ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF
\[ \Delta u + e^u = 0 \]
IN A PUNCTURED DISC

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ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF $\Delta u + e^u = 0$ IN A PUNCTURED DISC

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In this paper a representation formula for solutions of the equation

(*) $\Delta u + 2Ke^u = 0$, $K$ a constant,

in a punctured disc in terms of multi-valued meromorphic functions is found. As application it is deduced that a necessary and sufficient condition for a solution of (*) $K > 0$, being asymptotic radially symmetric is

$$\int e^u < \infty.$$ 

1. Introduction. In [3], L. A. Caffarelli, B. Gidas, and J. Spruck proved that non-negative smooth solutions of the conformally invariant equation

(1) $\Delta u + u^{(n+2)/(n-2)} = 0$, $u \geq 0$,

in a punctured $n$-dimensional ball, $n \geq 3$, with an isolated singularity at the origin, are asymptotically radial. More precisely, if $u$ is a solution of (1), then

$$u(x) = (1 + o(1))\psi(|x|) \text{ as } x \to 0,$$

for some radial singular solution $\psi(r)$.

Geometrically speaking, to solve equation (1) is to find locally a conformal metric on a conformally flat $n$-dimensional manifold with constant scalar curvature. Therefore, its two-dimensional analogue is

(2) $\Delta u + e^u = 0$.

In this paper, we shall establish a similar asymptotic radial symmetry result for a smooth solution $u$ of (2) in the punctured disc, $D^* = D \setminus \{0\}$, $D = \{z \in \mathbb{C}||z| < 1\}$, with an isolated singularity at the origin, under

(3) $\int_{D^*} e^u < +\infty$.

Unlike the higher dimensional case, as one will see, that the integrability condition (3) is necessary for $u$ being asymptotically radial.
We point out that the isolated singularities or the behaviour at infinity of (2) in a punctured ball $B_1(0) \setminus \{0\} = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$ in 3-dimensions have been studied by M. Bidaut-Véron and L. Véron [2].

2. Results. Our approach to this problem is based on a classical result of Liouville which gives a representation of solutions of equation (2) in a simply-connected domain by analytic functions. We extend this representation to a punctured disc, and then deduce the result from analytic function theory.

Let us first recall Liouville’s theorem.

**Theorem 1** (Liouville [6]; see also [1]). Let $\Omega$ be a simply-connected domain in $\mathbb{R}^2$. Then all real solutions of

$$
\Delta u + 2Ke^u = 0 \quad \text{in } \Omega, \quad K \text{ a constant},
$$

are of the form

$$
u = \log \frac{|f'|^2}{(1 + (K/4)|f|^2)^2},
$$

where $f(z)$ is a locally univalent meromorphic function in $\Omega$.

**Corollary 2.** All solutions of equation (4) in $\Omega = \mathbb{R}^2$ with $K > 0$ and

$$\int_{\mathbb{R}^2} e^u < \infty$$

are of the form

$$u(x) = \log \frac{16\lambda^2}{(4 + \lambda^2 K|x - x_0|^2)^2}, \quad \lambda > 0, \ x_0 \in \mathbb{R}^2.$$

**Proof.** Let $u$ and $f$ be given in (5). Observe that Theorem 1 implies that $e^u|dz|^2 = f^*g_K$, where $g_K$ denotes the standard metric on $S^2$ with curvature $K$. By the integrability assumption $f$ cannot have an essential singularity at infinity, for otherwise $f$ would cover $S^2$ (possibly except one point) infinitely many times near infinity, which is impossible. Therefore $\lim_{z \to \infty} f(z) = \infty$ or some $z_0 \in \mathbb{C}$. By compositing with an inversion, we may assume the former case holds. Then $f$ maps $S^2$ onto $S^2$. Since $\mathbb{C}$ cannot cover $S^2$ (notice that $f'(z) \neq 0$ for all $z \in \mathbb{C}$), $f$ does not have poles in $\mathbb{C}$. This means $f: \mathbb{C} \to \mathbb{C}$ is a covering map and therefore it assumes the form $f(z) = \alpha z + \beta$ for some $\alpha \neq 0$ and $\beta$ in $\mathbb{C}$. A substitution into (5) gives the desired conclusion. \qed
Corollary 2 was previously proved by Chen and Li [4] by the method of moving planes. From (5), one can see that the integrability condition is also necessary for asymptotic radial symmetry. All non-radial solutions, which arise from transcendental functions, satisfy \( \int e^u = \infty \).

Theorem 1 is, in general, not true for domains which are not simply-connected. For instance, the function \( u = -\log 4r(1 + \frac{K}{4}r)^2 \) is a solution of equation (4) in the punctured disc \( D^* \), with an isolated singularity at the origin. Yet it is easy to see that this solution is given by a multi-valued analytic function \( f(z) = z^{1/2} \) instead of a single-valued analytic function in the punctured disc via the formula (5).

We now give an extension of Liouville's theorem for the punctured disc.

**Theorem 3.** Real solutions of the equation (4) are of the form (5), with \( f \) a multi-valued locally univalent meromorphic function satisfying:

1. When \( K > 0 \), \( f(z) = g(z)z^\alpha, \alpha \in \mathbb{R}, \) or \( \varphi(\sqrt{z}) \),
2. when \( K = 0 \), \( f(z) = g(z)z^\alpha \) or \( g(z) + c\log z, \alpha \in \mathbb{R}, c \in \mathbb{C} \); and
3. when \( K < 0 \), \( f(z) = h(z)z^\beta, \beta \geq 0 \).

Here \( g, \varphi, \) and \( h \) are single-valued analytic functions in \( D^*, D^* \), and \( D \) respectively, \( \varphi(z)\varphi(-z) = 1, h(0) \neq 0, \) and \( |h(D)| < 1 \).

**Proof.** Consider the universal cover \( \tilde{D}^* = (0, 1] \times \mathbb{R} \) of the punctured disc. Let \( \pi(r, \theta) = re^{i\theta} \) be the projection and let \( \tilde{g} = dr^2 + \frac{1}{r^2}d\theta^2 = \pi^*|dx|^2 \). It follows from Theorem 1 that there exists a local univalent meromorphic function \( \tilde{h}(z) \) on \( \tilde{D}^* \) such that \( e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K \), where \( \tilde{u} = \pi^*u = u \circ \pi \) and now \( g_K \) denotes the standard metric on the two dimensional space form \( S_K \) with curvature \( K \). Let \( \tau: \tilde{D}^* \to \tilde{D}^* \) be the map \( \tau(r, \theta) = (r, \theta + 2\pi) \). Then

\[
\tau^*\tilde{h}^*g_K = \tau^*(e^{\tilde{u}}\tilde{g}) = e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K.
\]

Therefore, \( \tilde{h} \circ \tau \circ \tilde{h}^{-1} \) is a local isometry of \( S_K \). By a result in differential geometry (Corollary 6.4, p. 256 in [5]), \( \tilde{h} \circ \tau \circ \tilde{h}^{-1} \) can be extended uniquely to a global isometry of \( S_K \). Locally

\[
\tilde{h} \circ \tau = \rho \circ \tilde{h}, \quad \rho \in \text{Isom}(S_K).
\]

Since \( \tilde{D}^* \) is simply connected, this holds globally. Moreover, \( \rho \) is analytic since \( \tilde{h} \) and \( \tau \) are analytic. Therefore, there exists a locally
univalent multi-valued meromorphic function \( h(z) = \tilde{h}(\pi^{-1} z) \) satisfying \( h(ze^{2\pi i}) = \rho(h(z)), \rho \in \text{Isom}(S_K), \rho \) analytic, in \( D^* \) such that
\[
u = \log \frac{|h'|^2}{(1 + (K/4)|h|^2)^2}.
\]
Here \( h(ze^{2\pi i}) \) denotes the value of \( h \) after a turn along the circle centered at the origin with radius \( |z| \).

By a change of coordinates, we only need to prove the theorem for \( K = 4, K = 0, \) and \( K = -4 \), where now \( \rho \) is an analytic isometry of the standard unit sphere, the Euclidean plane, and the Poincaré disc respectively.

For \( K = 4 \), \( \rho \) is given by
\[
\frac{w - a}{1 + \bar{a}w} = \frac{e^{i\theta} z - a}{1 + \bar{a}z}
\]
and
\[
\frac{w - a}{1 + \bar{a}w} = \frac{e^{i\theta} 1 + \bar{a}z}{z - a}
\]
for some \( a \in \mathbb{C} \) and \( \theta \in [0, 2\pi) \). In the first case, let
\[
f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)}.
\]
Then \( f \) satisfies
\[
f(ze^{2\pi i}) = e^{i\theta} f(z), \quad \forall z \in D^*.
\]
Consider the function
\[
g(z) = f(z)z^{-\alpha}
\]
on \( D^* \), where \( \alpha = \theta/2\pi \). We have
\[
g(ze^{2\pi i}) = f(ze^{2\pi i})(ze^{2\pi i})^{-\alpha}
\]
\[
= f(z)e^{i\theta} z^{-\alpha} e^{-2\pi \alpha i} = g(z)
\]
for all \( z \in D^* \). Hence \( g(z) \) is a single-valued function and therefore analytic in \( D^* \). So \( f(z) \) takes the form \( g(z)z^\alpha \). Using the fact that
\[
w = (z - a)/(1 + \bar{a}z)
\]
is an isometry of the standard unit sphere,
\[
u = \log \frac{|h'|^2}{(1 + |h|^2)^2} = \log \frac{|f'|^2}{(1 + |f|^2)^2},
\]
which proves the first case.

In the second case, letting
\[
f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)},
\]
we have \( f(ze^{4\pi i}) = f(z) \). Hence there exists a single-valued analytic function \( \varphi \) in the punctured disc satisfying \( f(z) = e^{i\theta/2}\varphi(\sqrt{z}) \). The condition \( f(ze^{2\pi i})f(z) = e^{i\theta} \) implies \( \varphi(z)\varphi(-z) = 1 \). The proof of the positive case is completed.

For \( K = 0 \), we notice that analytic isometries of the Euclidean plane are of the form \( w = e^{i\theta}z + c \), which can be represented by \( w = e^{i\theta}(z - a) \) or \( w = z + c \). Similar argument as in the positive case gives us the desired result.

Finally, for \( K = -4 \), analytic isometries of the Poincaré disc are in one of the following forms:

\[
\begin{align*}
\frac{w - a}{1 - \bar{a}w} &= e^{i\theta}\frac{z - a}{1 - \bar{a}z}, \quad \text{with } |a| < 1, \\
\frac{w - e^{i\theta_1}}{w - e^{i\theta_2}} &= k\frac{z - e^{i\theta_1}}{z - e^{i\theta_2}}, \quad \text{with } k > 1, \theta_1 \neq \theta_2 \in \mathbb{R}, \\
\frac{w - e^{i\theta}}{w + e^{i\theta} + c} &= \frac{z - e^{i\theta}}{z + e^{i\theta}}, \quad \text{with } \theta \in \mathbb{R}, \ c \in \mathbb{C}.
\end{align*}
\]

Using the same argument as above one can show that \( f \) assumes one of the following forms:

1. \( g(z)z^\alpha \),
2. \( e^{i\theta_1}(e^{i\theta_2} - g(z)z^{i\alpha})/(e^{-i\theta_2} - g(z)z^{i\alpha}) \), and
3. \( e^{i\alpha}(1 + g(z) + \alpha \log z)/(1 - g(z) - \alpha \log z) \),

where \( g \) is analytic in \( D^* \), and \( \alpha, \theta_1, \theta_2, \theta \in \mathbb{R} \). Observe that in (5) \((K = -4) u \) becomes singular at \(|f| = 1\). Hence, by the analyticity of \( f \) and the regularity of \( u \), the image of \( f \) lies either inside or outside \( D \). Replacing \( f \) by \( 1/f \) if \(|f| > 1\), we may assume \( f(D^*) \) is contained in \( D \). This immediately implies that the expression in (i) can be rewritten as \( h(z)z^\beta \) where \( h(0) \neq 0 \) and \( \beta \geq 0 \).

In the following let \( h \) stand for an analytic function in \( D \) with \( h(0) \neq 0 \). We shall show that in (ii) and (iii) \( \alpha = 0 \) and \( g(z) = h(z) \), and consequently they are special cases of (i). To see this first observe that in case (ii) the image of \( D^* \) under the map \( g(z)z^{i\alpha} \) lies in a half plane, which, modulo a rotation, may be taken to be the upper half plane. We have

\[ 0 < \arg(g(z)z^{i\alpha}) = \arg g(z) + \alpha \log|z| < \pi \pmod{2\pi}. \]

Applying the maximum principle to \( \Im g(z) \) in the annulus \( r_j < |z| < r_{j_0}, \ r_j = e^{-2j\pi/|\alpha|}, \ j > j_0, \ j_0 \ \text{large}, \) we conclude that \( \Im g(z) > 0 \) for
all $z$ in a deleted neighborhood of 0. Hence 0 cannot be an essential singularity of $g$. Now we can write $g(z) = h(z)z^k$, $k \in \mathbb{Z}$. Then the inequality

$$0 < \arg(g(z)z^{i\alpha}) = \arg h(z) + \alpha \log|z| + k \arg z < \pi \quad (\text{mod } 2\pi)$$

implies $\alpha = k = 0$. Similarly one can show that in (iii) $\alpha = 0$ and $g(z) = h(z)$. This completes our proof of the theorem. \qed

Now we can deduce an asymptotic radial symmetry result for equation (4) from Theorem 3. First we need a lemma from complex analysis.

**Lemma 4.** Suppose that $g(z)$ is a holomorphic function in $D^*$ which has an essential singularity at the origin. Then the multi-valued function $f(z) = z^\alpha g(z)$, $\alpha \in \mathbb{R}$, takes all values infinitely many times except at most one value.

**Proof.** Consider the single-valued function $\phi(z) = z^k f(z) = z^k g(z)$, where $k$ is an integer such that $k > \alpha$. Since $g$ has an essential singularity at the origin, so has $\phi$. The sequence

$$\phi_n(z) = \phi\left(\frac{z}{2^n}\right)$$

is not a normal sequence on some annulus $\Gamma: r/4 < |z| < 2r$. In particular, the sequence is not a normal sequence on intersection $\Omega$ of $\Gamma$ with any sector: $|\arg z - \arg z_0| < \epsilon$, in the unit disc. Therefore the sequence

$$f_n(z) = f\left(\frac{z}{2^n}\right)$$

cannot be normal on $\Omega$. Now, applying the Montel theorem [7], we see that for any $a \in \mathbb{C}$, except at most one point, there exist infinitely many $n$ such that $f_n$ takes the value $a$ in $\Omega$. This implies that $f$ takes the value $a$ infinitely many times in the sector. \qed

**Theorem 5.** Let $u$ be a smooth real solution of the equation (4) with $K > 0$ in the punctured disc $D^*$. Then $u$ is asymptotically radial, more precisely,

$$u(z) = \alpha \log|z| + O(1) \quad \text{as } |z| \to 0, \alpha > -2,$$

if and only if

$$\int_{D^*} e^u < +\infty.$$
Proof. By Theorem 3, the metric \( e^u|dz|^2 \) is the pull-back of the spherical metric with curvature \( K \) via the holomorphic map \( f \). Moreover \( f \) is a covering map on \( D \setminus \{z < 0\} \) since \( f' \neq 0 \) for all \( z \in D^* \). If \( g \) takes the value \( \infty \) infinitely many times, then so does \( f \). This implies \( e^u|dz|^2 \) has infinite volume, i.e. \( \int_{D^*} e^u = +\infty \). So we may assume \( g \) takes \( \infty \) for finitely many times. Then \( g \) is holomorphic near the essential singularity and we can apply Lemma 4 (in case \( f(z) = g(z)z^a \)) to conclude that \( f \) covers the image of \( f \) in the sphere infinitely many times. Thus \( \int_{D^*} e^u = +\infty \). Therefore, the integrability condition implies that \( g \) at most has a pole at the origin. Simple calculation now establishes the asymptotic radial symmetry of the solution \( u \).

Remark. Theorem 5 no longer holds for \( K = 0 \). In fact, it is straightforward to show that \( \int e^u|dz|^2 < \infty \) for some deleted neighborhood of 0 if and only if \( f(z) = h(z)z^\alpha \), \( h(0) \neq 0 \) and \( \alpha > 0 \). In particular, all radially symmetric solutions corresponding to \( f(z) = h(z)z^k + c \log z \), \( k \in \mathbb{Z} \), \( c \neq 0 \), satisfy \( \int e^u|dz|^2 = \infty \) in any deleted neighborhood of 0.

On the other hand, Theorem 5 holds for \( K < 0 \). In fact, all solutions are asymptotic radially symmetric and satisfy \( \int e^u|dz|^2 < \infty \).

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