourselves with the following questions.

1. Suppose that \(K_1\) is unknotted and \(K_2\) is knotted in \(S^3\). Can \(f(K_1)\) be ambient isotopic to \(f(K_2)\) in \(S^3\) for some embedding \(f: V \hookrightarrow S^3\)?

2. Suppose that \(K_1\) and \(K_2\) are both unknotted in \(S^3\). How are \((V, K_1)\) and \((V, K_2)\) related if \(f(K_1)\) is ambient isotopic to \(f(K_2)\) in \(S^3\) for some embedding \(f: V \hookrightarrow S^3\)?

For two knots \(K_1\) and \(K_2\), we write \(K_1 \equiv K_2\) provided that there exists an orientation preserving self-homeomorphism of \(S^3\) carrying \(K_1\) to \(K_2\) (or equivalently, \(K_1\) and \(K_2\) are ambient isotopic in \(S^3\)). For two patterns \((V, K_1)\) and \((V, K_2)\), if there exists an orientation preserving self-homeomorphism \(h\) of \(V\) sending longitude to
±longitude which satisfies $h(K_1) = K_2$, then we write $(V, K_1) \sim (V, K_2)$. In addition if the homeomorphism $h$ sends longitude to longitude, then we write $(V, K_1) \cong (V, K_2)$. It is known that $(V, K_1) \cong (V, K_2)$ if and only if $K_1$ and $K_2$ are ambient isotopic in $V$. Throughout this paper longitude means preferred longitude.

The wrapping number of $K$ in $V$—the minimal geometric intersection number of $K$ with a meridian disk in $V$—is denoted by $\text{wrap}_V(K)$, and the winding number of $K$ in $V$—the algebraic intersection number of $K$ with a meridian disk in $V$—is denoted by $\text{wind}_V(K)$. (We may assume $\text{wind}_V(K) \geq 0$ by considering an appropriate orientation of $K$.)

Now our main result is stated as follows.

**Theorem 1.1.** Let $(V, K_i) (i = 1, 2)$ be a pattern. Suppose that $K_1$ is unknotted in $S^3$ and $\text{wind}_V(K_2) \neq 0$. If $f(K_1) \cong f(K_2)$ in $S^3$ for some embedding $f$ from $V$ into $S^3$, then $(V, K_1) \sim (V, K_2)$ holds.

**Remark 1.2.** (1) In this theorem the condition $\text{wind}_V(K_2) \neq 0$ is essential. The example below (Figure 2) demonstrates the necessity of such a condition.

In this example $K_1$ is unknotted in $S^3$ and $K_2$ is knotted (and hence $(V, K_1) \not\cong (V, K_2)$), but $\text{wind}_V(K_2) = 0$. From them, we can obtain the same knot $f(K_1) \cong f(K_2)$.

The modification to recognize that $f(K_1) \cong f(K_2)$ is given by Figure 3.
FIGURE 2

FIGURE 3

FIGURE 4
(2) If $K_1$ is knotted, even when $\text{wind}_V(K_2) \neq 0$, it is easy to construct the example such that $(V, K_1) \not\approx (V, K_2)$ but $f(K_1) \cong f(K_2)$ in $S^3$ (see Figure 4).

As consequences of the Theorem 1.1, we have Corollary 2.6 and Theorem 3.1. By these results together with Remark 1.2 and Theorem 3.3, we can answer the above questions depending upon whether $\text{wind}_V(K_2) = 0$ or not.

Throughout this paper $N(X)$, $\partial X$ and $\text{int} X$ denote the tubular neighborhood of $X$, the boundary of $X$ and the interior of $X$ respectively.

2. Isotopy between satellite knots and equivalence of patterns. To prove Theorem 1.1 we prepare some lemmas and give a necessary condition for a pattern $(V, K)$ so that $K$ is unknotted in $S^3$.

The next lemma is well known and we omit the proof here (see [7]).

**Lemma 2.1.** Let $W$ be a knotted solid torus in $S^3$ and $K$ a knot in $W$ with $\text{wrap}_W(K) \neq 0$. Then $K$ is knotted in $S^3$.

Consider a nontrivial knot exterior $E$ (i.e. $E$ is homeomorphic to $S^3 - \text{int} N(k)$ for some nontrivial knot $k$ in $S^3$) embedded in $V$. Since $\partial E$ is compressible in $V - \text{int} N(K)$, otherwise $V = (V - \text{int} E) \cup E$ has an incompressible torus $\partial E$, there exists an embedded disk $D$ in $V - \text{int} E$ such that $\partial D$ is essential in $\partial(V - \text{int} E) = \partial E$. Thus $D$ is a meridian disk for the solid torus $W = S^3 - \text{int} E$ and is contained in $V$. We call the disk $D$ a meridian disk for $E$ in $V$. The following lemma is a straightforward consequence of Lemma 2.1.

**Lemma 2.2.** Let $(V, K)$ be a pattern such that $K$ is unknotted in $S^3$ and $E$ a nontrivial knot exterior embedded in $V - K$. Then the algebraic intersection number of $K$ and a meridian disk for $E$ in $V$ is zero (see Figure 5).

Now we shall prove Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that $f(K_1) \cong f(K_2)$ in $S^3$ for two patterns $(V, K_1)$ and $(V, K_2)$. Then there is an orientation preserving homeomorphism $h$ of $S^3$ carrying $f(K_1)$ to $f(K_2)$. It suffices to show that by an isotopy of $S^3$ which leaves $f(K_2) = h(f(K_1))$ fixed, we can modify the homeomorphism $h$ so that $h(S^3 - \text{int} f(V)) = S^3 - \text{int} f(V)$. To do this we need the next lemma, which was proved by H. Schubert in [8], but since we rely heavily on this theorem, we give a proof here using torus decompositions.
LEMMA 2.3 ([8]). Let $V_t (i = 1, 2)$ be a knotted solid torus in $S^3$ which contains a knot $K$ with $\text{wrup}_V(K) \neq 0$. Then by an ambient isotopy of $S^3$ which leaves $K$ fixed, $V_2$ can be deformed so that one of the following holds.

1. $\partial V_1 \cap \partial V_2 = \emptyset$
2. there exist meridian disks $D_1$ and $D_2$ of both $V_1$ and $V_2$ such that the closure of one component of $V_1 - \bigcup_{i=1}^2 D_i$ is a knotted 3-ball in the closure of some component of $V_2 - \bigcup_{i=1}^2 D_i$ (see Figure 6).

Proof of Lemma 2.3. If $V_i - \text{int} N(K)$ is homeomorphic to $S^1 \times S^1 \times I$ for $i = 1$ or 2, then $K$ is a core of $V_i$ and we can deform $V_2$ so that (1) in Lemma 2.3 holds. Now we assume that $V_i - \text{int} N(K)$ is not homeomorphic to $S^1 \times S^1 \times I$ for $i = 1, 2$. Consider the torus decomposition of $S^3 - \text{int} V_i$ and $V_i - \text{int} N(K) (i = 1, 2)$ in the sense of Jaco-Shalen [4] and Johannson [5]. Then each piece is Seifert fibred or admits a complete hyperbolic structure of finite volume in its interior by Thurston's uniformization theorem [6]. Moreover, by
Theorem VI 3.4 in [4], the Seifert part is one of torus knot spaces, cable spaces and composing spaces. Let $M_i$ be the piece in $S^3-\text{int } V_i$ which contains $\partial V_i (= \partial (S^3- \text{int } V_i))$ and $N_i$ the piece in $V_i-\text{int } N(K)$ which contains $\partial V_i$. We divide into two cases depending upon whether one of $\partial V_i (i=1, 2)$ belongs to the minimal family of tori $J_i$ defining a torus decomposition of $S^3-\text{int } N(K) = (S^3-\text{int } V_i) \cup (V_i-\text{int } N(K))$ or not.

**Case (A).** At least one of $\partial V_i (i=1, 2)$ belongs to $J_i (i=1, 2)$.

In this case, by the uniqueness of the torus decomposition we can deform $\partial V_2$ so that $\partial V_1 \cap \partial V_2 = \emptyset$ or $\partial V_1 = \partial V_2$. If $\partial V_1 = \partial V_2$, then isotoping $\partial V_2$ slightly off $\partial V_1$ in the normal direction so that $\partial V_1 \cap \partial V_2 = \emptyset$. Thus (1) in Lemma 2.3 does hold.

**Case (B).** $\partial V_i (i=1, 2)$ does not belong to $J_i$.

Then it turns out that $M_i \cup N_i$ is a composing space in $S^3-\text{int } N(K) = (S^3-\text{int } V_i) \cup (V_i-\text{int } N(K))$. By the uniqueness of the torus decomposition, we can isotope $M_2 \cup N_2$ so that $M_2 \cup N_2$ is one of decomposing pieces defined by $J_1$. (Notationally we do not distinguish the original $M_2 \cup N_2$ and isotoped $M_2 \cup N_2$.) If $M_2 \cup N_2 \neq M_1 \cup N_1$, then (1) in Lemma 2.3 holds. Now suppose that $M_2 \cup N_2 = M_1 \cup N_1$. We note that $\partial V_1$ is a saturated torus (i.e. a union of fibres) in the composing space $M_1 \cup N_1$. Since a Seifert fibration of the composing space $M_1 \cup N_1$ is unique up to isotopy, we may modify $\partial V_2$ so that it is also saturated. Consider the orbit manifold of $M_1 \cup N_1$, which is a disk with holes. The image of $\partial V_i$ in this orbit manifold is an essential circle $C_i$. If we can modify $C_2$ so that $C_1 \cap C_2 = \emptyset$, then we can also modify $\partial V_2$ so that $\partial V_1 \cap \partial V_2 = \emptyset$ and (1) in Lemma 2.3 holds. Assume we can not isotope $C_2$ so that $C_1 \cap C_2 = \emptyset$. Then isotope $C_2$ so that the number of points of $C_1 \cap C_2$ is minimal. Let $T$ be the component of $\partial (M_1 \cup N_1)$ separating $\partial N(K)$ and $M_1 \cup N_1$. In $S^3$, $T$ bounds a solid torus $W$ containing $K$, whose meridian coincides with the regular fibre of $M_1 \cup N_1$. In the orbit manifold we can find arcs $A_1$ and $A_2$ joining a point in $C_1 \cap C_2$ and the boundary circle $C$ which is the image of $T$ (see Figure 7 (1)).

These arcs $A_1$ and $A_2$ are corresponding to saturated annuli $\tilde{A}_1$ and $\tilde{A}_2$. From $\tilde{A}_j$ and meridian disk $\tilde{D}_j$ of $W$, we can construct a meridian disk $A_j \cup \tilde{D}_j$ of both $V_1$ and $V_2$. Finally consider the boundary circle $C'$ of the orbit manifold depicted in Figure 7 (1), which corresponds to the torus boundary $T'$. Since $T'$ bounds a
nontrivial knot exterior in $V_2 - \text{int } V_1$ (see Figure 7 (2)), we get just a situation for (2) in Lemma 2.3, and this completes the proof.  

Let us study the relationship between two solid tori $h(f(V))$ and $f(V)$ using Lemma 2.3. By an ambient isotopy of $S^3$ which leaves $f(K_2) = h(f(K_1))$ fixed we can deform $h(f(V))$ into the position such that either $\partial h(f(V)) \cap \partial f(V) = \emptyset$ or there exist meridian disks $D_1$ and $D_2$ of both $f(V)$ and $h(f(V))$ such that one component of the closure of $f(V) - \bigcup_{i=1}^2 D_i$ is a knotted 3-ball in some component of the closure of $h(f(V)) - \bigcup_{i=1}^2 D_i$.

**Lemma 2.4.** We can deform $h(f(V))$ into the position such that $\partial h(f(V)) \cap \partial f(V) = \emptyset$.

**Proof of Lemma 2.4.** If not, the second situation in the above occurs. Then we get a solid torus $W'$ (⊂ $h(f(K_1)) = f(K_2)$) in int $h(f(V))$, whose core $C_{W'}$ satisfies $\text{wrap}_{h(f(V))}(C_{W'}) = 1$ and is not a core of $h(f(V))$. It follows that the solid torus $V$ also contains a knotted solid torus $W$ (⊂ $K_1$) in its interior such that $\text{wrap}_{V}(C_{W}) = 1$ for the core $C_{W}$ of $W$. Since $\text{wrap}_{W}(K_1) \neq 0$, $K_1$ cannot be unknotted in $S^3$ by Lemma 2.1 and this contradicts the assumption. Hence we can deform $h(f(V))$ so that $\partial h(f(V)) \cap \partial f(V) = \emptyset$.  

Now we have following three possibilities.

1. $h(S^3 - \text{int } f(V)) \subset \text{int } (S^3 - \text{int } f(V))$
2. $\text{int } h(S^3 - \text{int } f(V)) \supset S^3 - \text{int } f(V)$
3. $h(S^3 - \text{int } f(V)) \subset \text{int } f(V)$

In (1) (or (2), resp.), assume $(S^3 - i \text{int } f(V)) - \text{int}(h(S^3 - \text{int } f(V)))$ (or $h(S^3 - \text{int } f(V)) - \text{int}(S^3 - \text{int } f(V))$, resp.) is homeomorphic
to $S^1 \times S^1 \times I$. Then by an isotopy which leaves $f(K_2)$ fixed, we may modify $h$ so that $h(S^3 - \text{int } f(V)) = S^3 - \text{int } f(V)$ and also $h(f(V)) = f(V)$ with $h(f(K_1)) = f(K_2)$. For homological reasons, $h(\text{longitude}) = \pm \text{longitude}$. Hence $f^{-1} \circ h \circ f : V \to V$ is an orientation preserving homeomorphism carrying $K_1$ to $K_2$ and $\text{longitude}$ to $\pm \text{longitude}$. So we get $(V, K_1) \sim (V, K_2)$.

Let us consider the case where $h(S^3 - \text{int } f(V)) \subset \text{int}(S^3 - \text{int } f(V))$ and $(S^3 - \text{int } f(V)) - \text{int}(h(S^3 - \text{int } f(V)))$ is not homeomorphic to $S^1 \times S^1 \times I$. Then using the homeomorphism $h|_{S^3 - \text{int } f(V)}$ from $S^3 - \text{int } f(V)$ to $h(S^3 - \text{int } f(V))$, we get mutually nonparallel incompressible tori $\{h^n(\partial f(V))\}$ in $S^3 - \text{int } f(V)$ for any positive integer $n$. This contradicts Haken’s finiteness theorem. The similar argument can be applied in the case where $\text{int } h(S^3 - \text{int } f(V)) \supset \text{int } f(V)$ and $h(S^3 - \text{int } f(V)) - \text{int}(S^3 - \text{int } f(V))$ is not homeomorphic to $S^1 \times S^1 \times I$, and again we get a contradiction.

Let us consider the case (3). The assumption implies that the nontrivial knot exterior $E = h^{-1}(S^3 - \text{int } f(V))$ is contained in $\text{int } f(V)$. Since $\text{wind}_V(K_2) \neq 0$, we have $\text{wind}_{h^{-1}(f(V))}(h^{-1}(f(K_2))) \neq 0$ and so we have $\text{wind}_{h^{-1}(f(V))}(h^{-1}(f(K_2))) \neq 0$. It follows that $\text{wind}_{(S^3 - \text{int } E)}(f(K_1))$ is also not zero. On the other hand since $K_1$ is unknotted in $S^3$, by Lemma 2.2, the algebraic intersection number of $K_1$ and a meridian disk for the nontrivial knot exterior $f^{-1}(E)$ in $V$ must be zero. Hence we get $\text{wind}_{(S^3 - \text{int } E)}(f(K_1)) = 0$. This is a contradiction. □

In Theorem 1.1, if $f(C)$ is a noninvertible knot, where $C$ is a core of $V$, then more precisely we have the following.

**Theorem 2.5.** Let $(V, K_i)$ be a pattern. Suppose that $K_1$ is unknotted in $S^3$, and $\text{wind}_V(K_2) \neq 0$. If $f(K_1) \cong f(K_2)$ in $S^3$ for some embedding $f$ from $V$ into $S^3$ such that $f(C)$ is noninvertible, then $(V, K_1) \cong (V, K_2)$, that is $K_1$ and $K_2$ are ambient isotopic in $V$.

**Proof.** In the proof of Theorem 1.1, we have an orientation preserving homeomorphism $h$ of $S^3$ satisfying $h(f(V)) = f(V)$ and $h(f(K_1)) = f(K_2)$. For homological reasons, $h(\text{longitude}) = \pm \text{longitude}$. In addition since $f(C)$ is noninvertible, we get $h(\text{longitude}) = \text{longitude}$ (see 3.19. Proposition in [1]). It follows that $f^{-1} \circ h \circ f : V \to V$ is an orientation preserving homeomorphism carrying $K_1$ to $K_2$ and $\text{longitude}$ to $\text{longitude}$. Thus we conclude $(V, K_1) \cong (V, K_2)$.

□
As an application of Theorem 1.1, we have the following corollary.

**Corollary 2.6.** Let \((V, K_i)\) be a pattern. Suppose that \(K_1\) is unknotted and \(K_2\) is knotted in \(S^3\) and \(\text{wind}_V(K_2) \neq 0\). Then for any embedding \(f\) from \(V\) into \(S^3\), \(f(K_1) \not\cong f(K_2)\) in \(S^3\).

**Proof.** If \(f(K_1) \cong f(K_2)\) for some embedding \(f\) from \(V\) into \(S^3\), then \((V, K_1) \sim (V, K_2)\) must hold by Theorem 1.1. Extending the orientation preserving homeomorphism \(h\) of \(V\) to that of \(S^3\), we get \(K_1 \cong K_2\). This is a contradiction. \(\square\)

Concluding this section, we give the following proposition which is an implicit corollary of Soma's sum formula for the Gromov invariants [10]. We denote the Gromov invariant of \(X\) by \(\|X\|\).

**Proposition 2.7.** Let \((V, K_i)\) \((i = 1, 2)\) be a pattern such that \(\|V - \text{int} N(K_1)\| \neq \|V - \text{int} N(K_2)\|\). Then \(f(K_1) \not\cong f(K_2)\) for any embedding \(f\) from \(V\) into \(S^3\).

So we see that, with no conditions on \(K_1\) and \(K_2\), if \(f(K_1) \cong f(K_2)\) then the Gromov invariants of their complements in \(V\) are the same.

3. **Classification of satellite knots constructed from trivial knots.** As a special case of Theorem 1.1, we have

**Theorem 3.1.** Let \((V, K_i)\) be a pattern and \(K_i\) a trivial knot in \(S^3\) \((i = 1, 2)\). Suppose that \(\text{wind}_V(K_1) \neq 0\) or \(\text{wind}_V(K_2) \neq 0\). If \(f(K_1) \cong f(K_2)\) in \(S^3\) for some embedding \(f\) from \(V\) into \(S^3\), then \((V, K_1) \sim (V, K_2)\) holds.

The winding numbers and the wrapping numbers of knots in a solid torus are elementary invariants for them. Particularly for a faithful (i.e. sending longitude to longitude) embedding \(f : V \hookrightarrow S^3\), winding number of \(K\) in \(V\) has an important role for Alexander polynomial of \(f(K)\), as is shown by Seifert's formula [9] ([1]). However if \(K\) is unknotted and \(f(C)\) has a trivial Alexander polynomial, then \(f(K)\) has also a trivial one independent of \(\text{wind}_V(K)\). Moreover when \(K_1 \cong K_2\) and \(\text{wind}_V(K_1) = \text{wind}_V(K_2)\), \(f(K_1)\) and \(f(K_2)\) have the same Alexander polynomial.

As a consequence of Theorem 3.1, we have the following result for satellite knots constructed from trivial knots.
COROLLARY 3.2. Suppose $K_i$ is a trivial knot contained in a standardly embedded solid torus $V$ in $S^3$ $(i = 1, 2)$.

1. If $\text{wind}_V(K_1) \neq \text{wind}_V(K_2)$, then $f(K_1) \neq f(K_2)$ in $S^3$ for any embedding $f$ from $V$ into $S^3$.

2. When $\text{wind}_V(K_1) = \text{wind}_V(K_2) = 0$, if $\text{wrap}_V(K_1) \neq \text{wrap}_V(K_2)$, then $f(K_1) \neq f(K_2)$ in $S^3$ for any embedding $f$ from $V$ into $S^3$.

In the case $\text{wind}_V(K_1) = \text{wind}_V(K_2) = 0$, we have

THEOREM 3.3. For any faithful embedding $f$ from $V$ into $S^3$ (i.e. $f$ sends a longitude of $V$ to a longitude of $f(V)$), there exist patterns $(V, K_1)$ and $(V, K_2)$ such that both $K_1$ and $K_2$ are unknotted in $S^3$, which satisfy the following properties:

1. $\text{wind}_V(K_1) = \text{wind}_V(K_2) = 0$ and $(V, K_1) \not\sim (V, K_2)$.

2. $f(K_1) \cong f(K_2)$ in $S^3$.

Proof. Let us consider a 3-components Brunnian link $L = k \cup L_1 \cup L_2$ depicted in Figure 8.

![Figure 8](image.png)

We denote the meridian-longitude pair of $L_i$ by $(m_i, l_i)$ $(i = 1, 2)$. Let $t$ be a knot ambient isotopic to $f(C)$, where $C$ is a core of $V$, and $(m, l)$ a meridian-longitude pair of $t$. To obtain the required pattern, remove a tubular neighborhood $N(L_i)$ and glue the knot exterior $E(t) = S^3 - \text{int} N(t)$ so that $m_i = l$ and $l_i = m$. Then, for $i = 1, 2$, the result $(S^3 - \text{int} N(L_i)) \cup_{m_i = l} (S^3 - \text{int} N(t))$ is again $S^3$, and we have new knots $K_{3-i}$ and $\tilde{L}_{3-i}$ as the images of $k$ and $L_{3-i}$ respectively. It is easy to see that both $K_{3-i}$ and $\tilde{L}_{3-i}$ are unknotted in $S^3$. Thus the exterior $V$ of $\tilde{L}_{3-i}$ containing $K_{3-i}$ forms a pattern $(V, K_{3-i})$. In this way we get two patterns $(V, K_1)$ and $(V, K_2)$. By the construction, for the faithful
embedding $f : V \hookrightarrow S^3$, $f(K_1) \cong f(K_2)$ does hold in $S^3$. In fact, $f(K_1) \cong f(K_2)$ can be described as the knot obtained from $k$ in Figure 8 by simultaneously replacing a neighborhood of a meridian disk of each of $L_1$ and $L_2$ by a tube knotted according to the given knot $t$.

From now on we prove $(V, K_1) \not\cong (V, K_2)$ by showing $\text{wrap}_V(K_1) \neq \text{wrap}_V(K_2)$. Clearly $\text{wrap}_V(K_1) \leq 2$ and $\text{wrap}_V(K_2) \leq 4$. Since $\text{wind}_V(K_2) = 0$, $\text{wrap}_V(K_2)$ must be even. Now we assume $\text{wrap}_V(K_2) = 2$. Then there exists a disk $D_2$ in

$$V = (S^3 - \text{int} N(L_1)) \cup_{m \in \mathbb{Z}} (S^3 - \text{int} N(t)) - \text{int} N(L_2)$$

such that $D_2 \cap K_2 = D_2 \cap k$ consists of two points and $\partial D_2 = l_2$. Extending $D_2$, we may assume $\partial D_2 = L_2$. Let $D_k$ be the disk depicted in Figure 9 (1), such that $\partial D_k = k$. We remark that $D_k \cap L_2$ consists of four points $p_1, p_2, q_1, q_2$ (see Figure 9 (1)).

From the assumption we see that the boundary of arc components of $D_2 \cap D_k$ in $D_k$ consists of six points $p_1, p_2, q_1, q_2, x, y$ (see Figure 9 (2)). Considering the orientations, there exists an arc component $\alpha$ of $D_2 \cap D_k$ joining $p_i$ and $q_j$ for some $i, j$ (see Figure 9 (2)(3)). Let $\beta$ be an arc of $L_2$ connecting $p_i$ and $q_j$, and $D$ a disk in $D_2$ bounded by $\alpha \cup \beta$. Then $\alpha \cup \beta$ clearly has winding number one in the solid torus $S^3 - \text{int} N(L_1)$, which is knotted as $N(t)$ in $S^3 \supset V$. 

**Figure 9**
This contradicts, via Lemma 2.1, that \( \alpha \cup \beta = \partial D \). Hence we can conclude \( \text{wrap}_V(K_2) \neq 2 \), and applying the same argument we get also \( \text{wrap}_V(K_2) \neq 0 \). It follows that \( \text{wrap}_V(K_2) = 4 \). We see \( \text{wrap}_V(K_1) = 2 \) easily as follows. If \( \text{wrap}_V(K_1) \neq 2 \), then \( \text{wrap}_V(K_1) = 0 \). However this means \( \text{wrap}_V(K_2) = 0 \), thus \( \text{wrap}_V(K_1) = 2 \).

In this way we get the required patterns. \( \square \)

This result can be generalized to

**Corollary 3.4.** For any knot \( K \) in \( S^3 \) and any faithful embedding \( f \) from \( V \) into \( S^3 \), there exist patterns \( (V, K_1) \) and \( (V, K_2) \) such that \( K_i \cong K \) in \( S^3 \) \((i = 1, 2)\), which satisfy the following properties:

1. \( (V, K_1) \not\sim (V, K_2) \).
2. \( f(K_1) \cong f(K_2) \) in \( S^3 \) for the embedding \( f \) from \( V \) into \( S^3 \).

**Proof.** Let \( (V, k_1) \) and \( (V, k_2) \) be the patterns constructed in Theorem 3.3 depending upon the embedding \( f \). Since \( k_i \) \((i = 1, 2)\) is trivial in \( S^3 \), we can locally replace an unknotted arc of \( k_i \) by a knotted arc (with a suitable direction) so that the resulting knot \( K_i \) represents \( K \) in \( S^3 \). Then it follows from the choice of \( (V, k_1) \) and \( (V, k_2) \) that \( (V, K_1) \) and \( (V, K_2) \) are the required patterns. \( \square \)

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**References**


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DEPARTMENT OF MATHEMATICS
COLLEGE OF HUMANITIES & SCIENCES
NIHON UNIVERSITY
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TOKYO 156, JAPAN
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Correction to: “One-dimensional Nash groups”

James Joseph Madden