Pacific Journal of Mathematics

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Volume 162 No. 1 January 1994

PACIFIC JOURNAL OF MATHEMATICS
Vol. 162, No. 1, 1994

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The paper considers some sufficient conditions for a closed *-derivation of a C^*-algebra, implemented by a symmetric operator, to have a unique representational index.

1. Introduction. Let $\mathscr{A}$ be a C^*-subalgebra of the algebra $B(H)$ of all bounded operators on a Hilbert space H, and let a dense *-subalgebra $D(\delta)$ of $\mathscr{A}$ be the domain of a closed *-derivation δ from $\mathscr{A}$ into $B(H)$. A closed operator S on H *implements* δ if $D(S)$ is dense in H and if

$$AD(S) \subseteq D(S) \quad \text{and}$$
$$\delta(S)|_{D(S)} = i(SA - AS)|_{D(S)} \quad \text{for all } A \in D(\delta).$$

If S is symmetric (dissipative), it is called a *symmetric* (*dissipative*) *implementation* of δ. If a closed operator T extends S and also implements δ, then T is called a δ-extension of S. If S has no δ-extension, it is called a *maximal implementation* of δ.

If δ is implemented by a closed operator, it always has an infinite set $\mathscr{I}(\delta)$ of implementations. However, not much can be said about the structure of $\mathscr{I}(\delta)$. We do not even know whether it has maximal implementations. The subsets $\mathscr{S}(\delta)$ and $\mathscr{D}(\delta)$ of $\mathscr{I}(\delta)$ ($\mathscr{S}(\delta) \subseteq \mathscr{D}(\delta)$), which consist respectively of symmetric and of dissipative implementations of δ, are more interesting. In [4] it was shown that every symmetric implementation of δ extends to a *maximal* symmetric implementation of δ. Therefore if $\mathscr{S}(\delta) \neq \varnothing$, then $\mathscr{S}(\delta)$ as well as the set $\mathscr{MS}(\delta)$ of all maximal symmetric implementations of δ are infinite sets.

If $S \in \mathscr{MS}(\delta)$ and it is not selfadjoint, then the question arises as to whether S has *dissipative* δ-extensions and, if so, whether there exist maximal dissipative implementations of δ. This question was partly answered in [5] where it was established that, under some conditions on δ and S (for example, if $\max(n_-(S), n_+(S)) < \infty$), the maximal dissipative implementations of δ do exist.

Let $\mathscr{R}(\delta)$ be the set of all J-equivalence classes of J-symmetric representations of the algebra $D(\delta)$ on Krein spaces. In **[3]** and **[4]** it was shown that the deficiency space $N(S) = N_-(S) + N_+(S)$ of every operator $S \in \mathscr{S}(\delta)$ is a Krein space and that there exists a J-symmetric representation π_S^δ of $D(\delta)$ on $N(S)$. Thus there is a mapping of $\mathscr{S}(\delta)$ into $\mathscr{R}(\delta)$, and different symmetric implementations may have corresponding representations which are J-equivalent.

The structure of the representations π_S^δ can be extremely complicated, partly due to the fact that they may have neutral invariant subspaces. In **[4]** it was proved that π_S^δ has no neutral invariant subspaces if and only if S is a maximal symmetric implementation of δ. If $S \in \mathscr{MS}(\delta)$, we shall call the image of π_S^δ in $\mathscr{R}(\delta)$ a *representational index* of δ (relative to S), and denote it by i_S^δ. In this context the following problems naturally arise:

—finding simple characteristics of the representations π_S^δ;

—the description of the images of $\mathscr{S}(\delta)$ and $\mathscr{MS}(\delta)$ in $\mathscr{R}(\delta)$;

—finding conditions on δ such that the image of $\mathscr{MS}(\delta)$ in $\mathscr{R}(\delta)$ consists of only one element.

The simplest characterization π_S^δ is the pair $(n_+(S), n_-(S))$ of deficiency indices of the operator S. Different properties of these indices were considered in **[6–8]**. In particular, if $\mathscr{A}$ is unital, if $S \in \mathscr{MS}(\delta)$ and $\max(n_+(S), n_-(S)) < \infty$, then there are disjoint sets of irreducible *-representations $\{\pi_i\}_{i=1}^p$ and $\{\rho_j\}_{j=1}^q$ of $\mathscr{A}$ such that

$$n_+(S) = \sum_{i=1}^{p} \dim \pi_i \quad \text{and} \quad n_-(S) = \sum_{j=1}^{q} \dim \rho_j .$$

Arveson **[1]** and Powers **[12]** studied the case when δ is the generator of a semigroup α_t of endomorphisms of $B(H)$ which has semigroups of intertwining isometries. If d is a generator of a semigroup $U(t)$ of such isometries, then the operator $S = id$ implements δ, it is a symmetric operator, $N_-(S) = \{0\}$, and $N(S) = N_+(S)$ is a Hilbert space. In this case $S \in \mathscr{MS}(\delta)$, $n_+(S) = \infty$, and π_S^δ is a *-representation. Powers **[12]** defined the index of α_t (relative to $U(t)$) to be the multiplicity of π_S^δ. Arveson **[1]** gave another definition of the index of α_t and Powers and Price **[13]** proved that the Arveson's index is precisely the number of times the identity representation of $D(\delta)$ on H occurs in the representation π_S^δ.

Jorgensen and Price **[3]** studied the general case when $N(S)$ is not necessarily a Hilbert space. They introduced the V-index as the dimension of the Krein space of operators $V: H \to N(S)$, satisfying $VA = \pi_S^\delta(A)V$, $V \in D(\delta)$.

In [7] a sextuple $\operatorname{ind}(\delta, S)$ was associated with every pair (δ, S). All of its elements are either integers or infinity. If $N_-(S) = \{0\}$, one of the elements of $\operatorname{ind}(\delta, S)$ is the Powers' index. The sextuple is stable under perturbations of δ of the form $\sigma(A) = \delta(A) + i(BA - AB)$: $\operatorname{ind}(\sigma, S + B) = \operatorname{ind}(\delta, S)$, where $B = B^* \in B(H)$. Under some conditions on δ, $\operatorname{ind}(\delta, S) = \operatorname{ind}(\delta, T)$ for all $S, T \in \mathscr{MS}(\delta)$.

This paper studies the conditions on δ such that the image of $\mathscr{MS}(\delta)$ in $\mathscr{R}(\delta)$ consists of only one element, i.e., all representations π_S^δ, $S \in \mathscr{MS}(\delta)$ are J-equivalent. Obviously, only then can one speak about *the* representational index of δ. In §3 we consider the following problem: given a symmetric implementation S of δ, under what conditions on π_S^δ are all the representational indices i_T^δ, which correspond to different maximal symmetric δ-extensions T of S, equal? Theorem 3.2 gives a partial solution to this problem and shows that if the representation π_S^δ is finitely Π_-- or Π_+-decomposable, then all representations π_T^δ, $S \subseteq T$ and $T \in \mathscr{MS}(\delta)$, are J-equivalent, so that all the corresponding representational indices i_T^δ are equal.

As a corollary of this result, we obtain that if δ has a *minimal* implementation S, and if the representation π_S^δ is finitely Π_-- or Π_+-decomposable, then, for *all* maximal symmetric implementations T of δ (and not only for those which δ-extend S), the representations π_T^δ are J-equivalent, so that δ has a unique representational index.

Although the conditions imposed on δ are strong, the examples of §3 demonstrate that these conditions are justified. Without assuming the existence of a minimal symmetric implementation it is difficult to "compare" different representations π_T^δ and $\pi_{T_1}^\delta$, $T, T_1 \in \mathscr{MS}(\delta)$, and to establish whether they are J-equivalent. This is especially so if $D(T) \cap D(T_1) = \{0\}$, as in Example 2 (see **[13]**). In the cases studied in **[1]**, **[12]** and **[13]** (see Example 2), minimal symmetric implementations of the generators δ of semigroups of endomorphisms of $B(H)$ do not exist. Therefore the representational indices i_T^δ, $T = id$, where d are the generators of semigroups $U(t)$ of intertwining isometries, seem to depend on $U(t)$ **[13]**. On the other hand, in many interesting cases the derivations do have minimal symmetric implementations. This is so, for example, if $\mathscr{A}$ contains the ideal of all compact operators **[6]** (see Theorem 3.4 and Example 3).

The condition that π_S^δ (S is a minimal symmetric implementation of δ) is finitely Π-decomposable is crucial for our attempt to show that all representational indices of δ are equal. For every maximal symmetric implementation T of δ, there is a maximal neutral invariant subspace $L(T)$ in $N(S)$ such that the representation π_T^δ is J-equivalent to the quotient representation $(\Pi_S^\delta)^{L(T)}$ on $L(T)^{[\perp]}/L(T)$. Theorem 2.6 considers finitely Π-decomposable representations π and proves that, for all maximal neutral invariant subspaces L, the quotient representations π^L on $L^{[\perp]}/L$ are J-equivalent. Therefore it follows that all representational indices of δ are equal. Example 4 shows that if π_S^δ is Π-decomposable but not *finitely* Π-decomposable, the derivation δ may have an infinite number of distinct representational indices.

2. J-symmetric representations of *-algebras on Krein spaces.

2.1. *Preliminaries.* This section considers J-symmetric representations of *-algebras on Krein spaces. For the benefit of the reader and for the sake of being reasonably selfcontained, we provide some amount of detail about the theory of Krein spaces and J-symmetric representations.

Let H be a Hilbert space with a scalar product (x, y) and a norm $\|x\| = (x, x)^{1/2}$. Let $H = H_- \oplus H_+$ be a decomposition of H in the orthogonal sum of subspaces H_- and H_+. The involution $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ defines an indefinite form $[x, y] = (Jx, y)$ on H. The space H with this indefinite form $[\ ,\ ]$ is called a *Krein* space. Let $k_d = \dim H_d$, $d = \pm$, and $k = \min(k_-, k_+)$. If $k < \infty$, then H is called a Π_k*-space.*

Let H be a Krein space. A subspace L in H is called

(a) *nonnegative* if $[x, x] \geq 0$,

(b) *positive* if $[x, x] > 0$, $x \neq 0$,

(c) *uniformly positive* if there is $r > 0$ such that $[x, x] \geq r(x, x)$,

(d) *neutral* (*null*) if $[x, x] = 0$,

for all $x \in L$. The concept of *nonpositive, negative and uniformly negative* subspaces are introduced analogously.

A nonnegative subspace is called *maximal nonnegative* if it is not properly contained in any other nonnegative subspace. In the same way this concept of maximality can be introduced for all other types of subspaces.

Law of inertia [9]. If L is a maximal nonnegative (nonpositive) subspace in H, then $\dim L = \dim H_+$ $(\dim H_-)$.

The subspace

$$L^{[\perp]} = \{y \in H : [x, y] = 0 \text{ for all } x \in L\}$$

is called the *J-orthogonal complement* of L.

The geometry of Krein spaces is more complicated than the geometry of Hilbert spaces and the decomposition

$$H = L[+]L^{[\perp]} \tag{1}$$

does not always exist (the symbol $[+]$ means that the sum is direct and the summands are J-orthogonal).

THEOREM 2.1 [9]. (i) *Let L be a nonnegative (nonpositive) subspace of H. The decomposition (1) holds if and only if L is uniformly positive (negative).*

(ii) *If L is an indefinite space, then (1) holds if and only if L decomposes into a direct sum of two uniformly definite subspaces.*

(iii) ([9], *page* 118) *Let Q be the orthoprojection on L. The decomposition (1) holds if and only if the symmetric operator $G = QJQ$ has a bounded inverse.*

(iv) (*Iohvidov and Ginzburg, see* [9, *page* 118].) *Let $k_+ = \infty$. All positive subspaces of H are uniformly positive if and only if $k_- < \infty$.*

Every subspace L is decomposable into a simultaneously orthogonal and J-orthogonal direct sum

$$L = L_- \dotplus L_0 \dotplus L_+, \qquad L_0 = L \cap L^{[\perp]} \tag{2}$$

in which the summands are respectively negative, neutral and positive subspaces, or reduce to zero (see [9], p. 118).

A representation π of a *-algebra $\mathscr{A}$ on a Krein space H is called *J-symmetric* if

$$\pi(A^*) = J\pi(A)^*J, \text{ i.e., } [\pi(A)x, y] = [x, \pi(A)^*y], \qquad x, y \in H.$$

If a subspace L is invariant for π, then $L^{[\perp]}$ is also invariant for π. By π_L we shall denote the restriction of π to L.

Let N and P be respectively uniformly negative and uniformly positive subspaces of H invariant for π. Then

$$(x, y)_N = -[x, y], \qquad x, y \in N \quad \text{and}$$
$$(x, y)_P = [x, y], \qquad x, y \in P$$

are definite scalar products on N and P. Set

$$\|x\|_N^2 = (x, x)_N, \qquad x \in N, \quad \text{and} \quad \|x\|_P^2 = (x, x)_P, \qquad x \in P.$$

Since N and P are uniformly definite subspaces, the norms $\|\ \|_N$ and $\|\ \|_P$ are equivalent to the original norm $\|\ \|$ on H. Therefore N and

P are Hilbert spaces with respect to the scalar products $(\ ,\ )_N$ and $(\ ,\ )_P$. Then π_N and π_P are *-representations of $\mathscr{A}$ on N and P with respect to these scalar products.

Let G be a bounded selfadjoint operator on a Hilbert space H. Similar to the involution J, the operator G defines an indefinite metric on H

$$[x, y]_G = (Gx, y).$$

A representation π of a *-algebra $\mathscr{A}$ on H is called *G-symmetric* if

$$[\pi(A)x, y]_G = [x, \pi(A^*)y]_G, \text{ i.e., } G\pi(A^*) = \pi(A)^*G, \qquad x, y \in H.$$

LEMMA 2.2. *Let π be a G-symmetric representation of $\mathscr{A}$ on H.*

(i) ([**11**], *page* 77). *If G has a bounded inverse, then there are a new scalar product $(\ ,\ )_1$ and an involution J_G on H such that the norm $\|\ \|_1 = (\ ,\ )_1^{1/2}$ is equivalent to the original norm on H, that $[x, y]_G = (J_G x, y)_1$ and that π is a J_G-symmetric representation of $\mathscr{A}$.*

(ii) [**8**]. *Let Q be the orthoprojection on a subspace L invariant for π and let $G_1 = QGQ$. The representation π_L of $\mathscr{A}$ on L is G_1-symmetric and $[x, y]_{G_1} = [x, y]_G$.*

2.2. *Neutral invariant subspaces of J-symmetric representations.* In general the structure of neutral invariant subspaces of J-symmetric representations π of *-algebras on Krein spaces H can be very complicated. In some cases, however, it is possible to obtain some useful information about their structure.

Let π be a J-symmetric representation on H, let N be a uniformly negative (positive) invariant subspace and let $N^{[\perp]}$ be a Π_k-space. It is proved in [**8**, Lemma 3.2] that if L is a neutral invariant subspace in H, then there exist a nonnegative (nonpositive) invariant subspace P in $N^{[\perp]}$ and a bounded operator T from P onto an invariant subspace K of N such that

$$L = \{Tx + x : x \in P\} \quad \text{and} \quad \pi_K T = T\pi_P.$$

LEMMA 2.3. *Let N have no finite-dimensional invariant subspaces. If L is a maximal neutral invariant subspace in H, then* $\operatorname{Ker} T = L \cap N^{[\perp]}$ *is a maximal neutral invariant subspace in $N^{[\perp]}$.*

Proof. Let N be uniformly negative. Since L is a neutral space,

$$[x, x] + [Tx, Tx] = 0, \qquad x \in P.$$

Hence

$$\|Tx\|_N^2 = -[Tx, Tx] = [x, x]. \tag{3}$$

By (2), $P = P_0 \dot{+} P_+$, where P_0 and P_+ are neutral and positive subspaces. Since $N^{[\perp]}$ is a Π_k-space, it follows from Theorem 2.1 (iv) that P_+ is uniformly positive. By (3), $P_0 = \operatorname{Ker} T$ and $\|Tx\|_N = \|x\|_{P_+}$ for all $x \in P_+$, so that T is an isometry from P_+ onto K. Therefore

$$L = \{Tx + x : x \in P_+\}[+]P_0 \quad \text{and} \quad P_0 = L \cap N^{[\perp]}.$$

Since $\pi_K T = T\pi_P$, P_0 is a neutral invariant subspace in $N^{[\perp]}$ and we only have to prove that P_0 is a maximal invariant subspace in $N^{[\perp]}$.

Assume that there exists a neutral invariant subspace M in $N^{[\perp]}$ larger than P_0. Since P_+ is uniformly positive, by Theorem 2.1(i),

$$N^{[\perp]} = P_+[+]R \quad \text{and} \quad P_0 \subseteq R \cap M,$$

where R is the J-orthogonal complement of P_+ in $N^{[\perp]}$.

Let $P^{[\perp]}$ be the J-orthogonal complement of P in $N^{[\perp]}$. Then $P^{[\perp]} \subseteq R$ and $P^{[\perp]}$ is invariant for π, since P and $N^{[\perp]}$ are invariant for π. The subspace $R \cap M$ is J-orthogonal to P_+ and to P_0. Hence $R \cap M \subseteq P^{[\perp]}$, so that $R \cap M = P^{[\perp]} \cap M$. Thus $R \cap M$ is a neutral invariant subspace. If $P_0 \neq R \cap M$, then

$$L_1 = \{Tx + x : x \in P_+\}[+](R \cap M)$$

is a neutral invariant subspace in H larger than L. This contradiction shows that $P_0 = R \cap M$.

By Law of inertia, $\dim M \leq k$. Since $P_+ \cap M = \{0\}$, $M = \{z+y : y \in M_R,\ z \in M_P\}$, where M_R and M_P are finite-dimensional subspaces in R and P_+ respectively and where $y = 0$ implies $z = 0$. Since M is larger than P_0 and since $P_0 = R \cap M$, $M_P \neq \{0\}$ and $z = 0$ implies $y \in P_0$.

We shall show that the subspace $P_0[+]M_P$ is invariant for π. Since M is a neutral subspace and since $M_P \subseteq P_+$, every y in M_R is J-orthogonal to P. Therefore $M_R \subseteq P^{[\perp]}$. Since $P^{[\perp]}$ is invariant for π and since $P^{[\perp]} \subseteq R$, $\pi(A)y \in R$ for every $y \in M_R$ and $A \in \mathscr{A}$. Then, for all $z + y \in M$,

$$\pi(A)(z + y) = \pi(A)z + \pi(A)y = z_1 + y_1 \in M,$$

so that $\pi(A)z - z_1 = y_1 - \pi(A)y$. Since $\pi(A)z \in P$, since $y_1 - \pi(A)y \in R$ and since $P \cap R = P_0$, we have that $\pi(A)z - z_1 \in P_0$. Hence $\pi(A)z \in P_0[+]M_P$ and the subspace $P_0[+]M_P$ is invariant for π.

Since T is an isometry from P_+ onto K,

$$T(P_0[+]M_P) = TM_P = \{Tx : x \in M_P\}$$

is a finite-dimensional subspace in K. Since $\pi_K T = T\pi_P$, TM_P is a finite-dimensional invariant subspace in N which contradicts the assumption that N does not have such subspaces. Hence P_0 is a maximal neutral invariant subspace in $N^{[\perp]}$. The proof is complete.

The following lemma compares two maximal neutral invariant subspaces.

LEMMA 2.4. *Let L and K be maximal neutral invariant subspaces in H. Then $L \cap K = L \cap K^{[\perp]} = L^{[\perp]} \cap K$ and* $\dim L = \dim K$.

Proof. Set $M = L \cap K$. The subspace $L \cap K^{[\perp]}$ is neutral, invariant and J-orthogonal to K. If $L \cap K^{[\perp]} \not\subseteq K$, then $K + (L \cap K^{[\perp]})$ is a neutral invariant subspace larger than K. This contradiction shows that $L \cap K^{[\perp]} \subseteq K$. Therefore $M = L \cap K^{[\perp]}$. Similarly $M = L^{[\perp]} \cap K$.

If $M = \{0\}$, then $L \cap K^{[\perp]} = L^{[\perp]} \cap K = \{0\}$. Hence, for every $x \in L$ there is $y \in K$ such that $[x, y] \neq 0$ and vice versa. Therefore $\dim L = \dim K$.

If $M \neq \{0\}$, then

$$\dim L = \dim M + \dim(L/M) = \dim M + \dim(K/M) = \dim K,$$

since L/M and K/M are maximal neutral invariant subspaces in $M^{[\perp]}/M$ and since $(L/M) \cap (K/M) = \{0\}$. The lemma is proved.

2.3. *Quotient J-symmetric representations.* Let π be a J-symmetric representation of a *-algebra $\mathscr{A}$ on a Krein space H. For every neutral invariant subspace L, $L \subseteq L^{[\perp]}$ and we can consider the quotient representation π^L of $\mathscr{A}$ on the quotient space $\widehat{L} = L^{[\perp]}/L$. Making use of Phillips' approach ([**11**], Lemmas 4.2 and 4.3), it is easy to show that π^L is J-symmetric. Let L and M be different maximal neutral invariant subspaces in H. We shall investigate the question of when the representations π^L and π^M are equivalent. In order to answer this question we shall consider the following definition of equivalence of two representations.

DEFINITION. We say that a G-symmetric representation π of $\mathscr{A}$ on H is *J-equivalent* to a G_1-symmetric representation ρ of $\mathscr{A}$ on K $(\pi \sim \rho)$ if there is a bounded operator T from H *onto* K which has a bounded inverse and such that $T\pi = \rho T$ and that

$$[Tx, Ty]_{G_1} = [x, y]_G \text{ for all } x, y \text{ in } H, \text{ i.e., } T^* G_1 T = G.$$

If π and ρ are $*$-representations of $\mathscr{A}$, then $G = 1_H$, $G_1 = 1_K$ and J-equivalence becomes the usual equivalence of $*$-representations.

Let L be an invariant neutral subspace and let x and y in $L^{[\perp]}$ be representatives of classes $\hat{x}$ and $\hat{y}$ in $\widehat{L}$. Then the form

$$[\hat{x}, \hat{y}]\widehat{\ } = [x, y] \tag{4}$$

on $\widehat{L}$ does not depend on the choice of representatives.

It follows from Lemma 4.2 **[11]** that $L^{[\perp]}$ can be decomposed into three mutually orthogonal and J-orthogonal subspaces

$$L^{[\perp]} = L_+ \dotplus L \dotplus L_- \tag{5}$$

where $L_+ = H_+ \cap L^{[\perp]}$ and $L_- = H_- \cap L^{[\perp]}$. Thus the quotient space $\widehat{L} = L^{[\perp]}/L$ is isomorphic and isometric with $L_\pm = L_+ \dotplus L_-$. We shall denote by β the orthogonal projection of $L^{[\perp]}$ onto $L_\pm$. By (5), $\beta\pi(A)\beta = \beta\pi(A)$. Therefore

$$\pi_\beta(A)y = \beta\pi(A)y, \qquad A \in \mathscr{A}, \ y \in L_\pm,$$

is a representation of $\mathscr{A}$ on $L_\pm$ which is J-equivalent to π^L. We shall often identify π^L and π_β. The subspace $L_\pm$ is invariant for the involution J and the form $[\ ,\ ]$ does not degenerate on $L_\pm$, i.e., $[x, y] = 0$ for all y in $L_\pm$ implies $x = 0$.

LEMMA 2.5. (i) *The representation π^L of $\mathscr{A}$ on $\widehat{L}$ is J-symmetric. If L is a maximal neutral invariant subspace, then π^L has no neutral invariant subspaces.*

(ii) *If $\mathscr{L}$ is an invariant subspace in $L^{[\perp]}$ such that*

$$L^{[\perp]} = L + \mathscr{L} \quad \text{and} \quad L \cap \mathscr{L} = \{0\},$$

then the representations π^L and $\pi_{\mathscr{L}}$ are J-equivalent $(\pi^L \sim \pi_{\mathscr{L}})$.

Proof. Decomposing any y and z in $L^{[\perp]}$ according to (5):

$$y = y_+ + y_0 + y_-, \qquad z = z_+ + z_0 + z_-,$$

we see that $\beta y = y_+ + y_-$, $\beta z = z_+ + z_-$ and that

$$[y, z] = [\beta y, \beta z] = (y_+, z_+) - (y_-, z_-). \tag{6}$$

It follows from (6) that, for all A in $\mathscr{A}$ and $y, z \in L_{\pm}$,

$$[\pi_\beta(A)y, z] = [\beta\pi(A)y, \beta z] = [\pi(A)y, z] = [y, \pi(A^*)z]$$
$$= [\beta y, \beta\pi(A^*)z] = [y, \pi_\beta(A^*)z].$$

Therefore π_β is J-symmetric. Since $\pi^L \sim \pi_\beta$, π^L is J-symmetric. It follows from (4)–(6) that if L is maximal neutral invariant, then π^L has no neutral invariant subspaces. Part (i) is proved.

Let $\mathscr{L}$ be an invariant subspace in $L^{[\perp]}$ and let Q be the orthoprojection onto $\mathscr{L}$. Set $G = QJQ$. By Lemma 2.2 (ii), the representation $\pi_{\mathscr{L}}$ of $\mathscr{A}$ on $\mathscr{L}$ is G-symmetric.

Now assume that $L^{[\perp]} = L + \mathscr{L}$ and that $L \cap \mathscr{L} = \{0\}$. Let T be the restriction of the projection β to $\mathscr{L}$. Since $L^{[\perp]} = L + \mathscr{L}$, T is a bounded operator from $\mathscr{L}$ onto $L_{\pm}$. Since $L \cap \mathscr{L} = \{0\}$, we have that $\operatorname{Ker} T = \{0\}$. Therefore T has a bounded inverse. For every $y \in \mathscr{L}$,

$$\pi_\beta(A)Ty = \pi_\beta(A)\beta y = \beta\pi(A)\beta y = \beta\pi(A)y$$
$$= \beta\pi_{\mathscr{L}}(A)y = T\pi_{\mathscr{L}}(A)y,$$

since $\pi_{\mathscr{L}}(A)y \in \mathscr{L}$. From (6) and from Lemma 2.2 (ii) it follows that,

$$[Ty, Tz] = [\beta y, \beta z] = [y, z] = [y, z]_G,$$

for all $y, z \in \mathscr{L}$. Therefore $\pi^L \sim \pi_\beta \sim \pi_{\mathscr{L}}$. The lemma is proved.

It follows from the construction of the representation π^L that it depends heavily on the choice of a neutral invariant subspace L. Even if L and M are maximal neutral invariant subspaces in H, the representations π^L and π^M are not, generally speaking, J-equivalent. In Theorem 2.6 we shall show that if π satisfies a certain condition, then for all maximal neutral invariant subspaces L and M, the quotient representations π^L and π^M are J-equivalent.

Let M be a subspace of a Krein space H and let $H = M[+]M^{[\perp]}$. If Q is the orthoprojection on M, it follows from Theorem 2.1 (iii) that the operator $G = QJQ$ has a bounded inverse. By Lemma 2.2 (i), there are a scalar product $(\, , \,)_1$ and an involution J_G on M such that M decomposes into an orthogonal sum $M_- \oplus M_+$ of subspaces M_- and M_+ with respect to $(\, , \,)_1$ and such that $J_G = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ with respect to this decomposition. Hence M becomes a Krein space with respect to the form $[x, y]_G = (J_G x, y)_1$.

We shall now consider a special class of J-symmetric representations, which will play an important role in this paper.

DEFINITION. Let π be a J-symmetric representation of a $*$-algebra on a Krein space H. We say that π is *Π_--decomposable* if $H = K[+]K^{[\perp]}$ where K is a uniformly negative invariant subspace and where $K^{[\perp]}$ is a Π_k-space, $k = k_-$ (one of the summands can be zero). We say that π is *finitely Π_--decomposable* if, in addition, the $*$-representation π_K on K decomposes in a finite orthogonal sum of irreducible representations. Similarly we can define *Π_+-decomposable* and *finitely Π_+-decomposable* representations.

Let L be a neutral invariant subspace and let Z be an invariant subspace of L. Then

$$Z \subset L \subset L^{[\perp]} \subset Z^{[\perp]}$$

and the quotient space $L_1 = L/Z$ is contained in $\widehat{Z} = Z^{[\perp]}/Z$. It follows from (4) that $L_1^{[\perp]} = L^{[\perp]}/Z$, where $L_1^{[\perp]}$ is the J-orthogonal complement of L_1 in $\widehat{Z}$. Therefore the subspaces $L_1^{[\perp]}/L_1$ and $L^{[\perp]}/L$ are isomorphic and isometric and

$$\pi^L \sim (\pi^Z)^{(L/Z)}. \tag{7}$$

We shall now prove the main theorem of this section.

THEOREM 2.6. *Let π be a finitely Π-decomposable J-symmetric representation of a $*$-algebra $\mathscr{A}$ on a Krein space H. If L is a maximal neutral invariant subspace in H, then the representation π^L is finitely Π-decomposable. If K is another maximal neutral invariant subspace in H, then the quotient representations π^L and π^K on $L^{[\perp]}/L$ and on $K^{[\perp]}/K$ respectively are J-equivalent.*

Proof. Set $Z = L \cap K$. The quotient spaces $L_1 = L/Z$ and $K_1 = K/Z$ are contained in the quotient space $\widehat{Z} = Z^{[\perp]}/Z$, they are maximal neutral invariant subspaces for the representation π^Z and $L_1 \cap K_1 = \{0\}$. It follows from (7) that if we prove that the representations $(\pi^Z)^{L_1}$ and $(\pi^Z)^{K_1}$ are J-equivalent, we shall also obtain that the representations π^L and π^K are J-equivalent. Thus without loss of generality we may assume that $Z = L \cap K = \{0\}$.

We shall consider 3 cases.

Case 1. *Assume that H is a Π_k-space.*

Then $\widehat{L}$ is a Π_n-space, $n < k$, so that π^L is finitely Π-decomposable. By Lemma 2.4 and by Law of inertia, $\dim L = \dim K \leq k$. Set $N = L + K$. Then N is invariant for π and $\dim N \leq 2k$. Since $L \cap K = \{0\}$, it follows from Lemma 2.4 that $N \cap L^{[\perp]} = L$. Since

$N^{[\perp]} = L^{[\perp]} \cap K^{[\perp]}$, by Lemma 2.4,

$$N \cap N^{[\perp]} = N \cap (L^{[\perp]} \cap K^{[\perp]}) = L \cap K^{[\perp]} = L \cap K = \{0\}.$$

Hence, by (2), $N = N_- \dot{+} N_+$, where N_- and N_+ are respectively negative and positive finite-dimensional subspaces. Since every definite finite-dimensional subspace is also uniformly definite, it follows from Theorem 2.1 (ii), that $H = N[+]N^{[\perp]}$. Since $K \cap L^{[\perp]} = \{0\}$, we have that $L^{[\perp]} = L[+]N^{[\perp]}$. By Lemma 2.5 (ii), $\pi^L \sim \pi_{N^{[\perp]}}$. Similarly, $\pi^K \sim \pi_{N^{[\perp]}}$, so that $\pi^L \sim \pi^K$.

Case 2. Let H be finitely Π_--decomposable and let $H = M[+]M^{[\perp]}$ where M is an infinite dimensional uniformly negative invariant subspace such that $M^{[\perp]}$ is a Π_k-space, $k = k_-$, and such that π_M decomposes in a finite orthogonal sum of irreducible representations. *Assume that* $M^{[\perp]}$ *has no neutral invariant subspaces.*

In Lemma 3.2 **[8]** it is proved that in this case there exist a uniformly definite invariant subspace L_+ in $M^{[\perp]}$, a uniformly negative invariant subspace L_- in M and an isometry T from L_+ onto L_- ($\|Tx\|_{L_-} = \|x\|_{L_+}$) such that

$$\pi T|_{L_+} = T\pi|_{L_+} \quad \text{and} \quad L = \{Tx + x : x \in L_+\}.$$

By Theorem 3.5 **[8]**, $M^{[\perp]} = \mathscr{N}[+]\mathfrak{H}[+]P$ where $\mathscr{N}$ and P are maximal negative and maximal positive invariant subspaces in $M^{[\perp]}$ and where $\mathfrak{H}$ is an invariant Π_m-space, $m \leq k$, which has neither neutral nor definite invariant subspaces. It is also proved there that every positive invariant subspace of $M^{[\perp]}$ is contained in P. Hence $L_+ \subseteq P$.

The subspace $N = M[+]\mathscr{N}$ is uniformly negative, invariant and

$$H = N[+]\mathfrak{H}[+]P.$$

By Law of inertia, $\dim \mathscr{N} \leq k$. Since π_M decomposes in a finite orthogonal sum of irreducible representations, π_N also decomposes in a finite orthogonal sum of irreducible representations. Set

$$N_L = N \cap L^{[\perp]} \quad \text{and} \quad P_L = P \cap L^{[\perp]}.$$

Since N and P are uniformly definite subspaces, it follows that

$$N = N_L[+]L_-, \quad P = P_L[+]L_+ \quad \text{and} \quad L^{[\perp]} = N_L[+]L[+]P_L[+]\mathfrak{H}.$$

The subspaces N_L and P_L are invariant for π. Set

$$\mathscr{L} = N_L[+]P_L[+]\mathfrak{H}, \text{ so that } L^{[\perp]} = \mathscr{L}[+]L. \tag{8}$$

The subspace $\mathscr{L}$ is invariant for π. Therefore, by Lemma 2.5 (ii), $\pi^L \sim \pi_{\mathscr{L}}$. Since $N_L \subset N$, π_{N_L} decomposes in a finite orthogonal sum of irreducible representations. Since $P_L[+]\mathfrak{H}$ is a Π_m-space, $\pi_{\mathscr{L}}$ (and hence π^L) is a finitely Π_--decomposable representation.

Similarly,

$$\mathscr{K} = N_K[+]P_K[+]\mathfrak{H} \quad \text{and} \quad K^{[\perp]} = \mathscr{K}[+]K,$$

where $N_K = N \cap K^{[\perp]}$ and $P_K = P \cap K^{[\perp]}$, and $\pi^K \sim \pi_{\mathscr{K}}$.

In Theorem 2.6 [7] it is proved that in this case the *-representations π_{N_L} and π_{N_K} are equivalent and that the *-representations π_{P_L} and π_{P_K} are equivalent. Therefore there exist an isometry U from N_L onto N_K (in $\| \|_N$) and an isometry V from P_L onto P_K (in $\| \|_P$) such that

$$U\pi|_{N_L} = \pi U|_{N_L} \quad \text{and} \quad V\pi|_{P_L} = \pi V|_{P_L}.$$

Set $S = U[+]V[+]1_{\mathfrak{H}}$. Then S is a bounded operator from $\mathscr{L}$ onto $\mathscr{K}$ which has a bounded inverse.

Given x and y in $\mathscr{L}$ and decomposing them according to (8)

$$x = x_N + x_P + x_{\mathfrak{H}}, \qquad y = y_N + y_P + y_{\mathfrak{H}},$$

we obtain that

$$\begin{aligned}[Sx, Sy] &= [Ux_N + Vx_P + x_{\mathfrak{H}}, Uy_N + Vy_P + y_{\mathfrak{H}}] \\ &= [Ux_N, Uy_N] + [Vx_P, Vy_P] + [x_{\mathfrak{H}}, y_{\mathfrak{H}}] \\ &= [x_N, y_N] + [x_P, y_P] + [x_{\mathfrak{H}}, y_{\mathfrak{H}}] = [x, y].\end{aligned}$$

We also have that

$$\pi S|_{\mathscr{L}} = \pi U|_{N_L}[+]\pi V|_{P_L}[+]\pi|_{\mathfrak{H}} = U\pi|_{N_L}[+]V\pi|_{P_L}[+]\pi|_{\mathfrak{H}} = S\pi|_{\mathscr{L}}.$$

Therefore the representations $\pi_{\mathscr{L}}$ and $\pi_{\mathscr{K}}$ are J-equivalent, so that the representations π^L and π^K are J-equivalent.

Case 3 (general case). Let $H = M[+]M^{[\perp]}$, as in Case 2. *Assume that* $M^{[\perp]}$ *has neutral invariant subspaces.*

By the assumption of the theorem, the representation π_M decomposes in a finite orthogonal sum of irreducible representations. If all of them are finite-dimensional, then H is a Π_k-space and this was considered in Case 1. Let M_f be the subspace in M which contains all finite-dimensional irreducible subrepresentations. Then M_f is finite-dimensional and $(M[-]M_f)^{[\perp]} = M_f[+]M^{[\perp]}$ is a Π_n-space, $k_- \le n_-$. Considering $M[-]M_f$ instead of M, we may assume without loss of generality that π_M is a finite sum of *infinite dimensional* irreducible representations.

By Lemma 2.3, $L_M = L \cap M^{[\perp]}$ is a maximal neutral invariant subspace in $M^{[\perp]}$. Let $L_M^{[\perp]}$ be the J-orthogonal complement of L_M in H and let $\mathfrak{L} = L_M^{[\perp]} \cap M^{[\perp]}$. Then $\mathfrak{L}$ is the J-orthogonal complement of L_M in $M^{[\perp]}$,

$$L_M \subset \mathfrak{L} \quad \text{and} \quad \mathfrak{L}_M^{[\perp]} = M[+]L_M^{[\perp]}.$$

We also have that $L_M^{[\perp]}/L_M$ is isomorphic and isometric to $M[+](\mathfrak{L}/L_M)$ and that $\mathfrak{L}/L_M$ is a Π_n-space, $n < k$, which contains no neutral invariant subspaces. Therefore the representation π^{L_M} on $L_M^{[\perp]}/L_M$ is finitely Π_--decomposable. The subspace $\widetilde{L} = L/L_M$ is a maximal neutral invariant subspace in $L_M^{[\perp]}/L_M$ and $\widetilde{L} \cap (\mathfrak{L}/L_M) = \{0\}$. It follows from Case 2 that the representation $(\pi^{L_M})^{\widetilde{L}}$ is finitely Π_--decomposable. Since, by (7), $\pi^L \sim (\pi^{L_M})^{\widetilde{L}}$, π^L is also finitely Π_--decomposable. This concludes the proof that in all cases π^L is finitely Π_--decomposable.

Now let K be another maximal neutral invariant subspace in H such that $L \cap K = \{0\}$. Then $K_M = K \cap M^{[\perp]}$ is a maximal neutral invariant subspace in $M^{[\perp]}$ and $L_M \cap K_M = \{0\}$. Set

$$N = L_M + K_M$$

and let $N^{[\perp]}$ be the J-orthogonal complement of N in $M^{[\perp]}$. Since $M^{[\perp]}$ is a Π_k-space, we obtain, as in Case 1, that $N^{[\perp]}$ is a Π_n-space $n < k$, that it is invariant for π and that

$$M^{[\perp]} = N[+]N^{[\perp]} \quad \text{and} \quad \mathfrak{L} = L_M[+]N^{[\perp]}.$$

Set $H_1 = M[+]N^{[\perp]}$. Then H_1 is an invariant Π_--decomposable subspace of H and

$$L_M^{[\perp]} = M[+]\mathfrak{L} = M[+]L_M[+]N^{[\perp]} = L_M[+]H_1. \tag{9}$$

Therefore it follows from Lemma 2.5 (ii) that the quotient representation π^{L_M} is J-equivalent to the representation π_{H_1}. The subspace $\widetilde{L} = L/L_M$ is a maximal neutral invariant subspace for the representation π^{L_M}. Since $\pi^{L_M} \sim \pi_{H_1}$, there is a maximal neutral invariant subspace L_1 in H_1 such that $(\pi^{L_M})^{\widetilde{L}} \sim (\pi_{H_1})^{L_1}$. Therefore, by (7),

$$\pi^L \sim (\pi^{L_M})^{\widetilde{L}} \sim (\pi_{H_1})^{L_1}.$$

Similarly, there is a maximal neutral invariant subspace K_1 in H_1 such that $\pi^K \sim (\pi_{H_1})^{K_1}$.

Since $N^{[\perp]}$ is J-orthogonal to L_M and since L_M is a maximal neutral invariant subspace in $M^{[\perp]}$, $N^{[\perp]}$ has no neutral invariant subspaces. Hence the subspace H_1 and the representation π_{H_1} satisfy Case 2. Thus $(\pi_{H_1})^{L_1} \sim (\pi_{H_1})^{K_1}$, so that the representations π^L and π^K are J-equivalent which concludes the proof of the theorem.

The following example shows that if π is *not finitely* Π-decomposable, then Theorem 2.6 does not necessarily hold.

EXAMPLE 1. Let ρ be a *-representation of a *-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$, let

$$H_- = H_+ = \sum_{i=1}^{\infty} \bigoplus \mathfrak{H}_i, \quad \text{all } \mathfrak{H}_i = \mathfrak{H}.$$

Set $H = H_- \oplus H_+$. If $x = x_- + x_+$ and $y = y_- + y_+$, where $x_-, y_- \in H_-$ and $x_+, x_+ \in H_+$, set

$$[x, y] = -(x_-, y_-) + (x_+, y_+).$$

Then H becomes a Krein space and the representation

$$\pi = \left(\sum_{i=1}^{\infty} \bigoplus \rho_i\right) \oplus \left(\sum_{i=1}^{\infty} \bigoplus \rho_i\right), \quad \text{all } \rho_i = \rho,$$

on H is J-symmetric. Let $x_- = (x_1, \dots, x_i, \dots) \in H_-$, $x_i \in \mathfrak{H}_i$, and let T_n, $n = 0, 1, \dots$, be isometries from H_- into H_+ such that

$$T_n x_- = (y_1, \dots, y_i, \dots) \in H_+, \quad y_1 = \dots = y_n = 0 \text{ and } y_{n+i} = x_i.$$

The subspaces $L_n = \{x_- + T_n x_- : x_- \in H_-\}$ are maximal, neutral invariant subspaces and

$$L_n^{[\perp]} = L_n[+]\mathscr{L}_n \quad \text{where } \mathscr{L}_n = \sum_{i=1}^{n} \bigoplus \mathfrak{H}_i \subset H_+.$$

By Lemma 2.5 (ii), $\pi^{L_n} \sim \pi_{\mathscr{L}_n}$. Since all the representations $\pi_{\mathscr{L}_n}$ are different, we obtain that the quotient representations π^L depend on the choice of the maximal neutral invariant subspaces L.

3. Representational indices of derivations of C^*-algebras. In this section we apply the results of §2 to the investigation of derivations of C^*-algebras.

Let ρ be a *-representation of a C^*-algebra $\mathscr{A}$ on a Hilbert space H. A *derivation* δ *of* A *into* $B(H)$ *relative to* ρ is a linear mapping from a dense *-subalgebra $D(\delta)$ of $\mathscr{A}$ into $B(H)$ such that

(i) $\delta(AB) = \delta(A)\rho(B) + \rho(A)\delta(B)$, $A, B \in D(\delta)$;

(ii) $\delta(A^*) = \delta(A)^*$, $A \in D(\delta)$;
(iii) $\operatorname{Ker} \rho \subseteq D(\delta)$.

The derivation is *closed* if $A_n \in D(\delta)$, $A_n \to A$ and $\delta(A_n) \to B$ implies $A \in D(\delta)$ and $\delta(A) = B$. If δ is closed, then $D(\delta)$ is a *-normed algebra with respect to the norm

$$\|A\|_\delta = \|A\| + \|\delta(A)\|.$$

A symmetric operator S on H *implements* δ if its domain $D(S)$ is dense in H and if for all $A \in D(\delta)$

$$\rho(A)D(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|_{D(S)} = i(S\rho(A) - \rho(A)S)|_{D(S)}.$$

If T is a symmetric extension of S and if it also implements δ, we say that T is a *symmetric δ-extension of* S. If S has no symmetric δ-extensions, it is called a *maximal symmetric implementation of* δ.

We shall now consider briefly the link between derivations implemented by symmetric operators and J-symmetric representations on Krein spaces.

Let S be a symmetric operator and let S^* be its adjoint. Then $N_d(S) = \{x \in D(S^*) : S^*x = idx\}$, $d = \pm$, are the deficiency spaces of S and $n_d(S) = \dim N_d(S)$ are the deficiency indices of S. The scalar product

$$\langle x, y\rangle^S = (x, y) + (S^*x, S^*y), \qquad x, y \in D(S^*),$$

converts $D(S^*)$ into a Hilbert space and

$$D(S^*) = D(S)\langle+\rangle N_-(S)\langle+\rangle N_+(S)$$

is the orthogonal sum of the subspaces $D(S)$, $N_-(S)$ and $N_+(S)$. Set

$$N(S) = N_-(S)\langle+\rangle N_+(S).$$

Let Q and Q_+ be the projections onto $N(S)$ and onto $N_+(S)$ in $D(S^*)$. Then $J = 2Q_+ - Q$ is an involution on $N(S)$. The space $N(S)$ becomes a Krein space with respect to the indefinite form

$$[x, y]^S = \langle Jx, y\rangle^S, \qquad x, y \in N(S),$$

and it decomposes into a simultaneously J-orthogonal and orthogonal sum $N(S) = N_-(S) \dotplus N_+(S)$. We have that, for $x \neq 0$,

$$[x, x]^S = 2(x, x) > 0, \qquad x \in N_+(S), \quad \text{and}$$
$$[x, x]^S = -2(x, x) < 0, \qquad x \in N_-(S),$$

so that $N_+(S)$ and $N_-(S)$ are respectively uniformly positive and uniformly negative subspaces.

Now let S implement a derivation δ relative to ρ. Then it is easy to show that, for every A in $D(\delta)$,

$$\rho(A)D(S^*) \subseteq D(S^*) \quad \text{and} \quad \delta(A)|_{D(S^*)} = i(S^*\rho(A) - \rho(A)S^*)|_{D(S^*)}.$$

Set $\|x\|_S^2 = \langle x, x\rangle^S$ for $x \in D(S^*)$. In [3] and [5] it was shown that

$$\|\rho(A)x\|_S^2 \leq (\|\rho(A)\|^2 + \|\delta(A)\|^2)\|x\|_S^2 \leq \|A\|_\delta^2\|x\|_S^2.$$

Therefore $\rho(D(\delta))$ acts as an algebra of bounded operators on $D(S^*)$. Since $D(S)$ is invariant for $\rho(D(\delta))$, we define a representation π_S^δ of $D(\delta)$ on $N(S)$ by the formula:

$$\begin{aligned} &\pi_S^\delta(A) = Q\rho(A)Q, && A \in D(\delta), \text{ i.e.,} \\ &\pi_S^\delta(A)x = Q\rho(A)x, && x \in N(S). \end{aligned} \tag{10}$$

THEOREM 3.1 **[4]**. (i) (*cf.* **[3]**). *The representation* π_S^δ *of the algebra* $D(\delta)$ *on* $N(S)$ *is* J*-symmetric and bounded with respect to the norm* $\|\ \|_\delta$.

(ii) *There is a one-to-one correspondence between closed symmetric* δ*-extensions* T *of* S *and neutral subspaces* L *in* $N(S)$ *invariant for* $\pi_S^\delta : T = S^*|_{D(T)}$, *where* $D(T) = D(S)\langle+\rangle L$.

(iii) *There is a maximal symmetric implementation* T *of* δ *which extends* S. *The representation* π_T^δ *has no neutral invariant subspaces.*

If T is a symmetric extension of S and if $L(T)$ is the neutral subspace in $N(S)$ which corresponds to it, then, using Lemma 13 **[2]**, we obtain that

$$D(T^*) = D(S)\langle+\rangle L(T)^{[\perp]} \quad \text{and} \quad T^* = S^*|_{D(T^*)}, \tag{11}$$

where $L(T)^{[\perp]}$ is the J-orthogonal complement of $L(T)$ in $N(S)$.

Let S be a maximal symmetric implementation of a derivation δ. By Theorem 3.1 (iii), the representation π_S^δ of $D(\delta)$ on $N(S)$ has no neutral invariant subspaces. We shall call the class of all representations of $D(\delta)$ J-equivalent to π_S^δ a *representational index* of δ relative to S and denote it by i_S^δ.

3.2. *Uniqueness of representational indices.* By Theorem 3.1, every derivation δ implemented by a symmetric operator has a maximal symmetric implementation S. In fact, δ always has an infinite set $\mathscr{MS}(\delta)$ of maximal symmetric implementations, since, for example, for every selfadjoint operator B in the commutant $\rho(A)'$, the operator $S + B$ is also a maximal symmetric implementation of δ. In this context the following question arises: under what conditions on δ are all the representations π_S^δ, $S \in \mathcal{MS}(\delta)$, J-equivalent, so that δ has only one representational index?

Let S and T be maximal symmetric implementations of δ. For the case when $T = S + B$, $B \in \rho(A)'$, it was shown in [7] that the representations π_T^δ and π_S^δ are J-equivalent, so that $i_S^\delta = i_T^\delta$. It was also proved there that if S and T are isomorphic, i.e., there is a unitary operator V such that $VS = TV$, and if $V \in \rho(\mathscr{A})'$, then $i_S^\delta = i_T^\delta$.

We shall now prove the main theorem of this section.

THEOREM 3.2. *Let S be a symmetric implementation of a derivation δ and let the representation π_S^δ be finitely Π_-- or Π_+-decomposable. Then for all maximal symmetric δ-extensions T and T_1 of S, the representations π_T^δ and $\pi_{T_1}^\delta$ are J-equivalent, so that $i_T^\delta = i_{T_1}^\delta$.*

Proof. Let T be a maximal symmetric δ-extension of S. By Theorem 3.1, there is a maximal neutral subspace $L(T)$ in $N(S)$ invariant for π_S^δ such that $D(T) = D(S)\langle + \rangle L(T)$. By (11), $D(T^*) = D(S)\langle + \rangle L(T)^{[\perp]}$ where $L(T)^{[\perp]}$ is the J-orthogonal complement of $L(T)$ in $N(S)$. Since $T^* = S^*|_{D(T^*)}$, we have that $\langle x, y\rangle^T = \langle x, y\rangle^S$, $x, y \in D(T^*)$. Therefore $L(T)$ and $N(T)$ are J-orthogonal and orthogonal with respect to $\langle\ ,\ \rangle^S$. Since $D(T^*) = D(T)\langle + \rangle N(T)$,

$$L(T)^{[\perp]} = L(T) \dotplus N(T).$$

Let Q_S and Q_T be the orthoprojections onto $N(S)$ and $N(T)$ in $D(S^*)$ respectively. Then $Q_T \subset Q_S$ and, by (10),

$$\pi_S^\delta(A) = Q_S \rho(A) Q_S \quad \text{and}$$
$$\pi_T^\delta(A) = Q_T \rho(A) Q_T = Q_T \pi_S^\delta(A) Q_T, \qquad A \in D(\delta).$$

It follows from the discussion before Lemma 2.5 that the representation π_T^δ is J-equivalent to the quotient representation $(\pi_S^\delta)^{L(T)}$ of $D(\delta)$ on $L(T)^{[\perp]}/L(T)$. Since π_S^δ is finitely Π-decomposable, it follows from Theorem 2.6 that all quotient representations $(\pi_S^\delta)^L$ of $D(\delta)$ on $L^{[\perp]}/L$, where L are maximal neutral subspaces in $N(S)$ invariant for π_S^δ, are J-equivalent. Therefore all the representations π_T^δ, where T are maximal symmetric δ-extensions of S, are J-equivalent. The theorem is proved.

REMARK. The condition that the representation π_S^δ is finitely Π-decomposable is a strong one. If, however, π_S^δ is not Π-decomposable, there is hardly anything we can say about J-equivalence of the representations π_T^δ, $S \subsetneq T$ and $T \in \mathscr{MS}(\delta)$. Even if π_S^δ is Π-decomposable, but not *finitely* Π-decomposable, S may have an infinite number of maximal δ-extensions T such that the corresponding representations π_T^δ are all not J-equivalent, so that δ has an infinite number of different representational indices i_T^δ, $S \subsetneq T$ (see Example 4). On the other hand, in many interesting cases this condition is fulfilled. If, for example, $k = \min(n_+(S), n_-(S)) < \infty$, then $N(S)$ is a Π_k-space and π_S^δ is Π-decomposable. In the case, studied by Powers [12] and Arveson [1] (see Example 2), when δ is the generator of a semigroup of endomorphisms of $B(H)$ which has a semigroup of intertwining isometries, $k = 0$ and $N(S) = N_+(S)$ is a Hilbert space, so that π_S^δ is Π-decomposable. Below we consider derivations δ_S from C^*-subalgebras $\mathscr{A}_S$ of $B(H)$ into $B(H)$ generated by symmetric operators S on H. We also consider the restrictions δ of this derivation to some C^*-subalgebras of $\mathscr{A}_S$. If $\min(n_+(S), n_-(S)) < \infty$, then the representations π_S^δ are finitely Π-decomposable and Theorem 3.2 holds.

Let T and T_1 be maximal symmetric implementations of δ. If there exists a symmetric implementation S of δ such that $S \subsetneq T$ and $S \subsetneq T_1$ then Theorem 3.2 gives sufficient conditions for the representations π_T^δ and $\pi_{T_1}^\delta$ to be J-equivalent. If, however, such an implementation S does not exist, it becomes extremely difficult to establish whether π_T^δ and $\pi_{T_1}^\delta$ are J-equivalent. Therefore in order to be able to decide whether δ has a unique representational index or not, we have to impose another condition on δ which will allow us to "compare" different maximal symmetric implementations of δ.

DEFINITION. Let δ be a derivation of $\mathscr{A}$ relative to a representation ρ. We say that a symmetric implementation S of δ is *minimal* if, for every symmetric implementation T of δ, there is a selfadjoint operator B in the commutant $\rho(\mathscr{A})'$ such that $S \subseteq T + B$.

THEOREM 3.3. *Let S be a minimal symmetric implementation of a derivation δ of a C^*-algebra $\mathscr{A}$ relative to a representation ρ. If the representation π_S^δ is finitely Π-decomposable, then, for all maximal symmetric implementations T of δ, the representations π_T^δ are J-equivalent, so that δ has a unique representational index.*

Proof. Let R and T be maximal symmetric implementations of δ. Then there are $B, C \in \rho(\mathscr{A})'$ such that $S \subseteq R + B$ and $S \subseteq$

$T+C$. The operators $R+B$ and $T+C$ are also maximal symmetric implementations of δ. By Theorem 3.2, π^{δ}_{R+B} is J-symmetric to π^{δ}_{T+C}. By Theorem 3.6 [7], π^{δ}_{R} and π^{δ}_{R+B} are J-equivalent and π^{δ}_{T} and π^{δ}_{T+C} are J-equivalent. Hence π^{δ}_{R} and π^{δ}_{T} are J-equivalent.

REMARK. The existence of a minimal symmetric implementation is another strong condition imposed on δ. However, without this assumption it is difficult to test the representations π^{δ}_{S}, $S \in \mathscr{MS}(\delta)$, on J-equivalence. In Example 2 below a minimal symmetric implementation does not exist and, therefore, it is not clear whether the representations π^{δ}_{S} and $\pi^{\delta}_{S_1}$, $S, S_1 \in \mathscr{MS}(\delta)$, which correspond to different intertwining semigroups of isometries, are J-equivalent [13]. In many cases the derivations do have minimal symmetric implementations. In [6], for example, it was shown that if $\rho(\mathscr{A})$ contains the ideal $C(H)$ of all compact operators on H, then δ has a minimal symmetric implementation. Example 4 considers a derivation δ from $\mathscr{A}$ into $B(H)$ such that $\mathscr{A}$ does not contain $C(H)$ and that δ has a minimal symmetric implementation.

EXAMPLE 2. Powers [12] and Arveson [1] studied a special case when δ is the generator of a semigroup α_t of endomorphisms of $B(H)$ and when there exists a semigroup $U = \{U(t) \colon t \geq 0\}$ of isometries which intertwine α_t: $U(t)A = \alpha_t(A)U(t)$, $A \in B(H)$. Then δ is a *-derivation from a C^*-subalgebra $\mathscr{A}_\alpha$ of all $A \in B(H)$ such that

$$\|\alpha_t(A) - A\| \to 0 \quad \text{as } t \to 0^+$$

into $B(H)$. If d is the generator of U, then the operator $S = id$ implements δ, it is symmetric, $N_-(S) = \{0\}$ and $N(S) = N_+(S)$ is a Hilbert space. Therefore S is a maximal symmetric implementation of δ and the *-representation π^{δ}_{S} is Π_--decomposable. Powers and Price [13] showed that if $\{V(t) \colon t \geq 0\}$ is another semigroup of isometries which intertwine α_t and if d_1 is its generator, then $D(d) \cap D(d_1) = \{0\}$. In this case, obviously, δ has no minimal symmetric implementations and, therefore, there is no reason to think that the representational indices i^{δ}_{S} and $i^{\delta}_{S_1}$, where $S_1 = id_1$, are equal. From the above remark it also follows that $\mathscr{A}_\alpha$ does not contain $C(H)$.

We shall now consider derivations δ which have minimal symmetric implementations S such that the representations π^{δ}_{S} are finitely Π-decomposable, so that Theorem 3.3 holds.

Let S be a densely defined symmetric operator on a Hilbert space H. Set

$$\mathscr{B}_S = \{A \in B(H) : AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S) \text{ and } (SA - AS)|_{D(S)} \text{ extends to a bounded operator}\}.$$

Then $\mathscr{B}_S$ is a *-algebra. For every $x, y \in H$, we denote by $x \otimes y$ the rank-1 operator $z \to (z, x)y$. Then $(x \otimes y)^* = y \otimes x$ and if $x, y \in D(S)$, then $x \otimes y \in \mathscr{B}_S$. By $\mathscr{A}_S$ we denote the norm closure of $\mathscr{B}_S$. Then $\mathscr{A}_S$ is a C^*-algebra and it contains $C(H)$. The operator S defines a closed *-derivation from $\mathscr{A}_S$ into $B(H)$

$$\delta_S(A)|_{D(S)} = i(SA - AS)|_{D(S)}$$

and $D(\delta_S) = \mathscr{B}_S$. Since $C(H) \subseteq \mathscr{A}_S$, δ_S has a minimal implementation. In fact, S is a minimal implementation of δ_S. In order to prove this we assume that T also implements δ_S. Then for all $x, y \in D(s)$,

$$(x \otimes y)D(T) \subseteq D(T),$$

so that $D(S) \subseteq D(T)$. We also have that for all $z \in D(S)$,

$$(S - T)(x \otimes y)z = (x \otimes y)(S - T)z.$$

Therefore $T|_{D(S)} = (S + \lambda I)|_{D(S)}$, $\lambda \in \mathbb{C}$, so that S is a minimal implementation of δ_S.

Let $\mathscr{A}$ be a unital C^*-subalgebra of $\mathscr{A}_S$ which contains $C(H)$ and such that $\mathscr{B}_S \cap \mathscr{A}$ is dense in $\mathscr{A}$. Then δ_S generates a derivation $\delta = \delta_S|_{\mathscr{A}}$ on $\mathscr{A}$ and $D(\delta) = \mathscr{B}_S \cap \mathscr{A}$. Since all rank-1 operators $x \otimes y$, $x, y \in D(S)$, belong to $D(\delta)$, the operator S is still a minimal implementation of δ.

If $n_-(S) = 0$ or $n_+(S) = 0$, then S has no symmetric extensions at all and, therefore, S is a maximal symmetric implementation of δ_S and of any derivation δ generated by δ_S considered above. Another example of a symmetric operator S, which is also a maximal symmetric implementation of δ_S, was given in [7]:

$$S = i\frac{d}{dx} \text{ on } L_2(0, a),\quad a < \infty, \quad \text{and} \quad n_-(S) = n_+(S) = 1.$$

In general, however, we do not know whether S is a maximal implementation of δ_S or not. Even if S is a maximal symmetric implementation of δ_S, it is not necessarily a maximal symmetric implementation of a derivation $\delta = \delta_S|_{\mathscr{A}}$ generated by δ_S on a C^*-subalgebra $\mathscr{A}$ of $\mathscr{A}_S$ considered above. If $\min(n_+(S), n_-(S)) < \infty$, then $N(S)$ is a Π_k-space, so that the representation π_S^δ of $D(\delta)$ is finitely Π-decomposable. Therefore from Theorem 3.3 we obtain the following theorem.

THEOREM 3.4. *Let* $\min(n_+(S), n_-(S)) < \infty$. *Let* $\mathscr{A}$ *be a unital* C^**-subalgebra of* $\mathscr{A}_S$ *such that* $C(H) \subset \mathscr{A}$ *and* $\mathscr{B}_S \cap A$ *is dense in* $\mathscr{A}$. *Let* $\delta = \delta_S|_{\mathscr{A}}$ *and* $D(\delta) = \mathscr{B}_S \cap \mathscr{A}$. *Then for all maximal symmetric implementations* T *of* δ, *the representations* π_T^δ *of* $D(\delta)$ *are* J*-equivalent, so that there exists a unique representational index of* δ.

The following example illustrates Theorems 3.3 and 3.4.

EXAMPLE 3. Let $\min(n_+(S), n_-(S)) < \infty$, let $\mathscr{A} = C(H) + \mathbb{C}I_H$ and let $\delta = \delta_S|_{\mathscr{A}}$. Then $D(\delta) = \mathscr{B}_S \cap C(H) + \mathbb{C}I_H$. Assume that $n_-(S) \le n_+(S)$ and let T be a maximal symmetric implementation of δ. For the case when $n_+(S) < \infty$, it was proved in **[4]**, and, for the case when $n_+(S) = \infty$, it was proved in **[10]** that

$$n_-(T) = 0, \quad n_+(T) = n_+(S) - n_-(S) \quad \text{and} \quad \operatorname{Ker} \pi_S^\delta = \mathscr{B}_S \cap C(H).$$

From this it follows immediately that all representations π_T^δ, $T \in \mathscr{M}\mathscr{S}(\delta)$, of $D(\delta)$ are *-equivalent (since all the deficiency spaces $N(T) = N_+(T)$ are Hilbert spaces, J-equivalence coincides with *-equivalence). Thus δ has a unique representational index which fits well with Theorems 3.3 and 3.4.

The following example shows that if the representation π_S^δ (S is a minimal symmetric implementation of δ) is Π-decomposable but not *finitely* Π-decomposable, then δ may have an infinite number of distinct representational indices.

EXAMPLE 4. Let S_1 and S_2 be symmetric operators on H_1 and H_2 respectively. Set $H = H_1 \oplus H_2$ and $S = S_1 \oplus S_2$. Then $S^* = S_1^* \oplus S_2^*$,

$$D(S) = D(S_1) \oplus D(S_2) \quad \text{and} \quad D(S^*) = D(S_1^*) \oplus D(S_2^*).$$

For $x = x_1 + x_2$ and $y = y_1 + y_2$, $x_i, y_i \in D(S_i^*)$, $i = 1, 2$, let

$$\langle x, y\rangle^S = (x, y) + (S^*x, S^*y) = \langle x_1, y_1\rangle^{S_1} + \langle x_2, y_2\rangle^{S_2}.$$

Therefore

$$N(S) = N(S_1)\langle+\rangle N(S_2) \quad \text{and} \quad N_\pm(S) = N_\pm(S_1)\langle+\rangle N_\pm(S_2).$$

Let J_i, $i = 1, 2$, be the involutions on $N(S_i)$, as in §3.1, and let $J = J_1 \oplus J_2$. Then J is an involution on $N(S)$ and $N(S_1)$ and

$N(S_2)$ are J-orthogonal with respect to the form

$$[x, y]^S = \langle Jx, y\rangle^S.$$

Let $\mathscr{A} = (C(H_1)\oplus C(H_2))+\mathbb{C}I_H$. The operator S defines a derivation $\delta = \delta_S|_{\mathscr{A}}$ on $\mathscr{A}$ where

$$D(\delta) = \mathscr{B}_S \cap \mathscr{A} = [(B_{S_1} \cap C(H_1)) \oplus (\mathscr{B}_{S_2} \cap C(H_2))] + \mathbb{C}I_H$$

and

$$\delta(A)|_{D(S)} = i(SA - AS)|_{D(S)}, \qquad A \in D(\delta).$$

Since all operators $(x_1 \otimes y_1) \oplus (x_2 \otimes y_2)$, $x_i, y_i \in D(S_i)$, belong to $D(\delta)$, the operator S is a minimal symmetric implementation of δ. Assume now that

$$n_+(S_1) = n_-(S_2) = \infty \quad \text{and} \quad n_-(S_1) = n_+(S_2) = 0,$$

so that

$$N_+(S) = N_+(S_1) = N(S_1) \quad \text{and} \quad N_-(S) = N_-(S_2) = N(S_2).$$

Let $A_i \in \mathscr{B}_{S_i} \cap C(H_i)$. Then

$$\pi_S^\delta(A_1 \oplus A_2) = \pi_S^\delta(A_1)\langle+\rangle\pi_S^\delta(A_2).$$

Therefore $\pi_S^\delta|_{\mathscr{B}_{S_i}\cap C(H_i)}$ are $*$-representations of $\mathscr{B}_{S_i}\cap C(H_i)$ on Hilbert spaces $N(S_i)$ respectively. Hence they extend to $*$-representations π_i of $C(H_i)$ on $N(S_i)$. Let $x_i, y_i \in D(S_i)$. For every $z_i \in D(S_i^*)$

$$(x_i \otimes y_i)z_i = (z_i, x_i)y_i \in D(S_i).$$

Therefore

$$\pi_S^\delta((x_1 \otimes y_1) \otimes (x_2 \otimes y_2)) = 0.$$

Hence $\pi_i(x_i \otimes y_i) = 0$, for all $x_i, y_i \in D(S_i)$, and, therefore, $\pi_i = 0$. Thus

$$\pi_S^\delta((A_1 \oplus A_2) + tI_H) = tI_{N(S)}, \qquad t \in \mathbb{C},$$

for all $A_i \in \mathscr{B}_{S_i} \cap C(H_i)$. We shall now proceed as in Example 1. Let $\{e_i\}_{i=1}^\infty$ be a basis in $N_+(S)$ and let $\{f_j\}_{j=1}^\infty$ be a basis in $N_-(S)$. For $0 \le n < \infty$, set

$$L_n = \{e_i + f_{i+n} : 1 \le i < \infty\}.$$

Then L_n are maximal neutral subspaces invariant for π_S^δ and

$$L_n^{[\perp]} = L_n[+]\mathscr{L}_n, \text{ where } \mathscr{L}_n = \{f_k : 1 \le k \le n\}.$$

By Theorem 3.1, for every L_n, there is a maximal symmetric implementation T_n of δ. It follows from the discussion in Theorem 3.2

that the representations $\pi^{\delta}_{T_n}$ are J-equivalent to the quotient representations $(\pi^{\delta}_{S})^{L_n}$ of $D(\delta)$ on $L_n^{[\perp]}/L_n$. Since $\dim(L_n^{[\perp]}/L_n) = \dim \mathscr{L}_n = n$, all $\pi^{\delta}_{T_n}$ are not J-equivalent. Therefore δ has an infinite number of distinct representational indices.

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Received October 11, 1991 and in revised form April 13, 1992.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 162 No. 1 January 1994

0030-8730(1994)162:1;1-K