Dec GROUPS FOR ARBITRARILY HIGH EXPONENTS

Bharath Al Sethuraman
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B. A. Sethuraman

For each prime $p$ and each $n \geq 1$ ($n \geq 2$ if $p = 2$), examples are constructed of a Galois extension $K/F$ whose Galois group has exponent $p^n$ and a central simple $F$-algebra $A$ of exponent $p$ which is split by $K$ but is not in the Dec group of $K/F$.

1. Introduction. Let $K/F$ be an abelian Galois extension of fields, and let $G = \mathcal{G}(K/F)$. Let $G = G_1 \times G_2 \times \cdots \times G_k$ be a direct sum decomposition of $G$ into cyclic groups, with $G_i = \langle \sigma_i \rangle$ ($i = 1, \ldots, k$). Let $F_i$ be the fixed field of $G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_k$ ($i = 1, \ldots, k$). Thus, the $F_i$ are cyclic Galois extensions of $F$, with Galois group isomorphic to $G_i$. The group $\text{Dec}(K/F)$ is defined as the subgroup of $\text{Br}(K/F)$ generated by the subgroups $\text{Br}(F_i/F)$ ($i = 1, \ldots, k$). This group was introduced by Tignol ([T1]), where he shows that $\text{Dec}(K/F)$ is independent of the choice of the direct sum decomposition of $G$. If $p$ is a prime, we will write $p^n \text{Br}(K/F)$ and $p^n \text{Dec}(K/F)$ for the subgroups of $\text{Br}(K/F)$ and $\text{Dec}(K/F)$ consisting of all elements with exponent dividing $p^n$.

A key issue in several past constructions of division algebras has been the non-triviality of the factor group $p \text{Br}(K/F)/p \text{Dec}(K/F)$ for suitable abelian extensions $K/F$. For instance, the Amitsur-Rowen-Tignol construction of an algebra of index 8 with involution with no quaternion subalgebra ([ART]) depends crucially on the existence of a triquadratic extension $K/F$ for which $2 \text{Br}(K/F) \neq 2 \text{Dec}(K/F)$. Similarly, the constructions of indecomposable algebras of exponent $p$ by Tignol ([T2]) and Jacob ([J]) also depend on the existence of an (elementary) abelian extension $K/F$ for which $p \text{Br}(K/F) \neq p \text{Dec}(K/F)$.

The extension fields $K/F$ that occur in these examples above are all of exponent $p$, and it is an interesting question whether there exist abelian extensions $K/F$ whose Galois groups have arbitrarily high ($p$-power) exponents for which the factor group $p \text{Br}(K/F)/p \text{Dec}(K/F)$ is non-trivial. The purpose of this paper is to show that for each $n \geq 1$ ($n \geq 2$ if $p = 2$), there exists an abelian extension $K/F$ with Galois group $\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (and thus, of exponent $p^n$) and an algebra $A \in p \text{Br}(K/F)$ such that $A \notin p \text{Dec}(K/F)$. (Note that if $K/F$ is an...
Our field $F$ will be the rational function field in $3$ variables over a field $F_0$ of characteristic $0$ that contains sufficiently many roots of unity. (For instance, $F_0$ may be algebraically closed.) Our algebras will in fact be generalizations of the example given by Tignol in [T2]. Moreover, we will prove that for $A$, $K$, and $F$ as above, $A \otimes_F L \notin p \text{Dec}(K \cdot L/L)$ for any finite degree extension $L/F$ with $p \nmid [L:F]$.

The special case $n = 2$ (and $p$ odd) of these computations was done in [Se1], where the result was used to construct non-elementary abelian crossed products of index $p^3$ and exponent $p^2$.

We remark that using different techniques, Rowen and Tignol ([RT]) have shown that if the ground field is assumed to only contain a primitive $p^s$th root of unity but not a primitive $p^{s+1}$th root of unity for some $s \geq 1$, then examples of non-trivial factor groups $p \text{Br}(K/F)/p \text{Dec}(K/F)$ exist for suitable abelian extensions $K/F$ whose Galois groups have arbitrarily large ($p$-power) exponents. Using ultraproducts ([R]), their example can be extended to also cover the case where the ground field contains all primitive $p^s$th roots of unity ($i = 1, 2, \ldots$).

2. $p$-adic valuations on rational function fields. Let $p$ be a prime, which, for now, can be either odd or even. Let $F_0$ be a field of characteristic $0$. The subfield $Q$ of $F_0$ has a standard valuation $v: Q \to \mathbb{Z}$ obtained by writing any non-zero element in $Q$ as $p^n a/b$, where $n$, $a$, and $b$ are integers, and $p$ is relatively prime to $a$ and $b$, and defining $v(p^n a/b) = n$. We will refer to any valuation on $F_0$ that extends this distinguished valuation on $Q$ as a $p$-adic valuation. Since the residue field of $Q$ under $v$ is $\mathbb{Z}/p\mathbb{Z}$, the residue of $F_0$ under any $p$-adic valuation is of characteristic $p$.

Now let $F = F_0(x_1, x_2, \ldots, x_k)$ be the rational function field over $F_0$ in $k$ indeterminates ($k \geq 1$), and let $v$ be a fixed $p$-adic valuation on $F_0$. Then $v$ admits an extension $w$ to $F$ defined as follows: for any polynomial $f \in F_0[x_1, x_2, \ldots, x_k]$, $w(f)$ is the minimum of the values of the coefficients, and for $f$ and $g$ in $F_0[x_1, x_2, \ldots, x_k]$, $w(f/g) = w(f) - w(g)$. (It is easy to check that $w$ is indeed a valuation on $F$.). It can be shown that the residues $\bar{x}_i$ of the $x_i$ ($i = 1, \ldots, k$) are algebraically independent over the residue $\bar{F}_0$ of $F_0$; and that, moreover, $\bar{F}$ is precisely the rational function field $\bar{F}_0(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$. (It is also clear from the definition of $w$ that
Γ_F = Γ_{F_0}. We will refer to \( w \) as the standard extension of \( v \) to \( F \). Also, we will abuse notation and continue to write \( x_i \) for the residues \( \overline{x}_i \).

**Remark 2.1.** Furthermore, it can be shown that \( w \) is the unique extension of \( v \) to \( F \) with the property that the values of the \( x_i \) are 0, and the residues of the \( x_i \) are algebraically independent over \( \overline{F}_0 \). (See [B, §10, Proposition 2].)

The following is well known, but we include a proof here for convenience.

**Lemma 2.2.** Let \( p \) be any prime, and let \( F \) be a field of characteristic 0. Let \( v \) be a \( p \)-adic valuation on \( F \). Let \( K = F(\sqrt[p]{f}) \), where \( f \notin F^{*p} \), and \( v(f) = 0 \). Assume that \( f = f_0^p + \pi f_1 + \delta \), where \( v(f_0) = v(f_1) = 0 \), \( 0 < v(\pi) < (p/(p-1))v(p) \), and \( v(\delta) > v(\pi) \). Assume, too, that \( f_1 \notin \overline{F}^p \), and that there exists \( \theta \in F^* \) such that \( \theta^p = \pi \). Then \( v \) extends uniquely to \( K \), and \( K = \overline{F(f_1^{1/p})} \).

**Proof.** Let \( r \in K^* \) satisfy \( r^p = f \), and let \( s = (r - f_0)/\theta \). Then \( s + (f_0/\theta) = (r/\theta) \), so \( s \) satisfies

\[
\left(s + \frac{f_0}{\theta}\right)^p = \frac{f_0^p + \pi f_1 + \delta}{\theta^p}.
\]

Expanding the left-hand side of (1) and noting that \( \theta^p = \pi \), we find

\[
s^p + \sum_{i=1}^{p-1} \binom{p}{i} s^i \left(\frac{f_0}{\theta}\right)^{p-i} = f_1 + \left(\frac{\delta}{\theta^p}\right).
\]

Now for \( i = 1, \ldots, p-1 \), \( v(\binom{p}{i}) = v(p) \), while \( v(\theta^{p-i}) \leq v(\theta^{p-1}) = v(\pi^{(p-1)/p}) < v(p) \). (The last inequality is because \( v(\pi) < (p/(p-1))v(p) \).) From this, as well as the fact that \( v(f_0) = 0 \), we find that each of the expressions \( \binom{p}{i}(f_0/\theta)^{p-i} \) \( (i = 1, \ldots, p-1) \) has positive value. It follows that for any extension \( w \) of \( v \) from \( F \) to \( K \), if \( w(s) < 0 \), then the left-hand side of (2) would have the same value as \( s^p \). (Here we use the fact that if \( w(a) < w(b) \), then \( w(a + b) = w(a) \).) Since this contradicts the fact that the value of the right-hand side of (2) is 0 (note that \( v(f_1) = 0 \), while \( v(\delta/\theta^p) > 0 \), we must have \( w(s) \geq 0 \). Similarly, if \( w(s) > 0 \), then from \( w(a + b) \geq \min(w(a), w(b)) \), it follows that the left-hand side of (2) must have positive value. Hence \( w(s) = 0 \). Taking the residues of each term in (2) and noting again that all terms except \( s^p \) and \( f_1 \) have positive value, we find \( \overline{s^p} = \overline{f_1} \). Thus \( K \supset \overline{F(f_1^{1/p})} \).
Since \( f_1 \notin F^p \), and since \([K : F] = p\), we find by the fundamental inequality ([E, Corollary 17.5]) that \( w \) is unique, and \( \overline{K} = \overline{F}(f_1^{1/p}) \).

Now let \( F_0 \) be a field of characteristic 0. We will assume that \( F_0 \) contains \( p^{1/p^i} \) for all \( i (i = 1, 2, \ldots) \). Let \( F \) be the rational function field \( F_0(x_1, x_2, y) \). For each \( n (n \geq 0) \), let

\[
\phi_n = (x_1^{p^n} - y^{p^n})(x_2^{p^n} - y^{p^n}).
\]

Let \( H_n = F(\phi_n^{1/p}) \). Let \( v \) be the standard extension of any \( p \)-adic valuation on \( F_0 \) to \( F \). The manner in which \( v \) extends from \( F \) to \( H_n \) will be crucial to our Dec results, and the rest of §2 is devoted to this topic.

First, some notation. For \( p \) odd, and \( i = 1, 2, \ldots, p - 1 \), let

\[
\lambda_i = \frac{(-1)^{p-i} \binom{p}{i}}{p}
\]

(so each \( \lambda_i \) is an integer). For \( p \) odd, again, define \( g_n(x, y) \in \mathbb{Z}[x, y] (n = 0, 1, 2, \ldots) \) by

\[
g_n(x, y) = \sum_{i=1}^{p-1} \lambda_i (x^{p^n})^i (y^{p^n})^{p-i},
\]

so

\[
(x^{p^n} - y^{p^n})^p = x^{p^{n+1}} - y^{p^{n+1}} + pg_n(x, y).
\]

Now for \( p \) odd, define \( h_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y] (n = 0, 1, 2, \ldots) \) by

\[
h_n(x_1, x_2, y) = (x_1^{p^n} - y^{p^n})^p g_n(x_2, y) + (x_2^{p^n} - y^{p^n})^p g_n(x_1, y),
\]

and for \( p = 2 \), define \( h_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y] (n = 0, 1, 2, \ldots) \) by

\[
h_n(x_1, x_2, y) = (x_1^{2^n} + y^{2^n})^2 x_2^{2^n} y^{2^n} + (x_2^{2^n} + y^{2^n})^2 x_1^{2^n} y^{2^n}.
\]

**Remark 2.3.** We will abuse notation and continue to write \( g_n \) and \( h_n \) for the images of \( g_n \) and \( h_n \) in \( \mathbb{Z}/p\mathbb{Z}[x, y] \) and \( \mathbb{Z}/p\mathbb{Z}[x_1, x_2, y] \) (respectively).

The special case \( n = 1 \) (and \( p \) odd) of the following was proved in [T2, Lemma 3.7].
**Proposition 2.4.** For every prime $p$ and for all $n$ ($n \geq 1$), $v$ extends uniquely from $F$ to $H_n$, and $H_n = F(h_0(x_1, x_2, y)^{1/p})$.

Before proving Proposition 2.4, we need some further notation, as well as some easy lemmas.

For $p = 2$, define $e_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ (for $n \geq 1$) by

$$e_n(x_1, x_2, y) = y^{2^n}(x_1^{2^n} + x_2^{2^n}),$$

and $\psi_n(x_1, x_2, y) \in \mathbb{Z}[x_1, x_2, y]$ (for $n \geq 0$) by

$$\psi_n(x_1, x_2, y) = (x_1^{2^n} + y^{2^n})(x_2^{2^n} + y^{2^n}).$$

For $n \in \mathbb{Z}$ ($n \geq 1$), define $\alpha_n \in \mathbb{Q}$ by

$$\alpha_n = \begin{cases} 1, & \text{if } n = 1, \\ 1 + 1/p + 1/p^2 + \cdots + 1/p^{n-1}, & \text{if } n > 1. \end{cases}$$

Finally, for any $k \in \mathbb{Q}$, abbreviate the phrase "terms of value at least $v(p^k)$" by $[[p^k]]$.

**Remark 2.5.** Just as with $g_n$ and $h_n$, we will abuse notation and continue to write $e_n$ for the image of $e_n$ in $\mathbb{Z}/2\mathbb{Z}[x_1, x_2, y]$.

**Lemma 2.6.** Let $f$, $g$, $f_1$, and $g_1$ be polynomials in $\mathbb{Z}[x_1, x_2, y]$. Then, with respect to the restriction of $v$ to $\mathbb{Q}(x_1, x_2, y)$ (i.e., the standard extension of the $p$-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}(x_1, x_2, y)$),

1. If $f = g + [[p]]$, and $f_1 = g_1 + [[p]]$, then $f + f_1 = g + g_1 + [[p]]$ and $ff_1 = gg_1 + [[p]]$.
2. $(f + g)^p = f^p + g^p + [[p]]$.
3. Let $k \geq 1$, and suppose

$$f = \sum c_{i_1, i_2, i_3}(x_1^{p^k})^{i_1}(x_2^{p^k})^{i_2}(x_3^{p^k})^{i_3},$$

for some $c_{i_1, i_2, i_3} \in \mathbb{Z}$. Define $f^{1/p} \in \mathbb{Z}[x_1, x_2, y]$ by

$$f^{1/p} = \sum c_{i_1, i_2, i_3}(x_1^{p^{k-1}})^{i_1}(x_2^{p^{k-1}})^{i_2}(x_3^{p^{k-1}})^{i_3}.$$ 

Then $f = (f^{1/p})^p + [[p]]$.

**Proof.** Note that the values of $f$, $g$, $f_1$, and $g_1$ are non-negative. (1) and (2) are now elementary. (3) follows from (2) along with the fact that $a^p \equiv a \pmod{p}$ for any $a \in \mathbb{Z}$. □
LEMMA 2.7. With respect to the restriction of \( v \) to \( Q(x_1, x_2, y) \),
1. For \( n \geq 1 \) and for all \( p \), \( h_n = h_{n-1}^p + [[p]] \), and for \( n \geq 2 \) and \( p = 2 \), \( e_n = e_{n-1}^2 + [[2]] \).
2. For \( n \geq 1 \) and \( p \) odd, \( \phi_n = \phi_{n-1}^p - ph_{n-1} + [[p^2]] \).
3. For \( n \geq 1 \) and \( p = 2 \), \( \phi_n = \psi_n - 2e_n \), and \( \psi_n = \psi_{n-1}^2 - 2h_{n-1} + [[4]] \) (so \( \phi_n = \psi_{n-1}^2 - 2(h_{n-1} + e_n) + [[4]] \)).

Proof. (1) follows from the definitions of \( h_n \) and \( e_n \) and Lemma 2.6. For instance, for \( p \) odd (and \( n \geq 1 \)) we have
\[
(x_1^p - y^p)^p = ((x_1^{p-1} - y^{p-1})^p + [[p]])^p = (x_1^{p-1} - y^{p-1})^2 + [[p]].
\]
Also,
\[
g_n(x_2, y) = \sum_{i=1}^{p-1} \lambda_i (x_2^p)^i (y^p)^{p-i} = \left( \sum_{i=1}^{p-1} \lambda_i (x_2^{p-1})^i (y^{p-1})^{p-i} \right)^p + [[p]] = (g_{n-1}(x_2, y))^p + [[p]].
\]
Since similar relations hold for \( (x_2^p - y^p)^p \) and \( g_n(x_1, y) \), we find
\[
h_n = (x_1^{p-1} - y^{p-1})^p (g_{n-1}(x_2, y))^p + (x_2^{p-1} - y^{p-1})^p (g_{n-1}(x_1, y))^p + [[p]] = ((x_1^{p-1} - y^{p-1})^p g_{n-1}(x_2, y) + (x_2^{p-1} - y^{p-1})^p g_{n-1}(x_1, y))^p + [[p]]
\]
\[
= h_{n-1}^p + [[p]].
\]
The proof for \( p = 2 \) is similar. For (2), we have
\[
\phi_n = (x_1^p - y^p)(x_2^p - y^p) = ((x_1^{p-1} - y^{p-1})^p - p g_{n-1}(x_1, y))
\cdot ([x_2^{p-1} - y^{p-1})^p - p g_{n-1}(x_2, y)]
= [(x_1^{p-1} - y^{p-1})(x_2^{p-1} - y^{p-1})]^p
- p[(x_1^{p-1} - y^{p-1})^p g_{n-1}(x_2, y) + (x_2^{p-1} - y^{p-1})^p g_{n-1}(x_1, y)] + [[p^2]]
= \phi_{n-1}^p - ph_{n-1} + [[p^2]].
As for (3),
\[
\phi_n = (x_1^n - y^n)(x_2^n - y^n) \\
= (x_1^n + y^n - 2y^n)(x_2^n + y^n - 2y^n) \\
= (x_1^n + y^n)(x_2^n + y^n) - 2y^n(x_1^n + x_2^n) \\
= \psi_n - 2e_n.
\]

Also,
\[
\psi_n = (x_1^n + y^n)(x_2^n + y^n) \\
= [(x_1^{n-1} + y_1^{n-1})^2 - 2x_1^{n-1}y_1^{n-1}][(x_2^{n-1} + y_2^{n-1})^2 - 2x_2^{n-1}y_2^{n-1}] \\
= [(x_1^{n-1} + y_1^{n-1})(x_2^{n-1} + y_2^{n-1})]^2 \\
- 2[(x_1^{n-1} + y_1^{n-1})^2x_2^{n-1}y_2^{n-1} + (x_2^{n-1} + y_2^{n-1})^2x_1^{n-1}y_1^{n-1}] + [[4]] \\
= \psi_{n-1}^2 - 2h_{n-1} + [[4]].
\]

**Lemma 2.8.** For all \( p \) and for all \( k \geq 0 \), \( \alpha_{k+1} < \alpha_2 + 1/p \).

*Proof.* Since \( \alpha_1 < \alpha_2 < \alpha_2 + 1/p \), we may assume \( k > 2 \). Now \( \alpha_{k+1} = 1 + 1/p + \cdots + 1/p^k \) and \( \alpha_2 = 1 + 1/p \), so it is sufficient to prove that \( 1/p^2 + \cdots + 1/p^k < 1/p \). Multiplying both sides by \( p \), we need to prove that \( 1/p + \cdots + 1/p^{k-1} < 1 \). But this is clear, since
\[
1/p + \cdots + 1/p^{k-1} = 1/p(1 + 1/p + \cdots + 1/p^{k-2}) \\
< 1/p(1 + 1/p + 1/p^2 + \cdots) \\
= 1/(p-1) \leq 1.
\]

*Proof of Proposition 2.4.* We divide the proof according to whether \( p \) is odd or whether \( p = 2 \).

*Case 1 (Odd \( p \)).* If \( n = 1 \), this follows from Lemmas 2.7 and 2.2. For, by Lemma 2.7, \( \phi_1 = \phi_0^p - ph_0 + \delta \), for some \( \delta \in \mathbb{Z}[x_1, x_2, y] \) with \( v(\delta) \geq v(p^2) \). By assumption, \( p^{1/p} \in F_0 \). Clearly, \( -h_0 \notin \overline{F}^p = \overline{F}^p(x_1^p, x_2^p, y^p) \). Thus, by Lemma 2.2, \( v \) extends uniquely to \( H_1 \), and \( \overline{H}_1 = \overline{F}(\langle -h_0 \rangle^{1/p}) = \overline{F}(h_0^{1/p}) \).

In general, for \( n > 1 \), we have by Lemma 2.7,
Claim. For $2 < k < n - 1$, if

$$
\phi_n = \phi_{n-1}^p - ph_{n-1} + [[p^2]]
$$

then for some $\alpha_{k+1} \in F$ with $\alpha_{k+1} = \phi_{n-1} + [[p^1/p]],$

$$
S_k = a_k^p - p^{a_k+1} \phi_{n-k+1}^p h_{n-k} + [[p^{a_2+1/p}]],
$$
for some $a_k \in F$ with $a_k = \phi_{n-1} + [[p^1/p]],$ then

$$
S_k = a_{k+1}^p - p^{a_{k+1}+1} \phi_{n-k}^p h_{n-k+1} + [[p^{a_2+1/p}]],
$$
for some $a_{k+1} \in F$ with $a_{k+1} = \phi_{n-1} + [[p^1/p]].$

Proof of claim. For, by Lemma 2.7, $\phi_j = \phi_{j-1}^p + [[p]]$ and $h_j = h_{j-1}^p + [[p]]$ for all $j \geq 1$, so

$$
S_k = a_k^p - p^{a_k^p} (\phi_{n-2}^p + [[p]])^{p-1} (\phi_{n-3}^p + [[p]])^{p-1} \\
\cdots (\phi_{n-k}^p + [[p]])^{p-1} (h_{n-k-1}^p + [[p]]) + [[p^{a_2+1/p}]])
$$

$$
= a_k^p - p^{a_k^p} (\phi_{n-2}^p + [[p]]) (\phi_{n-3}^p + [[p]]) \\
\cdots (\phi_{n-k}^p + [[p]]) (h_{n-k-1}^p + [[p]]) + [[p^{a_2+1/p}]])
$$

$$
= a_k^p - p^{a_k^p} (\phi_{n-2}^p + (p^1/p) \phi_{n-3}^p + (p^1/p) h_{n-k-1}^p) \\
\cdots (\phi_{n-k}^p + (p^1/p) \phi_{n-k-1}^p + (p^1/p) h_{n-k-1}^p) + [[p^{a_2+1/p}]])
$$

$$
= a_k^p - (p^{a_k^p} (\phi_{n-2}^p (p-1) \cdots (\phi_{n-k}^p (p-1)h_{n-k-1}^p) + [[p^{a_2+1/p}]])
$$

(as $\alpha_2 + 1/p < \alpha_k + 1$)

$$
= (a_k^p - (p^{a_k^p} \phi_{n-2}^p \cdots (\phi_{n-k}^p h_{n-k-1}^p)^p)
$$

$$
- pg_0(a_k, p^{a_k^p} \phi_{n-2}^p \cdots (\phi_{n-k}^p h_{n-k-1}^p) + [[p^{a_2+1/p}]])
$$
Expanding $\exp(a_k, p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1})$ and considering the first two terms of lowest value, we find

$$S_k = (a_k - p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1})^p$$

$$- p^{1+\alpha_k/p} a_k^{p-1} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p + \left[[p^{1+(2\alpha_k/p)}]\right] + \left[[p^{\alpha_k+1/p}]\right].$$

Now $1 + \alpha_k/p = \alpha_{k+1}$. Also, $1 + (2\alpha_k/p) = \alpha_{k+1} + \alpha_k/p > \alpha_2 + 1/p$ (as $\alpha_{k+1} > \alpha_2$ and $\alpha_k > 1$ when $k \geq 2$). Thus,

$$S_k = (a_k - p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p)$$

$$- p^{\alpha_{k+1}} a_k^{p-1} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p + \left[[p^{\alpha_{k+1}/p}]\right].$$

Now recalling that $a_k = \phi_{n-1} + [[p^{1/p}]]$, we find $a_k^{p-1} = \phi_{n-1}^{p-1} + [[p^{1/p}]]$. Hence,

$$S_k = (a_k - p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p)$$

$$- p^{\alpha_{k+1}} a_k^{p-1} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p + \left[[p^{\alpha_{k+1}/p}]\right].$$

Since $\alpha_{k+1} > \alpha_2$, $\alpha_{k+1} + 1/p > \alpha_2 + 1/p$. Thus,

$$S_k = (a_k - p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p)$$

$$- p^{\alpha_{k+1}} a_k^{p-1} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1}^p + \left[[p^{\alpha_{k+1}/p}]\right].$$

Take $a_{k+1} = (a_k - p^{\alpha_k/p} \phi_{n-2}^{p-1} \cdots \phi_{n-k-1}^{p-1} h_{n-k-1})$. Since $a_k = \phi_{n-1} + [[p^{1/p}]]$ and since $1/p < \alpha_k/p$ (as $k \geq 2$), $a_{k+1} = \phi_{n-1} + [[p^{1/p}]]$. This proves the claim.

**Proof of Case 1 (continued).** We now use the claim above to inductively reduce (11) until it yields

$$\phi_n = a^p + p^{\alpha_n} b h_0 + \delta,$$

for some $a \in F$ with $v(a) = 0$, some $b \in F$ with $v(b) = 0$ and $b \in \overline{F}$, and some $\delta \in F$ with $v(\delta) > \alpha_n$. Since $p^{\alpha_n/p} = p^{1/p+1/p^2+\cdots+1/p^n} \in F_0$, it will follow immediately from Lemma 2.2 that $v$ extends uniquely from $F$ to $H_n$, and $H_n = \overline{F(h_0^{1/p})}$.

If $n = 2$, then (11) is already in the desired form, since $\overline{\phi_1} \in \overline{F}$. Otherwise, we write (11) as

$$\phi_n = S_2 + [[p^{\alpha_2+1/p}]].$$
with \( a_2 = \phi_{n-1} - p^{1/p}h_{n-2} \). By repeatedly applying the claim, we find

\[
\phi_n = S_n + \left[ [p^{\alpha_1 + 1/p}] \right],
\]

with \( S_n = a_n^p - p^{\alpha_n} \phi_{n-1}^{p-1} \cdots \phi_1^{p-1}h_0 \), for some \( a_n \in F \) with \( a_n = \phi_{n-1} + \left[ [p^{1/p}] \right] \). By Lemma 2.8, \( \alpha_n < \alpha_2 + 1/p \) for all \( n \geq 3 \). Observing that the residues of \( \phi_{n-1}, \ldots, \phi_1 \) are all \( p \)th powers in \( \overline{F} \), we find that \( \phi_n \) is now in the form (12), and we are done.

**Case 2** \((p = 2)\). The basic steps for the \( p = 2 \) case are the same as for the odd \( p \) case, the differences are only in the details.

If \( n = 1 \), then, by Lemma 2.7, \( \phi_1 = \psi_0^2 - 2(h_0 + e_1) + [4] \), so by Lemma 2.2, \( v \) extends uniquely to \( H_1 \), and \( \overline{H_1} = \overline{F}(\sqrt{(h_0 + e_1)}) \). But \( e_1 \) is already a square in \( \overline{F} \), so \( \overline{H_1} = \overline{F}(\sqrt{h_0}) \).

In general, for \( n > 1 \), we have, by Lemma 2.7

\[
\phi_n = \psi_{n-1}^2 - 2(h_{n-1} + e_n) + [4] \\
= \psi_{n-1}^2 - 2(h_{n-2}^2 + [2]) + e_{n-1}^2 + [2]) + [4] \\
= \psi_{n-1}^2 - 2(h_{n-2}^2 + e_{n-1}^2) + [4] \\
= \psi_{n-1}^2 - 2((h_{n-2} + e_{n-1})^2 + [2]) + [4] \\
= \psi_{n-1}^2 - 2(h_{n-2}^2 + e_{n-1}^2) + [4] \\
= \psi_{n-1}^2 + 2(h_{n-2} + e_{n-1})^2 - 4(h_{n-2} + e_{n-1})^2 + [4] \\
= \psi_{n-1}^2 + (2^{1/2}(h_{n-2} + e_{n-1}))^2 + [4] \\
= (\psi_{n-1} + (2^{1/2}(h_{n-2} + e_{n-1})))^2 \\
- 2(2^{1/2}\psi_{n-1}(h_{n-2} + e_{n-1}) + [4] \\
(13) = (\psi_{n-1} + (2^{1/2}(h_{n-2} + e_{n-1})))^2 \\
- 2^{\alpha_2}\psi_{n-1}(h_{n-2} + e_{n-1}) + [4]).
\]

**Claim.** For \( 2 \leq k \leq n - 1 \), let

\[
S_k = a_k^2 - 2^{\alpha_k}\psi_{n-1} \cdots \psi_{n-k+1}(h_{n-k} + e_{n-k+1}) + [4],
\]

for some \( a_k \in F \) with \( a_k = \psi_{n-1} + [2^{1/2}] \). Then,

\[
S_k = a_{k+1}^2 - 2^{\alpha_{k+1}}\psi_{n-1} \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) + [4],
\]

for some \( a_{k+1} \in F \) with \( a_{k+1} = \psi_{n-1} + [2^{1/2}] \).
Proof of Claim. We have

\[ S_k = a_k^2 - (2^{\alpha_k/2})^2(\psi_{n-2}^2 + [2]) \]
\[ \cdots (\psi_{n-k}^2 + [2])(h_{n-k-1}^2 + [2]) + e_{n-k}^2 + [2]) + [4]) \]
\[ = a_k^2 - (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}^2(h_{n-k-1}^2 + e_{n-k}^2) \]
\[ + [2^{1+2(\alpha_k/2)}] + [4]) \]
\[ = a_k^2 - (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}^2((h_{n-k-1} + e_{n-k})^2 + [2]) + [4]) \]
\[ = a_k^2 - (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 + [4]) \]
\[ = a_k^2 + (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 \]
\[ - 2(2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 + [4]) \]
\[ = a_k^2 + (2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 + [4]) \]
\[ = (a_k + 2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 \]
\[ - 2^{1+k/2}(\psi_{n-1}^2 + [2^{1/2}])\psi_{n-2} \]
\[ \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}) + [4]) \]
\[ = (a_k + 2^{\alpha_k/2})^2 \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 \]
\[ - 2^{1+k+1/2}(\psi_{n-1}^2 + [2^{1/2}])\psi_{n-2}^2 \]
\[ \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 + [4]) \]
\[ = a_{k+1}^2 - 2^{\alpha_{k+1}} \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k})^2 + [4]) \]

where

\[ a_{k+1} = a_k + 2^{\alpha_k/2} \psi_{n-2}^2 \cdots \psi_{n-k}(h_{n-k-1} + e_{n-k}), \]

(14) \[ \phi_n = a^2 + 2^{\alpha_n}b(h_0 + e_1) + \delta, \]

for some \( a \in F \) with \( v(a) = 0 \), some \( b \in F \) with \( v(b) = 0 \) and \( \overline{b} \in \overline{F}^2 \), and some \( \delta \in F \) with \( v(\delta) > \alpha_n \). Since \( 2^{\alpha_n/2} = 2^{1/2+1/2^2+\cdots+1/2^k} \in F_0 \), it will follow immediately from Lemma 2.2 that \( v \) extends uniquely from \( F \) to \( H_n \), and \( \overline{H_n} = \overline{F}(\sqrt{h_0 + e_1}) = \overline{F}(\sqrt{h_0}). \)

If \( n = 2 \), then (13) is already in the desired form, since \( \overline{\psi_1} \in \overline{F}^2 \).
Otherwise, we write (13) as
\[ \phi_n = S_2 + [[4]], \]
with \( a_2 = \psi_{n-1} + 2^{1/2}(h_{n-2} + e_{n-1}) \). By repeatedly applying the claim, we find
\[ \phi_n = S_n + [[4]], \]
with \( S_n = a_n^2 - (2\alpha_n)\psi_{n-1} \cdots \psi_1(h_0 + e_1) \), for some \( a_n \in F \) with \( a_n = \psi_{n-1} + [[2^{1/2}]] \). By Lemma 2.8 (or by more direct means), \( \alpha_n < 2 \) for all \( n \geq 3 \). Observing that the residues of \( \psi_{n-1}, \ldots, \psi_1 \) are all squares in \( \overline{F} \), we find that \( \phi_n \) is now in the form (15), and we are done.

3. The Dec results. Let \( F_0 \) be a field of characteristic 0 containing all primitive \( p^\prime \)-th roots of unity \( \omega_i (i = 1, 2, \ldots) \), chosen so that \( \omega_{i+1} = \omega_i \). (We will write \( \omega \) for \( \omega_1 \).) If \( L \supseteq F_0 \) is any field, and if \( a \) and \( b \) are in \( L^* \), then, as in [D, Chapter 11], \( (a, b; p^n, L, \omega_n) \) will denote the algebra generated over \( L \) by two symbols \( \alpha \) and \( \beta \) subject to \( \alpha^{p^n} = a \), \( \beta^{p^n} = b \), and \( \alpha \beta = \omega_n \alpha \beta \), and will be referred to as a symbol algebra. Now let \( F = F_0(x_1, x_2, y) \) be the rational function field over \( F_0 \) in the three indeterminates \( x_1, x_2, \) and \( y \). For each \( n \geq 1 \), define
\[ A_n = (x_1, x_1^{p^n} - y; p, F, \omega) \otimes_F (x_2, x_2^{p^n} - y; p, F, \omega). \]

Lemma 3.1. For each \( n \geq 1 \), \( A_n \) has index \( p^2 \) and exponent \( p \). Further,
\[ A_n \sim \left( y, \frac{(x_1^{p^n} - y)(x_2^{p^n} - y)}{x_1^{p^n} x_2^{p^n}}; p^{n+1}, F, \omega_{n+1} \right). \]

Proof. This is very similar to the proof of Proposition 2 in [Se2], and we only sketch the proof. The factor \( (x_1, x_1^{p^n} - y; p, F, \omega) \) is NSR with respect to the \( x_1 \)-adic valuation on \( F \), with residue isomorphic to \( F_0(x_2, z) \), where \( z = y^{1/p} \). The factor \( (x_2, x_2^{p^n} - y; p, F_0(x_2, z), \omega) \) (i.e., defined over \( F_0(x_2, z) \)) is NSR with respect to the \( x_2^{p^n-1} - z \) adic valuation (with residue isomorphic to \( F_0(x_2^{1/p}) \)). It follows from [JW, Theorem 5.15] that \( A_n \) has index \( p^2 \). It is clear that \( \exp(A_n) = p \). As for the final statement of the lemma, standard symbol algebra identities (e.g., [D, Chapter 11, pages 77–82]) along with the assumption
about the roots of unity in $F_0$ show that
\[ A_n \sim (x_1^{p^n}, x_1^{p^n} - y; p^{n+1}, F, \omega_{n+1}) \]
\[ \otimes_F (x_2^{p^n}, x_2^{p^n} - y; p^{n+1}, F, \omega_{n+1}) \]
\[ \sim \left( -y, \frac{x_1^{p^n} - y}{x_1^{p^n}}; p^{n+1}, F, \omega_{n+1} \right) \]
\[ \otimes_F \left( -y, \frac{x_2^{p^n} - y}{x_2^{p^n}}; p^{n+1}, F, \omega_{n+1} \right) \]
\[ \sim \left( y, \frac{(x_1^{p^n} - y)(x_2^{p^n} - y)}{x_1^{p^n} x_2^{p^n}}; p^{n+1}, F, \omega_{n+1} \right). \]

Now write $\phi_n$ for $(x_1^{p^n} - y)(x_2^{p^n} - y)$ (this notation will be seen to be consistent with that of §2), and write $K_n$ for the field $F(y^{1/p^n}, \phi_1^{1/p})$. Then $A_n \in \text{Br}(K_n/F)$. Tignol ([T2, Theorem 1]) showed that when $p$ is odd, $A_1 \notin \text{Dec}(K_1/F)$. We have

**Theorem 3.2.** 1. For $p$ odd and $n \geq 1$, or $p = 2$ and $n \geq 2$, $A_n \notin \text{Dec}(K_n/F)$. 2. More generally, for $p$ odd, $n \geq 1$, and $0 \leq l \leq n - 1$, or $p = 2$, $n \geq 2$, and $0 \leq l \leq n - 2$, let $F_l = F(y^{1/p^l})$ (so $F_l \subset K_n$). Then, $A_n \otimes_F F_l \notin \text{Dec}(K_n/F_l)$. 3. Further, let $E$ be any finite extension of $F$, with $p \nmid [E : F]$. For $p$ odd, $n \geq 1$, and $0 \leq l \leq n - 1$, or $p = 2$, $n \geq 2$, and $0 \leq l \leq n - 2$, let $E_l = E(y^{1/p^l})$ (so $E_l \subset K_n \cdot E$). Then, $A_n \otimes_F E_l \notin \text{Dec}(K_n \cdot E/E_l)$.

**Proof of Theorem 3.2.** It is clearly sufficient to prove (3). Moreover, it is sufficient to prove (3) for the case $l = n - 1$ (for $p$ odd) and $l = n - 2$ (for $p = 2$). For, assume that for $l < n - 1$ and $p$ odd, or for $l < n - 2$ and $p = 2$,

$A_n \otimes_F E_l \sim (y^{1/p^l}, b_1; p^{n-l}, E_l, \omega_{n-l}) \otimes_{E_l} (b_2, \phi_n; p, E_l, \omega)$,

for some $b_1$ and $b_2 \in E_l^*$. Then, extending scalars to $E_{n-1}$ (for $p$ odd) and $E_{n-2}$ (for $p = 2$), we find by standard symbol algebra identities

$A_n \otimes_F E_{n-1} \sim (y^{1/p^{n-1}}, b_1; p, E_{n-1}, \omega) \otimes_{E_{n-1}} (b_2, \phi_n; p, E_{n-1}, \omega)$

for $p$ odd, and

$A_n \otimes_F E_{n-2} \sim (y^{1/p^{n-2}}, b_1; p^2, E_{n-2}, \omega_2) \otimes_{E_{n-2}} (b_2, \phi_n; p, E_{n-2}, \omega)$.
for \( p = 2 \). Thus, we find that for \( p \) odd and \( l < n - 1 \), if
\[
A_n \otimes_F E_l \in \text{Dec}(K_n \cdot E/E_l)
\]
then
\[
A_n \otimes_F E_{n-1} \in \text{Dec}(K_n \cdot E/E_{n-1}),
\]
and for \( p = 2 \) and \( l < n - 2 \), if
\[
A_n \otimes_F E_l \in \text{Dec}(K_n \cdot E/E_l)
\]
then
\[
A_n \otimes_F E_{n-2} \in \text{Dec}(K_n \cdot E/E_{n-2}).
\]

We find it convenient at this point to divide the proof according to whether \( p \) is odd or even.

**Case 1 (p odd).** Assume that

\[
A_n \otimes_F E_{n-1} \sim (y^{1/p^{n-1}}, b_1; p, E_{n-1}, \omega) \otimes E_{n-1}(b_2, \phi_n; p, E_{n-1}, \omega),
\]
for some \( b_1 \) and \( b_2 \in E^{*}_n \). By Lemma 3.1 and standard symbol algebra identities,

\[
A_n \otimes_F E_{n-1} \sim \left(y^{1/p^{n-1}}, \frac{\phi_n}{x_1^{p^n} x_2^{p^n}}; p^2, E_{n-1}, \omega_2\right).
\]

Put \( z = y^{1/p^n} \). Then, extending scalars further to \( E_n = E(z) \), and noting that \( x_1^{p^n} \) and \( x_2^{p^n} \) are \( p \)th powers, we find

\[
(z, \phi_n; p, E_n, \omega) \sim (b, \phi_n; p, E_n, \omega),
\]
where we have written \( b \) for \( b_2 \). Hence,

\[
(z/b, \phi_n; p, E_n, \omega) \sim 1,
\]
so

\[
(15) \quad z/b = N(u)
\]
for some \( u \in E_n((\phi_n)^{1/p_n}) \), where \( N \) denotes the norm from \( E_n((\phi_n)^{1/p_n}) \) to \( E_n \). We will prove that it is impossible to find \( b \in E_{n-1} \) and \( u \in E_n((\phi_n)^{1/p_n}) \) such that (16) holds.

If \( \overline{F}_0 \) denotes the algebraic closure of \( F_0 \), then \( \overline{F}_0(x_1, x_2, y) \) is normal over \( F_0(x_1, x_2, y) \), so if \( E = F_0(x_1, x_2, y)(t) \) for some \( t \in E^* \), then it is standard that the degree of the minimum polynomial of \( t \) over \( \overline{F}_0(x_1, x_2, y) \) divides the degree of the minimum polynomial of \( t \) over \( F_0(x_1, x_2, y) \). Hence \( p \not| [E:\overline{F}_0(x_1, x_2, y): F_0(x_1, x_2, y)] \). Thus, while showing that (15) cannot hold, we may assume that \( F_0 \) is
algebraically closed. In particular, we may assume that $F_0$ contains $p^{1/p^i}$ for all $i (i = 1, 2, \ldots)$, so we may apply the machinery of §2.

Now write $\chi$ for $h_0(x_1, x_2, z)$, where $h_0$ is as in §2. As with the polynomial $h_0$, we will abuse notation and continue to write $\chi$ for the residue of $h_0$ under appropriate $p$-adic valuations. Observe that over $E_n$, $\phi_n = (x_1^{p^n} - z^{p^n})(x_2^{p^n} - z^{p^n})$, which, after renaming variables is indeed the same as the "$\phi_n$" of §2.

We first need an easy lemma:

**Lemma 3.3.** Let $p$ be a prime, and let $(F, v)$ be a valued field. Let $K$ be a finite dimensional separable extension of $F$ such that $p \nmid [K : F]$. Then for some extension of $v$ to $K$, $p \nmid [\overline{K} : \overline{F}]$.

**Proof.** Let $v_i (1 \leq i \leq s)$ be the extensions of $v$ to $K$, and let $(\overline{K})_i$ denote the residues of $K$ with respect to the valuations $v_i$. Let $F_h$ denote the henselization of $F$ with respect to $v_i$ (1 $\leq i \leq s$). Then (by [E, Theorem 17.17]) $[K : F] = \sum_{i=1}^s [K_{i, h} : F_h]$, so if $p \nmid [K : F]$, then $p \nmid [K_{i, h} : F_h]$ for some $i$. Now $K_{i, h} = (\overline{K})_i$ and $\overline{F_h} = \overline{F}$, so by Ostrowski's theorem ([O, Satz 4], see also [E, Theorem 20.21]), $((\overline{K})_i : \overline{F}) \mid [K_{i, h} : F_h]$. Hence, for this $i$, $p \nmid [((\overline{K})_i : \overline{F})].$

**Proof of Theorem 3.2 (continued).** Now let $L = F_0(x_1, x_2, z)$ and let $v$ be the standard extension of any $p$-adic valuation on $F_0$ to $L$ (so $L = \overline{F_0}(x_1, x_2, z)$). Let $L_1 = F_0(x_1, x_2, z^p)$, and let $v_{L_1}$ denote the restriction of $v$ to $L_1$. Choose an extension $w$ of $v_{L_1}$ to $E_n$ such that $p \nmid [E_{n-1} : \overline{L_1}]$. (Since $[E_{n-1} : L_1] = [E : F]$, the lemma above shows that such a choice is possible.) By Proposition 2.4 $v$ extends uniquely from $L$ to $L(\phi_n^{1/p})$, with residue $\overline{L}((\chi^{1/p})$. Since $p \nmid [E_{n-1} : \overline{L_1}]$, while $[L(\phi_n^{1/p}) : \overline{L_1}] = p^2$, it follows easily that $w$ extends uniquely from $E_{n-1}$ to $E_n(\phi_n^{1/p})$, with residue $\overline{E_n}((\chi^{1/p})$.

Now, continue to write $w$ for the (unique) extension of $w$ to $E_n(\phi_n^{1/p})$ and consider the relation (15). Since $v(z) = 0$, we get $w(b) + w(N(u)) = 0$. Since $\Gamma_{E_{n-1}} = \Gamma_{E_n(\phi_n^{1/p})}$, there is a $c \in E_{n-1}$ such that $w(c) = w(u)$. Then, $bN(u) = bc^pN(u/c)$, and $w(u/c) = 0$, $w(bc^p) = w(b) + p \cdot w(u) = w(b) + w(N(u)) = 0$, and of course, $bc^p \in E_{n-1}$. Hence, we may assume in (15) that $w(b) = w(u) = 0$.

Now let $\sigma$ be a generator of $\mathcal{G}(E_n(\phi_n^{1/p})/E_n)$, so

$$N(u) = u \cdot \sigma(u) \cdots \sigma^{p-1}(u).$$
Hence, $N(u) = \bar{u} \cdot \tilde{\sigma}(\bar{u}) \cdots \tilde{\sigma}^{p-1}(\bar{u})$, where $\tilde{\sigma}$ is the induced automorphism of $\overline{E_n(\chi^{1/p})/E_n}$ (i.e., $\tilde{\sigma}(\bar{x}) = \sigma(x)$ for all $\bar{x} \in \overline{E_n(\chi^{1/p})}$). Since the extension $\overline{E_n(\chi^{1/p})/E_n}$ is purely inseparable, $\tilde{\sigma}$ is just the identity, so find $N(u) = \bar{u}^p$. Thus, reducing the relation $z = bN(u)$ modulo the maximal ideal of the valuation ring of $w$, we find $z = \bar{b}\bar{u}^p$, where $\bar{b} \in \overline{E_{n-1}}$, and $\bar{u} \in \overline{E_n(\chi^{1/p})}$. We will show that such a relation is impossible.

Let $\overline{E_{n-1}} = \overline{L_1(\theta)}$, so that $1, \theta, \ldots, \theta^{s-1}$ form a basis for $\overline{E_{n-1}/L_1}$, with $s = [E_{n-1} : L_1]$. Since $p \nmid s$, it follows easily that $\overline{E_{n-1}} = \overline{L_1(\theta^p)}$, and $1, \theta^p, \ldots, \theta^{(s-1)p}$ also form a basis of $\overline{E_n/L_1}$. Likewise, $1, \theta, \ldots, \theta^{s-1}$, as well as $1, \theta^p, \ldots, \theta^{(s-1)p}$, are both bases of $\overline{E_n(\chi^{1/p})/L(\chi^{1/p})}$. Now let

$$1/\bar{b} = b_0 + b_1\theta^p + \cdots + b_{s-1}\theta^{(s-1)p},$$

where the $b_i \in \overline{L_1}$ ($i = 0, 1, \ldots, s - 1$). Similarly, let

$$\bar{u} = u_0 + u_1\theta + \cdots + u_{s-1}\theta^{s-1},$$

where the $u_i \in \overline{L(\chi^{1/p})}$ ($i = 0, 1, \ldots, s - 1$). Substituting the expressions above for $1/\bar{b}$ and $\bar{u}$ in $z/\bar{b} = \bar{u}^p$ and comparing like terms, we find

$$zb_0 = u_0^p,$$

where of course, $b_0 \in \overline{L_1}$ and $u_0 \in \overline{L(\chi^{1/p})}$. The impossibility of (16) above is just the impossibility of [T2, (23)], and follows immediately from the proof given there. However, for the sake of completeness, we will reprove this result here. Our proof will be different from that in [T2]; instead, it will be similar in spirit to the proof below of a corresponding result for $p = 2$.

Write $c$ for $1/b_0$ and $u$ for $u_0$, so we need to show that there do not exist $c \in \overline{L_1}$ ($= \overline{F_0(x_1, x_2, z^p)}$) and $u \in \overline{L(\chi^{1/p})}$ ($= \overline{F_0(x_1, x_2, z)(\chi^{1/p})}$) such that $z/c = u^p$. By considering the $z$-adic valuation on $\overline{L_1}$, it is easy to see that for any $c \in \overline{L_1}^*$, $z/c \notin \overline{L}^p$. Now assume that $z/c = u^p$ for some $c \in \overline{L_1}^*$ and some $u \in \overline{L(\chi^{1/p})}$. Then $\overline{L((z/c)^{1/p})} \subset \overline{L(\chi^{1/p})}$, so we find $\overline{L((z/c)^{1/p})} = \overline{L(\chi^{1/p})}$. Thus, there exist $f_i \in \overline{L^p}$ ($i = 0, 1, \ldots, p - 1$) such that

$$\chi := h_0(x_1, x_2, z) = \sum_{i=1}^{p-1} f_i(z/c)^i.$$
Since \(1, z, \ldots, z^{p-1}\) form a basis for \(L/L_1\), we may write

\[ h_0(x_1, x_2, z) = \sum_{i=0}^{p-1} e_i z^i \quad \text{for } e_i \in L_1, \]

where the values of the \(e_i\) may be derived from the definition of \(h_0\) in (6). Then, (17) takes the form

\[ c^{p-1} \left( \sum_{i=0}^{p-1} e_i z^i \right) = \sum_{i=0}^{p-1} c^{p-1-i} f_i z^i. \]

Now \(c \in L_1\), and \(L^p \subset L_1\). Hence, comparing the coefficients of \(z^i\) in (18), we find \(c^i e_i = f_i \) (\(i = 0, 1, \ldots, p-1\)). In particular, we find \(e_1 e_{p-1} = f_1 f_{p-1} / c^p\). Since \(f_1, f_{p-1}\), and \(c^p \in \overline{L}^p\), this shows \(e_1 e_{p-1} \in \overline{L}^p\). Now from (6), it is easy to see that

\[ e_1 = -[(x_1^p - z^p)x_2^{p-1} + (x_2^p - z^p)x_1^{p-1}], \]
\[ e_{p-1} = [(x_1^p - z^p)x_2 + (x_2^p - z^p)x_1]. \]

Multiplying out, we find \(x_2 x_2^{p-1} + x_2 x_1^{p-1} \in \overline{L}^p = F_0^p(x_1^p, x_2^p, z^p)\). Since \(p > 2\) (so \(x_1 x_2^{p-1} + x_2 x_1^{p-1} \neq 0\)), this is clearly impossible.

**Case 2 \((p = 2)\).** Assume that

\[ A_n \otimes_F E_{n-2} \sim (y^{1/2^{n-2}}, b_1; 2^2, E_{n-2}, \omega_2) \]
\[ \otimes E_{n-2}(b_2, \phi_n; 2, E_{n-2}, -1), \]

for some \(b_1\) and \(b_2 \in E_{n-2}^*\). Then, letting \(z = y^{1/2^n}\) and \(E_n = E(z)\), we find, exactly as in the \(p\) odd case, that \(z/b = N(u)\) for some \(b \in E_{n-2}^*\) and \(u \in E_n(\sqrt{\phi_n})\), where \(N\) denotes the norm from \(E_n(\sqrt{\phi_n})\) to \(E_n\). Letting \(\chi = h_0(x_1, x_2, z)\), assuming \(F_0\) is algebraically closed, and considering the standard extension of any 2-adic valuation on \(F_0\) to \(F_0(x_1, x_2, \ldots, z)\), we find, just as in the \(p\) odd case that for some \(b_0 \in F_0(x_1, x_2, z^4)\) and \(u_0 \in F_0(x_1, x_2, z)(\sqrt{\chi})\),

\[ zb_0 = u_0^2. \]

We will show that (19) is impossible.

Write \(L\) for the field \(F_0(x_1, x_2, z)\), \(L_1\) for the field \(F_0(x_1, x_2, z^2)\), and \(L_2\) for the field \(F_0(x_1, x_2, z^4)\). Assume that (19) holds for some \(b_0 \in L_2\) and \(u_0 \in L(\sqrt{\chi})\). By considering the \(z\)-adic valuation on \(L\) and noting that \(b_0 \in L_2\), it is easy to see that \(zb_0 \notin L_2^2\). Hence,
zb_0 = u_0^2$, then $L(\sqrt{x}) = L(\sqrt{zb_0})$. From this, as well as the definition of $h_0$ in (7), it follows that
\[ z((x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1) = f_0^2 + f_1^2 zb_0, \]
for some $f_0$ and $f_1 \in L$. Since 1 and $z$ form a basis for $L$ as an $L_1$ vector space, and since $f_0^2$, $f_1^2$, $(x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1$, and $b_0$ are all in $L_1$, we find
\[ (x_1^2 + z^2)x_2 + (x_2^2 + z^2)x_1 = f_1^2 b_0. \]
We write this as
\begin{equation}
(20) \quad \frac{x_1^2 x_2 + x_2^2 x_1}{b_0} + \frac{z^2(x_2 + x_1)}{b_0} = f_1^2.
\end{equation}
Now $f_1^2 \in L^2 = L_1^2(z^2)$. Thus $f_1^2 = g_0^2 + g_1^2 z^2$ for some $g_0$ and $g_1 \in L_1$. Substituting this in (20), we find
\begin{equation}
(21) \quad \frac{x_1^2 x_2 + x_2^2 x_1}{b_0} + \frac{z^2(x_2 + x_1)}{b_0} = g_0^2 + g_1^2 z^2.
\end{equation}
Now $x_1^2 x_2 + x_2^2 x_1$, $x_2 + x_1$, and $b_0$ (note!) are all in $L_2$. Moreover, $L_1^2 \subset L_2$. Since 1 and $z^2$ form a basis of $L_1$ as an $L_2$ vector space, we find on viewing (21) as an equation in $L_1$ that
\[ \frac{x_1^2 x_2 + x_2^2 x_1}{b_0} = g_0^2, \]
and
\[ \frac{x_2 + x_1}{b_0} = g_1^2. \]
Dividing, we find $x_1 x_2 = (g_0/g_1)^2$ for some $g_0$ and $g_1 \in L_1$. But $x_1 x_2$ is clearly not a square in $L_1$, and we are done. \[\square\]

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**California State University, Northridge**

Northridge, CA 91330
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