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For each prime p and each $n \geq 1$ ($n \geq 2$ if $p = 2$), examples are constructed of a Galois extension K/F whose Galois group has exponent p^n and a central simple F-algebra A of exponent p which is split by K but is not in the Dec group of K/F.

1. Introduction. Let K/F be an abelian Galois extension of fields, and let $G = \mathscr{G}(K/F)$. Let $G = G_1 \times G_2 \times \cdots \times G_k$ be a direct sum decomposition of G into cyclic groups, with $G_i = \langle \sigma_i \rangle$ ($i = 1, \ldots, k$). Let F_i be the fixed field of $G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_k$ ($i = 1, \ldots, k$). Thus, the F_i are cyclic Galois extensions of F, with Galois group isomorphic to G_i. The group $\mathrm{Dec}(K/F)$ is defined as the subgroup of $\mathrm{Br}(K/F)$ generated by the subgroups $\mathrm{Br}(F_i/F)$ ($i = 1, \ldots, k$). This group was introduced by Tignol (**[T1]**), where he shows that $\mathrm{Dec}(K/F)$ is independent of the choice of the direct sum decomposition of G. If p is a prime, we will write ${}_{p^n}\mathrm{Br}(K/F)$ and ${}_{p^n}\mathrm{Dec}(K/F)$ for the subgroups of $\mathrm{Br}(K/F)$ and $\mathrm{Dec}(K/F)$ consisting of all elements with exponent dividing p^n.

A key issue in several past constructions of division algebras has been the non-triviality of the factor group ${}_p\mathrm{Br}(K/F)/{}_p\mathrm{Dec}(K/F)$ for suitable abelian extensions K/F. For instance, the Amitsur-Rowen-Tignol construction of an algebra of index 8 with involution with no quaternion subalgebra (**[ART]**) depends crucially on the existence of a triquadratic extension K/F for which ${}_2\mathrm{Br}(K/F) \neq {}_2\mathrm{Dec}(K/F)$. Similarly, the constructions of indecomposable algebras of exponent p by Tignol (**[T2]**) and Jacob (**[J]**) also depend on the existence of an (elementary) abelian extension K/F for which ${}_p\mathrm{Br}(K/F) \neq {}_p\mathrm{Dec}(K/F)$.

The extension fields K/F that occur in these examples above are all of exponent p, and it is an interesting question whether there exist abelian extensions K/F whose Galois groups have arbitrarily high (p-power) exponents for which the factor group ${}_p\mathrm{Br}(K/F)/{}_p\mathrm{Dec}(K/F)$ is non-trivial. The purpose of this paper is to show that for each $n \geq 1$ ($n \geq 2$ if $p = 2$), there exists an abelian extension K/F with Galois group $\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (and thus, of exponent p^n) and an algebra $A \in {}_p\mathrm{Br}(K/F)$ such that $A \notin {}_p\mathrm{Dec}(K/F)$. (Note that if K/F is an

$\mathbb{Z}/2\times\mathbb{Z}/2$ extension, then ${}_2\operatorname{Br}(K/F)$ is always equal to ${}_2\operatorname{Dec}(K/F)$, see **[T3]** for instance.)

Our field F will be the rational function field in 3 variables over a field F_0 of characteristic 0 that contains sufficiently many roots of unity. (For instance, F_0 may be algebraically closed.) Our algebras will in fact be generalizations of the example given by Tignol in **[T2]**. Moreover, we will prove that for A, K, and F as above, $A\otimes_F L \notin {}_p\operatorname{Dec}(K\cdot L/L)$ for any finite degree extension L/F with $p\nmid [L:F]$.

The special case $n=2$ (and p odd) of these computations was done in **[Se1]**, where the result was used to construct non-elementary abelian crossed products of index p^3 and exponent p^2.

We remark that using different techniques, Rowen and Tignol (**[RT]**) have shown that if the ground field is assumed to only contain a primitive p^sth root of unity but *not* a primitive p^{s+1}th root of unity for some $s\geq 1$, then examples of non-trivial factor groups

$$ {}_p\operatorname{Br}(K/F)/{}_p\operatorname{Dec}(K/F)$$

exist for suitable abelian extensions K/F whose Galois groups have arbitrarily large (p-power) exponents. Using ultraproducts (**[R]**), their example can be extended to also cover the case where the ground field contains all primitive p^ith roots of unity ($i=1,2,\dots$).

2. p-adic valuations on rational function fields. Let p be a prime, which, for now, can be either odd or even. Let F_0 be a field of characteristic 0. The subfield $\mathbb{Q}$ of F_0 has a standard valuation $v:\mathbb{Q}\to\mathbb{Z}$ obtained by writing any non-zero element in $\mathbb{Q}$ as $p^n a/b$, where n, a, and b are integers, and p is relatively prime to a and b, and defining $v(p^n a/b)=n$. We will refer to any valuation on F_0 that extends this distinguished valuation on $\mathbb{Q}$ as a *p-adic valuation.* Since the residue field of $\mathbb{Q}$ under v is $\mathbb{Z}/p\mathbb{Z}$, the residue of F_0 under any p-adic valuation is of characteristic p.

Now let $F=F_0(x_1,x_2,\dots,x_k)$ be the rational function field over F_0 in k indeterminates ($k\geq 1$), and let v be a fixed p-adic valuation on F_0. Then v admits an extension w to F defined as follows: for any polynomial $f\in F_0[x_1,x_2,\dots,x_k]$, $w(f)$ is the minimum of the values of the coefficients, and for f and g in $F_0[x_1,x_2,\dots,x_k]$, $w(f/g)=w(f)-w(g)$. (It is easy to check that w is indeed a valuation on F.) It can be shown that the residues $\overline{x}_i$ of the x_i ($i=1,\dots,k$) are algebraically independent over the residue $\overline{F}_0$ of F_0; and that, moreover, $\overline{F}$ is precisely the rational function field $\overline{F}_0(\overline{x_1},\overline{x_2},\dots,\overline{x_k})$. (It is also clear from the definition of w that

$\Gamma_F = \Gamma_{F_0}$.) We will refer to w as the *standard extension of* v *to* F. Also, we will abuse notation and continue to write x_i for the residues $\overline{x}_i$.

REMARK 2.1. Furthermore, it can be shown that w is the *unique* extension of v to F with the property that the values of the x_i are 0, and the residues of the x_i are algebraically independent over $\overline{F_0}$. (See [**B**, §10, Proposition 2].)

The following is well known, but we include a proof here for convenience.

LEMMA 2.2. *Let* p *be any prime, and let* F *be a field of characteristic* 0. *Let* v *be a p-adic valuation on* F. *Let* $K = F(f^{1/p})$, *where* $f \notin F^{*p}$, *and* $v(f) = 0$. *Assume that* $f = f_0^p + \pi f_1 + \delta$, *where* $v(f_0) = v(f_1) = 0$, $0 < v(\pi) < (p/(p-1))v(p)$, *and* $v(\delta) > v(\pi)$. *Assume, too, that* $\overline{f_1} \notin \overline{F}^p$, *and that there exists* $\theta \in F^*$ *such that* $\theta^p = \pi$. *Then* v *extends uniquely to* K, *and* $\overline{K} = \overline{F}(\overline{f_1}^{1/p})$.

Proof. Let $r \in K^*$ satisfy $r^p = f$, and let $s = (r - f_0)/\theta$. Then $s + (f_0/\theta) = (r/\theta)$, so s satisfies

$$\left(s + \frac{f_0}{\theta}\right)^p = \frac{f_0^p + \pi f_1 + \delta}{\theta^p}. \tag{1}$$

Expanding the left-hand side of (1) and noting that $\theta^p = \pi$, we find

$$s^p + \sum_{i=1}^{p-1} \binom{p}{i} s^i \left(\frac{f_0}{\theta}\right)^{p-i} = f_1 + \left(\frac{\delta}{\theta^p}\right). \tag{2}$$

Now for $i = 1, \dots, p-1$, $v(\binom{p}{i}) = v(p)$, while $v(\theta^{p-i}) \le v(\theta^{p-1}) = v(\pi^{(p-1)/p}) < v(p)$. (The last inequality is because $v(\pi) < (p/(p-1))v(p)$.) From this, as well as the fact that $v(f_0) = 0$, we find that each of the expressions $\binom{p}{i}(f_0/\theta)^{p-i}$ $(i = 1, \dots, p-1)$ has positive value. It follows that for any extension w of v from F to K, if $w(s) < 0$, then the left-hand side of (2) would have the same value as s^p. (Here we use the fact that if $w(a) < w(b)$, then $w(a+b) = w(a)$.) Since this contradicts the fact that the value of the right-hand side of (2) is 0 (note that $v(f_1) = 0$, while $v(\delta/\theta^p) > 0$), we must have $w(s) \ge 0$. Similarly, if $w(s) > 0$, then from $w(a+b) \ge \min(w(a), w(b))$, it follows that the left-hand side of (2) must have positive value. Hence $w(s) = 0$. Taking the residues of each term in (2) and noting again that all terms except s^p and f_1 have positive value, we find $\overline{s}^p = \overline{f}_1$. Thus $\overline{K} \supset \overline{F}(\overline{f_1}^{1/p})$.

Since $\overline{f_1} \notin \overline{F}^p$, and since $[K : F] = p$, we find by the fundamental inequality ([**E**, Corollary 17.5]) that w is unique, and $\overline{K} = \overline{F}(\overline{f_1}^{1/p})$. $\square$

Now let F_0 be a field of characteristic 0. We will assume that F_0 contains p^{1/p^i} for all i $(i = 1, 2, \dots)$. Let F be the rational function field $F_0(x_1, x_2, y)$. For each n $(n \geq 0)$, let

$$\phi_n = (x_1^{p^n} - y^{p^n})(x_2^{p^n} - y^{p^n}). \tag{3}$$

Let $H_n = F(\phi_n^{1/p})$. Let v be the standard extension of any p-adic valuation on F_0 to F. The manner in which v extends from F to H_n will be crucial to our Dec results, and the rest of §2 is devoted to this topic.

First, some notation. For p odd, and $i = 1, 2, \dots, p-1$, let

$$\lambda_i = \frac{(-1)^{p-i}}{p}\binom{p}{i} \tag{4}$$

(so each λ_i is an integer). For p odd, again, define $g_n(x, y) \in \mathbb{Z}[x, y]$ $(n = 0, 1, 2, \dots)$ by

$$g_n(x, y) = \sum_{i=1}^{p-1} \lambda_i (x^{p^n})^i (y^{p^n})^{p-i}, \tag{5}$$

so

$$(x^{p^n} - y^{p^n})^p = x^{p^{n+1}} - y^{p^{n+1}} + p g_n(x, y).$$

Now for p odd, define $h_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ $(n = 0, 1, 2, \dots)$ by

$$h_n(x_1, x_2, y) = (x_1^{p^n} - y^{p^n})^p g_n(x_2, y) + (x_2^{p^n} - y^{p^n})^p g_n(x_1, y), \tag{6}$$

and for $p = 2$, define $h_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ $(n = 0, 1, 2, \dots)$ by

$$h_n(x_1, x_2, y) = (x_1^{2^n} + y^{2^n})^2 x_2^{2^n} y^{2^n} + (x_2^{2^n} + y^{2^n})^2 x_1^{2^n} y^{2^n}. \tag{7}$$

REMARK 2.3. We will abuse notation and continue to write g_n and h_n for the images of g_n and h_n in $\mathbb{Z}/p\mathbb{Z}[x, y]$ and $\mathbb{Z}/p\mathbb{Z}[x_1, x_2, y]$ (respectively).

The special case $n = 1$ (and p odd) of the following was proved in [**T2**, Lemma 3.7].

PROPOSITION 2.4. *For every prime p and for all n $(n \geq 1)$, v extends uniquely from F to H_n, and $\overline{H_n} = \overline{F}(h_0(x_1, x_2, y)^{1/p})$.*

Before proving Proposition 2.4, we need some further notation, as well as some easy lemmas.

For $p = 2$, define $e_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ (for $n \geq 1$) by

$$e_n(x_1, x_2, y) = y^{2^n}(x_1^{2^n} + x_2^{2^n}), \tag{8}$$

and $\psi_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ (for $n \geq 0$) by

$$\psi_n(x_1, x_2, y) = (x_1^{2^n} + y^{2^n})(x_2^{2^n} + y^{2^n}). \tag{9}$$

For $n \in \mathbb{Z}$ $(n \geq 1)$, define $\alpha_n \in \mathbb{Q}$ by

$$\alpha_n = \begin{cases} 1, & \text{if } n = 1, \\ 1 + 1/p + 1/p^2 + \cdots + 1/p^{n-1}, & \text{if } n > 1. \end{cases} \tag{10}$$

Finally, for any $k \in \mathbb{Q}$, abbreviate the phrase "terms of value at least $v(p^k)$" by $[[p^k]]$.

REMARK 2.5. Just as with g_n and h_n, we will abuse notation and continue to write e_n for the image of e_n in $\mathbb{Z}/2\mathbb{Z}[x_1, x_2, y]$.

LEMMA 2.6. *Let f, g, f_1, and g_1 be polynomials in $\mathbb{Z}[x_1, x_2, y]$. Then, with respect to the restriction of v to $\mathbb{Q}(x_1, x_2, y)$ (i.e., the standard extension of the p-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}(x_1, x_2, y)$),*

1. *If $f = g + [[p]]$, and $f_1 = g_1 + [[p]]$, then $f + f_1 = g + g_1 + [[p]]$ and $ff_1 = gg_1 + [[p]]$.*
2. *$(f+g)^p = f^p + g^p + [[p]]$.*
3. *Let $k \geq 1$, and suppose*

$$f = \sum c_{i_1, i_2, i_3}(x_1^{p^k})^{i_1}(x_2^{p^k})^{i_2}(x_3^{p^k})^{i_3},$$

for some $c_{i_1, i_2, i_3} \in \mathbb{Z}$. Define $f^{1/p} \in \mathbb{Z}[x_1, x_2, y]$ by

$$f^{1/p} = \sum c_{i_1, i_2, i_3}(x_1^{p^{k-1}})^{i_1}(x_2^{p^{k-1}})^{i_2}(x_3^{p^{k-1}})^{i_3}.$$

Then $f = (f^{1/p})^p + [[p]]$.

Proof. Note that the values of f, g, f_1, and g_1 are non-negative. (1) and (2) are now elementary. (3) follows from (2) along with the fact that $a^p \equiv a \pmod p$ for any $a \in \mathbb{Z}$. □

Lemma 2.7. *With respect to the restriction of* v *to* $\mathbb{Q}(x_1, x_2, y)$,

1. *For* $n \geq 1$ *and for all* p, $h_n = h_{n-1}^p + [[p]]$, *and for* $n \geq 2$ *and* $p = 2$, $e_n = e_{n-1}^2 + [[2]]$.
2. *For* $n \geq 1$ *and* p *odd,* $\phi_n = \phi_{n-1}^p - ph_{n-1} + [[p^2]]$.
3. *For* $n \geq 1$ *and* $p = 2$, $\phi_n = \psi_n - 2e_n$, *and* $\psi_n = \psi_{n-1}^2 - 2h_{n-1} + [[4]]$ (*so* $\phi_n = \psi_{n-1}^2 - 2(h_{n-1} + e_n) + [[4]]$).

Proof. (1) follows from the definitions of h_n and e_n and Lemma 2.6. For instance, for p odd (and $n \geq 1$) we have

$$(x_1^{p^n} - y^{p^n})^p = ((x_1^{p^{n-1}} - y^{p^{n-1}})^p + [[p]])^p = (x_1^{p^{n-1}} - y^{p^{n-1}})^{p^2} + [[p]].$$

Also,

$$\begin{aligned} g_n(x_2, y) &= \sum_{i=1}^{p-1} \lambda_i (x_2^{p^n})^i (y^{p^n})^{p-i} \\ &= \left(\sum_{i=1}^{p-1} \lambda_i (x_2^{p^{n-1}})^i (y^{p^{n-1}})^{p-i} \right)^p + [[p]] \\ &= (g_{n-1}(x_2, y))^p + [[p]]. \end{aligned}$$

Since similar relations hold for $(x_2^{p^n} - y^{p^n})^p$ and $g_n(x_1, y)$, we find

$$\begin{aligned} h_n &= (x_1^{p^{n-1}} - y^{p^{n-1}})^{p^2} (g_{n-1}(x_2, y))^p \\ &\quad + (x_2^{p^{n-1}} - y^{p^{n-1}})^{p^2} (g_{n-1}(x_1, y))^p + [[p]] \\ &= ((x_1^{p^{n-1}} - y^{p^{n-1}})^p g_{n-1}(x_2, y) \\ &\qquad + (x_2^{p^{n-1}} - y^{p^{n-1}})^p g_{n-1}(x_1, y))^p + [[p]] \\ &= h_{n-1}^p + [[p]]. \end{aligned}$$

The proof for $p = 2$ is similar. For (2), we have

$$\begin{aligned} \phi_n &= (x_1^{p^n} - y^{p^n})(x_2^{p^n} - y^{p^n}) \\ &= [(x_1^{p^{n-1}} - y^{p^{n-1}})^p - p g_{n-1}(x_1, y)] \\ &\quad \cdot [(x_2^{p^{n-1}} - y^{p^{n-1}})^p - p g_{n-1}(x_2, y)] \\ &= [(x_1^{p^{n-1}} - y^{p^{n-1}})(x_2^{p^{n-1}} - y^{p^{n-1}})]^p \\ &\quad - p[(x_1^{p^{n-1}} - y^{p^{n-1}})^p g_{n-1}(x_2, y) \\ &\qquad + (x_2^{p^{n-1}} - y^{p^{n-1}})^p g_{n-1}(x_1, y)] + [[p^2]] \\ &= \phi_{n-1}^p - p h_{n-1} + [[p^2]]. \end{aligned}$$

As for (3),

$$\begin{aligned}\phi_n &= (x_1^{2^n} - y^{2^n})(x_2^{2^n} - y^{2^n}) \\ &= (x_1^{2^n} + y^{2^n} - 2y^{2^n})(x_2^{2^n} + y^{2^n} - 2y^{2^n}) \\ &= (x_1^{2^n} + y^{2^n})(x_2^{2^n} + y^{2^n}) - 2y^{2^n}(x_1^{2^n} + x_2^{2^n}) \\ &= \psi_n - 2e_n.\end{aligned}$$

Also,

$$\begin{aligned}\psi_n &= (x_1^{2^n} + y^{2^n})(x_2^{2^n} + y^{2^n}) \\ &= [(x_1^{2^{n-1}} + y^{2^{n-1}})^2 - 2x_1^{2^{n-1}}y^{2^{n-1}}][(x_2^{2^{n-1}} + y^{2^{n-1}})^2 - 2x_2^{2^{n-1}}y^{2^{n-1}}] \\ &= [(x_1^{2^{n-1}} + y^{2^{n-1}})(x_2^{2^{n-1}} + y^{2^{n-1}})]^2 \\ &\quad - 2[(x_1^{2^{n-1}} + y^{2^{n-1}})^2 x_2^{2^{n-1}}y^{2^{n-1}} + (x_2^{2^{n-1}} + y^{2^{n-1}})^2 x_1^{2^{n-1}}y^{2^{n-1}}] + [[4]] \\ &= \psi_{n-1}^2 - 2h_{n-1} + [[4]].\end{aligned}$$

□

LEMMA 2.8. *For all p and for all $k \geq 0$, $\alpha_{k+1} < \alpha_2 + 1/p$.*

Proof. Since $\alpha_1 < \alpha_2 < \alpha_2 + 1/p$, we may assume $k > 2$. Now $\alpha_{k+1} = 1 + 1/p + \cdots + 1/p^k$ and $\alpha_2 = 1 + 1/p$, so it is sufficient to prove that $1/p^2 + \cdots + 1/p^k < 1/p$. Multiplying both sides by p, we need to prove that $1/p + \cdots + 1/p^{k-1} < 1$. But this is clear, since

$$\begin{aligned}1/p + \cdots + 1/p^{k-1} &= 1/p(1 + 1/p + \cdots + 1/p^{k-2}) \\ &< 1/p(1 + 1/p + 1/p^2 + \cdots) \\ &= 1/(p-1) \leq 1.\end{aligned}$$

□

Proof of Proposition 2.4. We divide the proof according to whether p is odd or whether $p = 2$.

Case 1 (*Odd* p). If $n = 1$, this follows from Lemmas 2.7 and 2.2. For, by Lemma 2.7, $\phi_1 = \phi_0^p - ph_0 + \delta$, for some $\delta \in \mathbb{Z}[x_1, x_2, y]$ with $v(\delta) \geq v(p^2)$. By assumption, $p^{1/p} \in F_0$. Clearly, $-h_0 \notin \overline{F}^p = \overline{F_0}^p(x_1^p, x_2^p, y^p)$. Thus, by Lemma 2.2, v extends uniquely to H_1, and $\overline{H_1} = \overline{F}((-h_0)^{1/p}) = \overline{F}(h_0^{1/p})$.

In general, for $n > 1$, we have by Lemma 2.7,

$$\begin{aligned}
(11)\quad \phi_n &= \phi_{n-1}^p - ph_{n-1} + [[p^2]] \\
&= \phi_{n-1}^p - p(h_{n-2}^p + [[p]]) + [[p^2]] \\
&= \phi_{n-1}^p - ph_{n-2}^p + [[p^2]] \\
&= \phi_{n-1}^p - (p^{1/p}h_{n-2})^p + [[p^2]] \\
&= (\phi_{n-1} - p^{1/p}h_{n-2})^p - pg_0(\phi_{n-1}, p^{1/p}h_{n-2}) + [[p^2]] \\
&= (\phi_{n-1} - p^{1/p}h_{n-2})^p - p^{\alpha_2}\phi_{n-1}^{p-1}h_{n-2} + [[p^{\alpha_2+1/p}]].
\end{aligned}$$

(For the last equality, note that

$$pg_0(\phi_{n-1}, p^{1/p}h_{n-2}) = p(p^{1/p})h_{n-2}\phi_{n-1}^{p-1} + \binom{p}{2}(p^{1/p}h_{n-2})^2\phi_{n-1}^{p-2} + \cdots .$$

Also, note that $p^{1+1/p} = p^{\alpha_2}$, and $p^{1+2/p} = p^{\alpha_2+1/p}$. Finally, note that since $p \geq 3$, $1 + 2/p < 2$.)

Claim. For $2 \leq k \leq n-1$, if

$$S_k = a_k^p - p^{\alpha_k}\phi_{n-1}^{p-1}\phi_{n-2}^{p-1}\cdots\phi_{n-k+1}^{p-1}h_{n-k} + [[p^{\alpha_2+1/p}]],$$

for some $a_k \in F$ with $a_k = \phi_{n-1} + [[p^{1/p}]]$, then

$$S_k = a_{k+1}^p - p^{\alpha_{k+1}}\phi_{n-1}^{p-1}\phi_{n-2}^{p-2}\cdots\phi_{n-k}^{p-1}h_{n-k-1} + [[p^{\alpha_2+1/p}]],$$

for some $a_{k+1} \in F$ with $a_{k+1} = \phi_{n-1} + [[p^{1/p}]]$.

Proof of claim. For, by Lemma 2.7, $\phi_j = \phi_{j-1}^p + [[p]]$ and $h_j = h_{j-1}^p + [[p]]$ for all $j \geq 1$, so

$$\begin{aligned}
S_k &= a_k^p - p^{\alpha_k}(\phi_{n-2}^p + [[p]])^{p-1}(\phi_{n-3}^p + [[p]])^{p-1} \\
&\quad \cdots(\phi_{n-k}^p + [[p]])^{p-1}(h_{n-k-1}^p + [[p]]) + [[p^{\alpha_2+1/p}]] \\
&= a_k^p - p^{\alpha_k}(\phi_{n-2}^{p(p-1)} + [[p]])(\phi_{n-3}^{p(p-1)} + [[p]]) \\
&\quad \cdots(\phi_{n-k}^{p(p-1)} + [[p]])(h_{n-k-1}^p + [[p]]) + [[p^{\alpha_2+1/p}]] \\
&= a_k^p - p^{\alpha_k}(\phi_{n-2})^{p(p-1)}\cdots(\phi_{n-k})^{p(p-1)}h_{n-k-1}^p \\
&\quad + [[p^{\alpha_k+1}]] + [[p^{\alpha_2+1/p}]] \\
&= a_k^p - (p^{\alpha_k/p}(\phi_{n-2})^{(p-1)}\cdots(\phi_{n-k})^{(p-1)}h_{n-k-1})^p + [[p^{\alpha_2+1/p}]] \\
&\qquad\qquad\qquad\qquad (\text{as } \alpha_2 + 1/p < \alpha_k + 1) \\
&= (a_k - p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})^p \\
&\quad - pg_0(a_k, p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1}) + [[p^{\alpha_2+1/p}]].
\end{aligned}$$

Expanding $pg_0(a_k, p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})$ and considering the first two terms of lowest value, we find

$$S_k = (a_k - p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})^p \\ - p^{1+\alpha_k/p}a_k^{p-1}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1} + [[p^{1+(2\alpha_k/p)}]] + [[p^{\alpha_2+1/p}]].$$

Now $1+\alpha_k/p = \alpha_{k+1}$. Also, $1+(2\alpha_k/p) = \alpha_{k+1}+\alpha_k/p > \alpha_2 + 1/p$ (as $\alpha_{k+1} > \alpha_2$ and $\alpha_k > 1$ when $k \geq 2$). Thus,

$$S_k = (a_k - p^{\alpha_k/p}\phi_{N-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})^p \\ - p^{\alpha_{k+1}}a_k^{p-1}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1} + [[p^{\alpha_2+1/p}]].$$

Now recalling that $a_k = \phi_{n-1}+[[p^{1/p}]]$, we find $a_k^{p-1} = \phi_{n-1}^{p-1}+[[p^{1/p}]]$. Hence,

$$S_k = (a_k - p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})^p \\ - p^{\alpha_{k+1}}\phi_{n-1}^{p-1}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1} + [[p^{\alpha_{k+1}+1/p}]] + [[p^{\alpha_2+1/p}]].$$

Since $\alpha_{k+1} > \alpha_2$, $\alpha_{k+1}+1/p > \alpha_2+1/p$. Thus,

$$S_k = (a_k - p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})^p \\ - p^{\alpha_{k+1}}\phi_{n-1}^{p-1}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1} + [[p^{\alpha_2+1/p}]].$$

Take $a_{k+1} = (a_k - p^{\alpha_k/p}\phi_{n-2}^{p-1}\cdots\phi_{n-k}^{p-1}h_{n-k-1})$. Since $a_k = \phi_{n-1} + [[p^{1/p}]]$ and since $1/p < \alpha_k/p$ (as $k \geq 2$), $a_{k+1} = \phi_{n-1} + [[p^{1/p}]]$. This proves the claim.

Proof of Case 1 (*continued*). We now use the claim above to inductively reduce (11) until it yields

$$\phi_n = a^p + p^{\alpha_n}bh_0 + \delta, \tag{12}$$

for some $a \in F$ with $v(a) = 0$, some $b \in F$ with $v(b) = 0$ and $\overline{b} \in \overline{F}^p$, and some $\delta \in F$ with $v(\delta) > \alpha_n$. Since $p^{\alpha_n/p} = p^{1/p+1/p^2+\cdots+1/p^n} \in F_0$, it will follow immediately from Lemma 2.2 that v extends uniquely from F to H_n, and $\overline{H_n} = \overline{F}(h_0^{1/p})$.

If $n = 2$, then (11) is already in the desired form, since $\overline{\phi_1} \in \overline{F}^p$. Otherwise, we write (11) as

$$\phi_n = S_2 + [[p^{\alpha_2+1/p}]],$$

with $a_2 = \phi_{n-1} - p^{1/p} h_{n-2}$. By repeatedly applying the claim, we find

$$\phi_n = S_n + [[p^{\alpha_2+1/p}]],$$

with $S_n = a_n^p - p^{\alpha_n}\phi_{n-1}^{p-1}\cdots\phi_1^{p-1}h_0$, for some $a_n \in F$ with $a_n = \phi_{n-1} + [[p^{1/p}]]$. By Lemma 2.8, $\alpha_n < \alpha_2 + 1/p$ for all $n \geq 3$. Observing that the residues of $\phi_{n-1}, \ldots, \phi_1$ are all pth powers in $\overline{F}$, we find that ϕ_n is now in the form (12), and we are done.

Case 2 ($p = 2$). The basic steps for the $p = 2$ case are the same as for the odd p case, the differences are only in the details.

If $n = 1$, then, by Lemma 2.7, $\phi_1 = \psi_0^2 - 2(h_0 + e_1) + [[4]]$, so by Lemma 2.2, v extends uniquely to H_1, and $\overline{H_1} = \overline{F}(\sqrt{(\overline{h_0 + e_1})})$. But e_1 is already a square in $\overline{F}$, so $\overline{H_1} = \overline{F}(\sqrt{\overline{h_0}})$.

In general, for $n > 1$, we have, by Lemma 2.7

$$\begin{aligned}
\phi_n &= \psi_{n-1}^2 - 2(h_{n-1} + e_n) + [[4]] \\
&= \psi_{n-1}^2 - 2(h_{n-2}^2 + [[2]] + e_{n-1}^2 + [[2]]) + [[4]] \\
&= \psi_{n-1}^2 - 2(h_{n-2}^2 + e_{n-1}^2) + [[4]] \\
&= \psi_{n-1}^2 - 2((h_{n-2} + e_{n-1})^2 + [[2]]) + [[4]] \\
&= \psi_{n-1}^2 - 2(h_{n-2} + e_{n-1})^2 + [[4]] \\
&= \psi_{n-1}^2 + 2(h_{n-2} + e_{n-1})^2 - 4(h_{n-2} + e_{n-1})^2 + [[4]] \\
&= \psi_{n-1}^2 + (2^{1/2}(h_{n-2} + e_{n-1}))^2 + [[4]] \\
&= (\psi_{n-1} + (2^{1/2}(h_{n-2} + e_{n-1})))^2 \\
&\quad - 2(2)^{1/2}\psi_{n-1}(h_{n-2} + e_{n-1}) + [[4]] \\
&= (\psi_{n-1} + (2^{1/2}(h_{n-2} + e_{n-1})))^2 \\
&\quad - 2^{\alpha_2}\psi_{n-1}(h_{n-2} + e_{n-1}) + [[4]].
\end{aligned} \tag{13}$$

Claim. For $2 \leq k \leq n-1$, let

$$S_k = a_k^2 - 2^{\alpha_k}\psi_{n-1}\cdots\psi_{n-k+1}(h_{n-k} + e_{n-k+1}) + [[4]],$$

for some $a_k \in F$ with $a_k = \psi_{n-1} + [[2^{1/2}]]$. Then,

$$S_k = a_{k+1}^2 - 2^{\alpha_k+1}\psi_{n-1}\cdots\psi_{n-k}(h_{n-k-1} + e_{n-k}) + [[4]],$$

for some $a_{k+1} \in F$ with $a_{k+1} = \psi_{n-1} + [[2^{1/2}]]$.

Proof of Claim. We have

$$\begin{aligned}
S_k &= a_k^2 - (2^{\alpha_k/2})^2(\psi_{n-2}^2 + [[2]]) \\
&\quad \cdots (\psi_{n-k}^2 + [[2]])(h_{n-k-1}^2 + [[2]] + e_{n-k}^2 + [[2]]) + [[4]] \\
&= a_k^2 - (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}^2 (h_{n-k-1}^2 + e_{n-k}^2) \\
&\quad + [[2^{1+2(\alpha_k/2)}]] + [[4]] \\
&= a_k^2 - (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}^2((h_{n-k-1} + e_{n-k})^2 + [[2]]) + [[4]] \\
&= a_k^2 - (2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 + [[4]] \\
&= a_k^2 + (2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 \\
&\quad - 2(2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 + [[4]] \\
&= a_k^2 + (2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 + [[4]] \\
&= (a_k + 2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 \\
&\quad - 2(2^{\alpha_k/2})a_k\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) + [[4]] \\
&= (a_k + 2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 \\
&\quad - 2^{1+\alpha k/2}(\psi_{n-1} + [[2^{1/2}]])\psi_{n-2} \\
&\quad \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) + [[4]] \\
&= (a_k + 2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}))^2 \\
&\quad - 2^{\alpha_{k+1}}\psi_{n-1}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) \\
&\quad + [[2^{\alpha_{k+1}+1/2}]] + [[4]] \\
&= a_{k+1}^2 - 2^{\alpha_{k+1}}\psi_{n-1}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) + [[4]],
\end{aligned}$$

where

$$a_{k+1} = a_k + 2^{\alpha_k/2}\psi_{n-2} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}),$$

(so $a_{k+1} = \psi_{n-1} + [[2^{1/2}]] + [[2^{\alpha_k/2}]] = \psi_{n-1} + [[2^{1/2}]])$.

Proof of Case 2 (*continued*). We now use the claim above to inductively reduce (13) until it yields

$$\phi_n = a^2 + 2^{\alpha_n}b(h_0 + e_1) + \delta, \tag{14}$$

for some $a \in F$ with $v(a) = 0$, some $b \in F$ with $v(b) = 0$ and $\overline{b} \in \overline{F}^2$, and some $\delta \in F$ with $v(\delta) > \alpha_n$. Since $2^{\alpha_n/2} = 2^{1/2+1/2^2+\cdots+1/2^n} \in F_0$, it will follow immediately from Lemma 2.2 that v extends uniquely from F to H_n, and $\overline{H_n} = \overline{F}(\sqrt{h_0 + e_1}) = \overline{F}(\sqrt{h_0})$.

If $n = 2$, then (13) is already in the desired form, since $\overline{\psi_1} \in \overline{F}^2$.

Otherwise, we write (13) as

$$\phi_n = S_2 + [[4]],$$

with $a_2 = \psi_{n-1} + 2^{1/2}(h_{n-2}+e_{n-1})$. By repeatedly applying the claim, we find

$$\phi_n = S_n + [[4]],$$

with $S_n = a_n^2 - (2^{\alpha_n})\psi_{n-1}\cdots\psi_1(h_0+e_1)$, for some $a_n \in F$ with $a_n = \psi_{n-1} + [[2^{1/2}]]$. By Lemma 2.8 (or by more direct means), $\alpha_n < 2$ for all $n \geq 3$. Observing that the residues of $\psi_{n-1}, \ldots, \psi_1$ are all squares in $\overline{F}$, we find that ϕ_n is now in the form (15), and we are done. □

3. The Dec results. Let F_0 be a field of characteristic 0 containing all primitive p^ith roots of unity ω_i ($i = 1, 2, \ldots$), chosen so that $\omega_{i+1}^p = \omega_i$. (We will write ω for ω_1.) If $L \supseteq F_0$ is any field, and if a and b are in L^*, then, as in [**D**, Chapter 11], $(a, b; p^n, L, \omega_n)$ will denote the algebra generated over L by two symbols α and β subject to $\alpha^{p^n} = a$, $\beta^{p^n} = b$, and $\alpha\beta = \omega_n\alpha\beta$, and will be referred to as a *symbol algebra.* Now let $F = F_0(x_1, x_2, y)$ be the rational function field over F_0 in the three indeterminates x_1, x_2, and y. For each $n \geq 1$, define

$$A_n = (x_1, x_1^{p^n} - y; p, F, \omega) \otimes_F (x_2, x_2^{p^n} - y; p, F, \omega).$$

LEMMA 3.1. *For each $n \geq 1$, A_n has index p^2 and exponent p. Further,*

$$A_n \sim \left(y, \frac{(x_1^{p^n} - y)(x_2^{p^n} - y)}{x_1^{p^n}x_2^{p^n}}; p^{n+1}, F, \omega_{n+1}\right).$$

Proof. This is very similar to the proof of Proposition 2 in [**Se2**], and we only sketch the proof. The factor $(x_1, x_1^{p^n} - y; p, F, \omega)$ is NSR with respect to the x_1-adic valuation on F, with residue isomorphic to $F_0(x_2, z)$, where $z = y^{1/p}$. The factor $(x_2, x_2^{p^n} - y; p, F_0(x_2, z), \omega)$ (i.e., defined over $F_0(x_2, z)$) is NSR with respect to the $x_2^{p^{n-1}} - z$ adic valuation (with residue isomorphic to $F_0(x_2^{1/p})$). It follows from [**JW**, Theorem 5.15] that A_n has index p^2. It is clear that $\exp(A_n) = p$. As for the final statement of the lemma, standard symbol algebra identities (e.g., [**D**, Chapter 11, pages 77–82]) along with the assumption about the roots of unity in F_0 show that

$$\begin{aligned} A_n &\sim (x_1^{p^n}, x_1^{p^n} - y; p^{n+1}, F, \omega_{n+1}) \\ &\quad \otimes_F (x_2^{p^n}, x_2^{p^n} - y; p^{n+1}, F, \omega_{n+1}) \\ &\sim \left(-y, \frac{x_1^{p^n} - y}{x_1^{p^n}}; p^{n+1}, F, \omega_{n+1}\right) \\ &\quad \otimes_F \left(-y, \frac{x_2^{p^n} - y}{x_2^{p^n}}; p^{n+1}, F, \omega_{n+1}\right) \\ &\sim \left(y, \frac{(x_1^{p^n} - y)(x_2^{p^n} - y)}{x_1^{p^n} x_2^{p^n}}; p^{n+1}, F, \omega_{n+1}\right). \qquad \square \end{aligned}$$

Now write ϕ_n for $(x_1^{p^n} - y)(x_2^{p^n} - y)$ (this notation will be seen to be consistent with that of §2), and write K_n for the field $F(y^{1/p^n}, \phi_n^{1/p})$. Then $A_n \in \mathrm{Br}(K_n/F)$. Tignol ([**T2**, Theorem 1]) showed that when p is odd, $A_1 \notin \mathrm{Dec}(K_1/F)$. We have

THEOREM 3.2. 1. *For p odd and $n \geq 1$, or $p = 2$ and $n \geq 2$, $A_n \notin \mathrm{Dec}(K_n/F)$.*

2. *More generally, for p odd, $n \geq 1$, and $0 \leq l \leq n-1$, or $p = 2$, $n \geq 2$, and $0 \leq l \leq n-2$, let $F_l = F(y^{1/p^l})$ (so $F_l \subset K_n$). Then, $A_n \otimes_F F_l \notin \mathrm{Dec}(K_n/F_l)$.*

3. *Further, let E be any finite extension of F, with $p \nmid [E:F]$. For p odd, $n \geq 1$, and $0 \leq l \leq n-1$, or $p = 2$, $n \geq 2$, and $0 \leq l \leq n-2$, let $E_l = E(y^{1/p^l})$ (so $E_l \subset K_n \cdot E$). Then, $A_n \otimes_F E_l \notin \mathrm{Dec}(K_n \cdot E/E_l)$.*

Proof of Theorem 3.2. It is clearly sufficient to prove (3). Moreover, it is sufficient to prove (3) for the case $l = n-1$ (for p odd) and $l = n-2$ (for $p = 2$). For, assume that for $l < n-1$ and p odd, or for $l < n-2$ and $p = 2$,

$$A_n \otimes_F E_l \sim (y^{1/p^l}, b_1; p^{n-l}, E_l, \omega_{n-l}) \otimes_{E_l} (b_2, \phi_n; p, E_l, \omega),$$

for some b_1 and $b_2 \in E_l^*$. Then, extending scalars to E_{n-1} (for p odd) and E_{n-2} (for $p = 2$), we find by standard symbol algebra identities

$$A_n \otimes_F E_{n-1} \sim (y^{1/p^{n-1}}, b_1; p, E_{n-1}, \omega) \otimes_{E_{n-1}} (b_2, \phi_n; p, E_{n-1}, \omega)$$

for p odd, and

$$A_n \otimes_F E_{n-2} \sim (y^{1/p^{n-2}}, b_1; p^2, E_{n-2}, \omega_2) \otimes_{E_{n-2}} (b_2, \phi_n; p, E_{n-2}, \omega)$$

for $p = 2$. Thus, we find that for p odd and $l < n - 1$, if

$$A_n \otimes_F E_l \in \mathrm{Dec}(K_n \cdot E/E_l)$$

then

$$A_n \otimes_F E_{n-1} \in \mathrm{Dec}(K_n \cdot E/E_{n-1}),$$

and for $p = 2$ and $l < n - 2$, if

$$A_n \otimes_F E_l \in \mathrm{Dec}(K_n \cdot E/E_l)$$

then

$$A_n \otimes_F E_{n-2} \in \mathrm{Dec}(K_n \cdot E/E_{n-2}).$$

We find it convenient at this point to divide the proof according to whether p is odd or even.

Case 1 (p *odd*). Assume that

$$A_n \otimes_F E_{n-1} \sim (y^{1/p^{n-1}}, b_1; p, E_{n-1}, \omega) \otimes E_{n-1}(b_2, \phi_n; p, E_{n-1}, \omega),$$

for some b_1 and $b_2 \in E^*_{n-1}$. By Lemma 3.1 and standard symbol algebra identities,

$$A_n \otimes_F E_{n-1} \sim \left(y^{1/p^{n-1}}, \frac{\phi_n}{x_1^{p^n} x_2^{p^n}}; p^2, E_{n-1}, \omega_2\right).$$

Put $z = y^{1/p^n}$. Then, extending scalars further to $E_n = E(z)$, and noting that $x_1^{p^n}$ and $x_2^{p^n}$ are pth powers, we find

$$(z, \phi_n; p, E_n, \omega) \sim (b, \phi_n; p, E_n, \omega),$$

where we have written b for b_2. Hence,

$$(z/b, \phi_n; p, E_n, \omega) \sim 1,$$

so

$$z/b = N(u) \tag{15}$$

for some $u \in E_n((\phi_n)^{1/p})$, where N denotes the norm from $E_n((\phi_n)^{1/p})$ to E_n. We will prove that it is impossible to find $b \in E_{n-1}$ and $u \in E_n((\phi_n)^{1/p})$ such that (16) holds.

If $\overline{F_0}$ denotes the algebraic closure of F_0, then $\overline{F_0}(x_1, x_2, y)$ is normal over $F_0(x_1, x_2, y)$, so if $E = F_0(x_1, x_2, y)(t)$ for some $t \in E^*$, then it is standard that the degree of the minimum polynomial of t over $\overline{F_0}(x_1, x_2, y)$ divides the degree of the minimum polynomial of t over $F_0(x_1, x_2, y)$. Hence $p \nmid [E \cdot \overline{F_0}(x_1, x_2, y) : \overline{F_0}(x_1, x_2, y)]$. Thus, while showing that (15) cannot hold, we may assume that F_0 is algebraically closed. In particular, we may assume that F_0 contains p^{1/p^i} for all i ($i = 1, 2, \dots$), so we may apply the machinery of §2.

Now write χ for $h_0(x_1, x_2, z)$, where h_0 is as in §2. As with the polynomial h_0, we will abuse notation and continue to write χ for the residue of h_0 under appropriate p-adic valuations. Observe that over E_n, $\phi_n = (x_1^{p^n} - z^{p^n})(x_2^{p^n} - z^{p^n})$, which, after renaming variables is indeed the same as the "ϕ_n" of §2.

We first need an easy lemma:

Lemma 3.3. *Let p be a prime, and let (F, v) be a valued field. Let K be a finite dimensional separable extension of F such that $p \nmid [K : F]$. Then for some extension of v to K, $p \nmid [\overline{K} : \overline{F}]$.*

Proof. Let v_i $(1 \le i \le s)$ be the extensions of v to K, and let $(\overline{K})_i$ denote the residues of K with respect to the valuations v_i. Let F_h denote the henselization of F with respect to v, and let $K_{i,h}$ denote the henselization of K with respect to v_i $(1 \le i \le s)$. Then (by [**E**, Theorem 17.17]) $[K : F] = \sum_{i=1}^{s}[K_{i,h} : F_h]$, so if $p \nmid [K : F]$, then $p \nmid [K_{i,h} : F_h]$ for some i. Now $\overline{K_{i,h}} = (\overline{K})_i$ and $\overline{F_h} = \overline{F}$, so by Ostrowski's theorem ([**O**, Satz 4], see also [**E**, Theorem 20.21]), $[(\overline{K})_i : \overline{F}] \mid [K_{i,h} : F_h]$. Hence, for this i, $p \nmid [(\overline{K})_i : \overline{F}]$.

Proof of Theorem 3.2 (*continued*). Now let $L = F_0(x_1, x_2, z)$ and let v be the standard extension of any p-adic valuation on F_0 to L (so $\overline{L} = \overline{F_0}(x_1, x_2, z)$). Let $L_1 = F_0(x_1, x_2, z^p)$, and let v_{L_1} denote the restriction of v to L_1. Choose an extension w of v_{L_1} to E_{n-1} such that $p \nmid [\overline{E_{n-1}} : \overline{L_1}]$. (Since $[E_{n-1} : L_1] = [E : F]$, the lemma above shows that such a choice is possible.) By Proposition 2.4 v extends uniquely from L to $L(\phi_n^{1/p})$, with residue $\overline{L}(\chi^{1/p})$. Since $p \nmid [\overline{E_{n-1}} : \overline{L_1}]$, while $[\overline{L(\phi_n^{1/p})} : \overline{L_1}] = p^2$, it follows easily that w extends uniquely from E_{n-1} to $E_n(\phi_n^{1/p})$, with residue $\overline{E_n}(\chi^{1/p})$.

Now, continue to write w for the (unique) extension of w to $E_n(\phi_n^{1/p})$ and consider the relation (15). Since $v(z) = 0$, we get $w(b) + w(N(u)) = 0$. Since $\Gamma_{E_{n-1}} = \Gamma_{E_n(\phi_n^{1/p})}$, there is a $c \in E_{n-1}$ such that $w(c) = w(u)$. Then, $bN(u) = bc^p N(u/c)$, and $w(u/c) = 0$, $w(bc^p) = w(b) + p \cdot w(u) = w(b) + w(N(u)) = 0$, and of course, $bc^p \in E_{n-1}$. Hence, we may assume in (15) that $w(b) = w(u) = 0$.

Now let σ be a generator of $\mathscr{G}(E_n(\phi_n^{1/p})/E_n)$, so

$$N(u) = u \cdot \sigma(u) \cdots \sigma^{p-1}(u).$$

Hence, $\overline{N(u)} = \overline{u} \cdot \overline{\sigma}(\overline{u}) \cdots \overline{\sigma}^{p-1}(\overline{u})$, where $\overline{\sigma}$ is the induced automorphism of $\overline{E_n}(\chi^{1/p})/\overline{E_n}$ (i.e., $\overline{\sigma}(\overline{x}) = \overline{\sigma(x)}$ for all $\overline{x} \in \overline{E_n}(\chi^{1/p})$). Since the extension $\overline{E_n}(\chi^{1/p})/\overline{E_n}$ is purely inseparable, $\overline{\sigma}$ is just the identity, so find $\overline{N(u)} = \overline{u}^p$. Thus, reducing the relation $z = bN(u)$ modulo the maximal ideal of the valuation ring of w, we find $z = \overline{b}\overline{u}^p$, where $\overline{b} \in \overline{E_{n-1}}$, and $\overline{u} \in \overline{E_n}(\chi^{1/p})$. We will show that such a relation is impossible.

Let $\overline{E_{n-1}} = \overline{L_1}(\theta)$, so that $1, \theta, \ldots, \theta^{s-1}$ form a basis for $\overline{E_{n-1}}/\overline{L_1}$, with $s = [\overline{E_{n-1}} : \overline{L_1}]$. Since $p \nmid s$, it follows easily that $\overline{E_{n-1}} = \overline{L_1}(\theta^p)$, and $1, \theta^p, \ldots, \theta^{(s-1)p}$ also form a basis of $\overline{E_{n_1}}/\overline{L_1}$. Likewise, $1, \theta, \ldots, \theta^{s-1}$, as well as $1, \theta^p, \ldots, \theta^{(s-1)p}$, are both bases of $\overline{E_n}(\chi^{1/p})/\overline{L}(\chi^{1/p})$. Now let

$$1/\overline{b} = b_0 + b_1\theta^p + \cdots + b_{s-1}\theta^{(s-1)p},$$

where the $b_i \in \overline{L_1}$ $(i = 0, 1, \ldots, s-1)$. Similarly, let

$$\overline{u} = u_0 + u_1\theta + \cdots + u_{s-1}\theta^{s-1},$$

where the $u_i \in \overline{L}(\chi^{1/p})$ $(i = 0, 1, \ldots, s-1)$. Substituting the expressions above for $1/\overline{b}$ and $\overline{u}$ in $z/\overline{b} = \overline{u}^p$ and comparing like terms, we find

$$zb_0 = u_0^p, \tag{16}$$

where of course, $b_0 \in \overline{L_1}$ and $u_0 \in \overline{L}(\chi^{1/p})$. The impossibility of (16) above is just the impossibility of [**T2**, (23)], and follows immediately from the proof given there. However, for the sake of completeness, we will reprove this result here. Our proof will be different from that in [**T2**]; instead, it will be similar in spirit to the proof below of a corresponding result for $p = 2$.

Write c for $1/b_0$ and u for u_0, so we need to show that there do not exist $c \in \overline{L_1}$ $(= \overline{F_0}(x_1, x_2, z^p))$ and $u \in \overline{L}(\chi^{1/p})$ $(= \overline{F_0}(x_1, x_2, z)(\chi^{1/p}))$ such that $z/c = u^p$. By considering the z-adic valuation on $\overline{L_1}$, it is easy to see that for any $c \in \overline{L_1}^*$ $z/c \notin \overline{L}^{*p}$. Now assume that $z/c = u^p$ for some $c \in \overline{L_1}^*$ and some $u \in \overline{L}(\chi^{1/p})$. Then $\overline{L}((z/c)^{1/p}) \subset \overline{L}(\chi^{1/p})$, so we find $\overline{L}((z/c)^{1/p}) = \overline{L}(\chi^{1/p})$. Thus, there exist $f_i \in \overline{L}^p$ $(i = 0, 1, \ldots, p-1)$ such that

$$\chi\ (= h_0(x_1, x_2, z)) = \sum_{i=1}^{p-1} f_i(z/c)^i. \tag{17}$$

Since $1, z, \dots, z^{p-1}$ form a basis for $L/\overline{L_1}$, we may write

$$h_0(x_1, x_2, z) = \sum_{i=0}^{p-1} e_i z^i \quad \text{for } e_i \in \overline{L_1},$$

where the values of the e_i may be derived from the definition of h_0 in (6). Then, (17) takes the form

$$c^{p-1}\left(\sum_{i=0}^{p-1} e_i z^i\right) = \sum_{i=0}^{p-1} c^{p-1-i} f_i z^i. \tag{18}$$

Now $c \in \overline{L_1}$, and $\overline{L}^p \subset \overline{L_1}$. Hence, comparing the coefficients of z^i in (18), we find $c^i e_i = f_i$ $(i = 0, 1, \dots, p-1)$. In particular, we find $e_1 e_{p-1} = f_1 f_{p-1}/c^p$. Since f_1, f_{p-1}, and $c^p \in \overline{L}^p$, this shows $e_1 e_{p-1} \in \overline{L}^p$. Now from (6), it is easy to see that

$$\begin{aligned} e_1 &= -[(x_1^p - z^p)x_2^{p-1} + (x_2^p - z^p)x_1^{p-1}], \\ e_{p-1} &= [(x_1^p - z^p)x_2 + (x_2^p - z^p)x_1]. \end{aligned}$$

Multiplying out, we find $x_2 x_2^{p-1} + x_2 x_1^{p-1} \in \overline{L}^p = \overline{F_0}^p(x_1^p, x_2^p, z^p)$. Since $p > 2$ (so $x_1 x_2^{p-1} + x_2 x_1^{p-1} \neq 0$), this is clearly impossible.

Case 2 $(p = 2)$. Assume that

$$\begin{aligned} A_n \otimes_F E_{n-2} \sim (y^{1/2^{n-2}}, b_1; 2^2, E_{n-2}, \omega_2) \\ \otimes E_{n-2}(b_2, \phi_n; 2, E_{n-2}, -1), \end{aligned}$$

for some b_1 and $b_2 \in E_{n-2}^*$. Then, letting $z = y^{1/2^n}$ and $E_n = E(z)$, we find, exactly as in the p odd case, that $z/b = N(u)$ for some $b \in E_{n-2}^*$ and $u \in E_n(\sqrt{\phi_n})$, where N denotes the norm from $E_n(\sqrt{\phi_n})$ to E_n. Letting $\chi = h_0(x_1, x_2, z)$, assuming F_0 is algebraically closed, and considering the standard extension of any 2-adic valuation on F_0 to $F_0(x_1, x_2, \dots, z)$, we find, just as in the p odd case that for some $b_0 \in \overline{F_0}(x_1, x_2, z^4)$ and $u_0 \in \overline{F_0}(x_1, x_2, z)(\sqrt{\chi})$,

$$z b_0 = u_0^2. \tag{19}$$

We will show that (19) is impossible.

Write L for the field $\overline{F_0}(x_1, x_2, z)$, L_1 for the field $\overline{F_0}(x_1, x_2, z^2)$, and L_2 for the field $\overline{F_0}(x_1, x_2, z^4)$. Assume that (19) holds for some $b_0 \in L_2$ and $u_0 \in L(\sqrt{\chi})$. By considering the z-adic valuation on L and noting that $b_0 \in L_2$, it is easy to see that $z b_0 \notin L^2$. Hence,

$zb_0 = u_0^2$, then $L(\sqrt{\chi}) = L(\sqrt{zb_0})$. From this, as well as the definition of h_0 in (7), it follows that

$$z((x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1) = f_0^2 + f_1^2 z b_0,$$

for some f_0 and $f_1 \in L$. Since 1 and z form a basis for L as an L_1 vector space, and since f_0^2, f_1^2, $(x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1$, and b_0 are all in L_1, we find

$$(x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1 = f_1^2 b_0.$$

We write this as

$$\frac{x_1^2 x_2 + x_2^2 x_1}{b_0} + \frac{z^2(x_2 + x_1)}{b_0} = f_1^2. \tag{20}$$

Now $f_1^2 \in L^2 = L_1^2(z^2)$. Thus $f_1^2 = g_0^2 + g_1^2 z^2$ for some g_0 and $g_1 \in L_1$. Substituting this in (20), we find

$$\frac{x_1^2 x_2 + x_2^2 x_1}{b_0} + \frac{z^2(x_2 + x_1)}{b_0} = g_0^2 + g_1^2 z^2. \tag{21}$$

Now $x_1^2 x_2 + x_2^2 x_1$, $x_2 + x_1$, and b_0 (note!) are all in L_2. Moreover, $L_1^2 \subset L_2$. Since 1 and z^2 form a basis of L_1 as an L_2 vector space, we find on viewing (21) as an equation in L_1 that

$$\frac{x_1^2 x_2 + x_2^2 x_1}{b_0} = g_0^2,$$

and

$$\frac{x_2 + x_1}{b_0} = g_1^2.$$

Dividing, we find $x_1 x_2 = (g_0/g_1)^2$ for some g_0 and $g_1 \in L_1$. But $x_1 x_2$ is clearly not a square in L_1, and we are done. □

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