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**ERRATA TO: "THE SET OF PRIMES DIVIDING THE LUCAS
NUMBERS HAS DENSITY $2/3$ "**

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 THE SET OF PRIMES DIVIDING THE LUCAS
 NUMBERS HAS DENSITY $2/3$

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Theorem C of my paper [2] states an incorrect density for the set of primes that divide the terms W_n of a recurrence of Laxton [3], due to a slip in the proof. A corrected statement and proof are given.

The corrected version of Theorem C of [2] is:

THEOREM C. *Let W_n denote the recurrence defined by $W_0 = 1$, $W_1 = 2$ and $W_n = 5W_{n-1} - 7W_{n-2}$. Then the set*

$$S_W = \{p: p \text{ is prime and } p \text{ divides } W_n \text{ for some } n \geq 0\}$$

has density $3/4$.

The proof below proceeds along the general lines of §4 of [2].

Proof. One has

$$W_n = \left(\frac{3 + \sqrt{-3}}{6}\right) \left(\frac{5 + \sqrt{-3}}{2}\right)^n + \left(\frac{3 - \sqrt{-3}}{6}\right) \left(\frac{5 - \sqrt{-3}}{2}\right)^n.$$

If

$$\alpha = \frac{3 + \sqrt{-3}}{6} \quad \text{and} \quad \phi = \frac{5 + \sqrt{-3}}{5 - \sqrt{-3}} = \frac{11 + 5\sqrt{-3}}{14}$$

then

$$W_n \equiv 0 \pmod{p} \Leftrightarrow \phi^n \equiv -\frac{\bar{\alpha}}{\alpha} \pmod{(p)} \quad \text{in } \mathbb{Z}\left[\frac{1 + \sqrt{-3}}{2}\right],$$

where $-\frac{\bar{\alpha}}{\alpha} = \frac{-1 + \sqrt{-3}}{2}$ is a cube root of unity. Consequently

$$(1.1) \quad p \text{ divides } W_n \text{ for some } n \geq 0 \Leftrightarrow \text{ord}_{(p)}\phi \equiv 0 \pmod{3}.$$

The argument now depends on whether the prime ideal (p) splits or remains inert in the ring of integers $\mathbb{Z}\left[\frac{1 + \sqrt{-3}}{2}\right]$ of $\mathbb{Q}(\sqrt{-3})$.

Case 1. $p \equiv 1 \pmod{3}$, so that $p = \pi\bar{\pi}$ in $\mathbb{Z}\left[\frac{1 + \sqrt{-3}}{2}\right]$. Since $\text{ord}_{(\pi)}\phi = \text{ord}_{(\bar{\pi})}\phi$, one has

$$\text{ord}_{(p)}\phi \equiv 0 \pmod{3} \Leftrightarrow \text{ord}_{(\pi)}\phi \equiv 0 \pmod{3}.$$

Now suppose that $3^j \parallel (p - 1)$, in which case

$$(1.2) \quad \text{ord}_{(\pi)} \phi \not\equiv 0 \pmod{3} \Leftrightarrow \phi^{(p-1)/3^j} \equiv 1 \pmod{(\pi)}.$$

Set

$$\zeta_j := \exp\left(\frac{2\pi i}{3^j}\right), \quad \phi_j := \sqrt[j]{\phi},$$

and define the fields $F_j = \mathbb{Q}(\zeta_j, \phi_j)$ and $F_j^* = \mathbb{Q}(\zeta_{j+1}, \phi_j) = F_j(\zeta_{j+1})$. The last equivalence holds since F_j and F_j^* are normal extensions of \mathbb{Q} . Both F_j and F_j^* are normal extensions of \mathbb{Q} , because ϕ has norm one, so that the complex conjugate $\bar{\phi} = \phi^{-1}$, and $\bar{\phi}_j = \phi_j^{-1} \in F_j$. Now

$$(1.3) \quad \begin{aligned} &3^j \parallel p - 1 \text{ and } \phi^{\frac{p-1}{3^j}} \equiv 1 \pmod{(\pi)} \\ &\Leftrightarrow (\pi) \text{ splits completely in } F_j/\mathbb{Q}(\sqrt{-3}) \text{ and not completely in } \\ &F_j^*/\mathbb{Q}(\sqrt{-3}) \\ &\Leftrightarrow (p) \text{ splits completely in } F_j/\mathbb{Q} \text{ but not completely in } F_j^*/\mathbb{Q}. \end{aligned}$$

Applying the prime ideal theorem for the fields F_j and F_j^* , the density of primes such that (1.3) holds is

$$[F_j : \mathbb{Q}]^{-1} - [F_j^* : \mathbb{Q}]^{-1} = (2 \cdot 3^{2j-1})^{-1} - (2 \cdot 3^{2j})^{-1} = 3^{-2j}.$$

Hence the density of primes d_j having $3^j \parallel p - 1$ and $p \mid W_n$ for some n , which are those for which (1.3) doesn't hold, is $d_j = 3^{-j} - 3^{-2j}$ and the total density of primes $p \equiv 1 \pmod{3}$ dividing some W_n is $D_1 = \sum_{j=1}^{\infty} d_j = \frac{3}{8}$.

Case 2. $p \equiv 2 \pmod{3}$, so (p) is inert in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Since (p) is inert

$$\phi^{p^2-1} \equiv 1 \pmod{(p)}.$$

Assuming that $3^j \parallel (p + 1)$, one has

$$(1.4) \quad \text{ord}_{(p)} \phi \not\equiv 0 \pmod{3} \Leftrightarrow \phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{(p)}.$$

Now for $3^j \parallel (p + 1)$,

$$(1.5) \quad \begin{aligned} &\phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{(p)} \\ &\Leftrightarrow \text{The inert prime ideal } (p) \text{ in } \mathbb{Q}(\sqrt{-3}) \text{ splits completely in } \\ &F_j \text{ but not completely in } F_j^*. \end{aligned}$$

This latter condition is characterized as exactly those primes whose Artin symbol $[\frac{F_j^*/\mathbb{Q}}{(p)}]$ lies in certain conjugacy classes of the Galois

group $G^* = \text{Gal}(F_j^*/\mathbb{Q})$. (More generally such a characterization exists for any set of primes p determined by prime-splitting conditions on (p) in the subfields of a finite extension of \mathbb{Q} , see [1], Theorem 1.2.) To specify the conjugacy classes, we use the following facts. The group G^* is of order $2 \cdot 3^j$ with generators σ_1, σ_2 given by

$$\begin{aligned} \sigma_1(\zeta_{j+1}) &= \zeta_{j+1}^2, & \sigma_1(\phi_j) &= \bar{\phi}_j, & \sigma_1(\bar{\phi}_j) &= \phi_j, \\ \sigma_2(\zeta_{j+1}) &= \zeta_{j+1}, & \sigma_2(\phi_j) &= \zeta_j \phi_j, & \sigma_2(\bar{\phi}_j) &= \zeta_j^{-1} \bar{\phi}_j, \end{aligned}$$

where $\bar{\phi}_j = \phi_j^{-1}$ is the complex conjugate of ϕ_j . A general element of G^* is denoted $[k, l]$ where $\sigma = [k, l]$ acts by

$$\sigma(\zeta_{j+1}) = \zeta_{j+1}^{2^k}, \quad \sigma(\phi_j) = \zeta_j^l \phi_j^{(-1)^k}, \quad \sigma(\bar{\phi}_j) = \zeta_j^{-l} \phi_j^{(-1)^{k+1}}.$$

Here k is taken $(\text{mod } 2 \cdot 3^j)$ and $l (\text{mod } 3^j)$, and the group law is

$$[k, l] \circ [k', l'] = [k + k', l(-1)^{k'} + l'2^k].$$

Note that $\tau = \sigma_1^{3^j} = [3^j, 0]$ is complex conjugation. We claim that

$$(1.6) \quad 3^j \mid (p+1) \text{ and } \phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{p}$$

$$\Leftrightarrow \text{The Artin symbol } \left[\frac{F_j^*/\mathbb{Q}}{(p)} \right] \text{ is either } \langle \sigma_1^{3^{j-1}} \rangle \text{ or } \langle \sigma_1^{-3^{j-1}} \rangle.$$

One easily checks that the conjugacy classes containing $\sigma_1^{3^{j-1}}$ and $\sigma_1^{-3^{j-1}}$ each consist of one element. To prove the \Rightarrow implication in (1.6), note first that the condition that $3^j \mid (p+1)$ implies that the Artin symbol $\left[\frac{F_j^*/\mathbb{Q}}{(p)} \right]$ contains only elements of G^* of the form $\sigma_1^{\pm 3^{j+1}} \sigma_2^k$. Indeed, consider the action of an automorphism σ in $\left[\frac{F_j^*/\mathbb{Q}}{(p)} \right]$ restricted to the subfield $\mathbb{Q}(\zeta_{j+1})$. Now $\text{Gal}(\mathbb{Q}(\zeta_{j+1})/\mathbb{Q})$ is isomorphic to the subgroup generated by σ_1 and the restriction map sends $\sigma_1 \rightarrow \sigma_1$ and $\sigma_2 \rightarrow (\text{identity})$. Then $3^j \mid (p+1)$ says that σ restricted to $\mathbb{Q}(\zeta_j)$ is complex conjugation, but is not complex conjugation on $\mathbb{Q}(\zeta_{j+1})$. Hence $\sigma = [\pm 3^{j-1}, l]$ for some l . Next, any element σ of $\left[\frac{F_j^*/\mathbb{Q}}{(p)} \right]$ when restricted to acting on the subfield F_j has order equal to the degree over \mathbb{Q} of the prime ideals in F_j lying over (p) , which is 2. The group $G = \text{Gal}(F_j/\mathbb{Q})$ is isomorphic to the subgroup generated by σ_1^3 and σ_2 , with the restriction map $\Omega: G^* \rightarrow G$ sending $\sigma_1 \rightarrow \sigma_1^3$ and $\sigma_2 \rightarrow \sigma_2$. Thus $\Omega(\sigma) = [3^j, l]$ for some l . However the group law gives

$$[3^j, l] \circ [3^j, l] = [0, -2l].$$

Thus $[3^j, l]$ is of order 2 only if $l = 0$, and this proves the right

side of (1.6) holds. For the reverse direction, if $\sigma = [\pm 3^{j-1}, 0]$, then σ restricted to acting on F_j is $\Omega(\sigma) = [3^j, 0]$, which is complex conjugation τ , hence of order 2, so that

$$x^{p^2} \equiv x^{\sigma^2} = x \pmod{\mathfrak{p}}$$

for all prime ideals \mathfrak{p} in F_j lying over (p) , for all algebraic integers x in F_j . Thus

$$x^{p^2-1} \equiv 1 \pmod{(\mathfrak{p})}$$

for all such x , such that $(x, (p)) = 1$, including ϕ_j , and the left side of (1.6) holds.

Now the set of primes satisfying (1.6) has density $2[F_j^* : \mathbb{Q}]^{-1} = 3^{-2j}$, by the Chebotarev density theorem. The density of primes with $p^j \mid (p+1)$ and $p \mid W_n$ for some n then is $d_j^* = 3^{-j} - 3^{-2j}$, and the total density of primes $p \equiv 2 \pmod{3}$ with p dividing some W_n is

$$D_2 = \sum_{j=1}^{\infty} d_j = \frac{3}{8}.$$

Finally $D_1 + D_2 = \frac{3}{4}$, completing the proof. \square

REMARK. Of the 1228 primes less than 10^4 , one finds:

$$\#\{p: p \equiv 1 \pmod{3}, p \text{ divides some } W_n\} = 450,$$

$$\#\{p: p \equiv 2 \pmod{3}, p \text{ divides some } W_n\} = 466,$$

$$\#\{p: p \text{ does not divide any } W_n\} = 312.$$

These give frequencies of 36.6%, 37.3%, 25.4%, which may be compared with the asymptotic densities $3/8$, $3/8$, $1/4$, respectively, predicted by the proof of Theorem C.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 162 No. 2 February 1994

On the existence of convex classical solutions to multilayer fluid problems in arbitrary space dimensions	201
ANDREW FRENCH ACKER	
Extremal functions and the Chang-Marshall inequality	233
VALENTIN V. ANDREEV and ALEC LANE MATHESON	
Productive polynomials	247
RICHARD ARENS	
On factor representations of discrete rational nilpotent groups and the Plancherel formula	261
LAWRENCE JAY CORWIN and CAROLYN PFEFFER JOHNSTON	
Commutants of Toeplitz operators on the Bergman space	277
ZELJKO CUCKOVIC	
When L^1 of a vector measure is an AL-space	287
GUILLERMO P. CURBERA	
A convexity theorem for semisimple symmetric spaces	305
KARL-HERMANN NEEB	
Ideals of finite codimension in free algebras and the fc-localization	351
AMNON ROSENMAN and SHMUEL ROSSET	
Dec groups for arbitrarily high exponents	373
BHARATH AL SETHURAMAN	
Errata to: "The set of primes dividing the Lucas numbers has density $2/3$ "	393
JEFFREY C. LAGARIAS	



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