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**ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF
 $\Delta u + e^u = 0$ IN A PUNCTURED DISC**

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ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF $\Delta u + e^u = 0$ IN A PUNCTURED DISC

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In this paper a representation formula for solutions of the equation

$$(*) \quad \Delta u + 2Ke^u = 0, \quad K \text{ a constant,}$$

in a punctured disc in terms of multi-valued meromorphic functions is found. As application it is deduced that a necessary and sufficient condition for a solution of $(*)$, $K > 0$, being asymptotic radially symmetric is

$$\int e^u < \infty.$$

1. Introduction. In [3], L. A. Caffarelli, B. Gidas, and J. Spruck proved that non-negative smooth solutions of the conformally invariant equation

$$(1) \quad \Delta u + u^{(n+2)/(n-2)} = 0, \quad u \geq 0,$$

in a punctured n -dimensional ball, $n \geq 3$, with an isolated singularity at the origin, are asymptotically radial. More precisely, if u is a solution of (1), then

$$u(x) = (1 + o(1))\psi(|x|) \quad \text{as } x \rightarrow 0,$$

for some radial singular solution $\psi(r)$.

Geometrically speaking, to solve equation (1) is to find locally a conformal metric on a conformally flat n -dimensional manifold with constant scalar curvature. Therefore, its two-dimensional analogue is

$$(2) \quad \Delta u + e^u = 0.$$

In this paper, we shall establish a similar asymptotic radial symmetry result for a smooth solution u of (2) in the punctured disc, $D^* = D \setminus \{0\}$, $D = \{z \in \mathbb{C} \mid |z| < 1\}$, with an isolated singularity at the origin, under

$$(3) \quad \int_{D^*} e^u < +\infty.$$

Unlike the higher dimensional case, as one will see, that the integrability condition (3) is necessary for u being asymptotically radial.

We point out that the isolated singularities or the behaviour at infinity of (2) in a punctured ball $B_1(0) \setminus \{0\} = \{x \in \mathbb{R}^3: 0 < |x| < 1\}$ in 3-dimensions have been studied by M. Bidaut-Véron and L. Véron [2].

2. Results. Our approach to this problem is based on a classical result of Liouville which gives a representation of solutions of equation (2) in a simply-connected domain by analytic functions. We extend this representation to a punctured disc, and then deduce the result from analytic function theory.

Let us first recall Liouville's theorem.

THEOREM 1 (Liouville [6]; see also [1]). *Let Ω be a simply-connected domain in \mathbb{R}^2 . Then all real solutions of*

$$(4) \quad \Delta u + 2Ke^u = 0 \quad \text{in } \Omega, \quad K \text{ a constant,}$$

are of the form

$$(5) \quad u = \log \frac{|f'|^2}{(1 + (K/4)|f|^2)^2},$$

where $f(z)$ is a locally univalent meromorphic function in Ω .

COROLLARY 2. *All solutions of equation (4) in $\Omega = \mathbb{R}^2$ with $K > 0$ and*

$$\int_{\mathbb{R}^2} e^u < \infty$$

are of the form

$$u(x) = \log \frac{16\lambda^2}{(4 + \lambda^2 K|x - x_0|^2)^2}, \quad \lambda > 0, \quad x_0 \in \mathbb{R}^2.$$

Proof. Let u and f be given in (5). Observe that Theorem 1 implies that $e^u |dz|^2 = f^* g_K$, where g_K denotes the standard metric on \mathbb{S}^2 with curvature K . By the integrability assumption f cannot have an essential singularity at infinity, for otherwise f would cover \mathbb{S}^2 (possibly except one point) infinitely many times near infinity, which is impossible. Therefore $\lim_{z \rightarrow \infty} f(z) = \infty$ or some $z_0 \in \mathbb{C}$. By compositing with an inversion, we may assume the former case holds. Then f maps \mathbb{S}^2 onto \mathbb{S}^2 . Since \mathbb{C} cannot cover \mathbb{S}^2 (notice that $f'(z) \neq 0$ for all $z \in \mathbb{C}$), f does not have poles in \mathbb{C} . This means $f: \mathbb{C} \rightarrow \mathbb{C}$ is a covering map and therefore it assumes the form $f(z) = \alpha z + \beta$ for some $\alpha \neq 0$ and β in \mathbb{C} . A substitution into (5) gives the desired conclusion. \square

Corollary 2 was previously proved by Chen and Li [4] by the method of moving planes. From (5), one can see that the integrability condition is also necessary for asymptotic radial symmetry. All non-radial solutions, which arise from transcendental functions, satisfy $\int e^u = \infty$.

Theorem 1 is, in general, not true for domains which are not simply-connected. For instance, the function $u = -\log 4r(1 + \frac{K}{4}r)^2$ is a solution of equation (4) in the punctured disc D^* , with an isolated singularity at the origin. Yet it is easy to see that this solution is given by a multi-valued analytic function $f(z) = z^{1/2}$ instead of a single-valued analytic function in the punctured disc via the formula (5).

We now give an extension of Liouville's theorem for the punctured disc.

THEOREM 3. *Real solutions of the equation (4) are of the form (5), with f a multi-valued locally univalent meromorphic function satisfying:*

1. *When $K > 0$, $f(z) = g(z)z^\alpha$, $\alpha \in \mathbb{R}$, or $\varphi(\sqrt{z})$,*
 2. *when $K = 0$, $f(z) = g(z)z^\alpha$ or $g(z) + c \log z$, $\alpha \in \mathbb{R}$, $c \in \mathbb{C}$;*
- and*
3. *when $K < 0$, $f(z) = h(z)z^\beta$, $\beta \geq 0$.*

Here g , φ , and h are single-valued analytic functions in D^ , D^* , and D respectively, $\varphi(z)\varphi(-z) = 1$, $h(0) \neq 0$, and $|h(D)| < 1$.*

Proof. Consider the universal cover $\tilde{D}^* = (0, 1] \times \mathbb{R}$ of the punctured disc. Let $\pi(r, \theta) = re^{i\theta}$ be the projection and let $\tilde{g} = dr^2 + \frac{1}{r^2}d\theta^2 = \pi^*|dx|^2$. It follows from Theorem 1 that there exists a local univalent meromorphic function $\tilde{h}(z)$ on \tilde{D}^* such that $e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K$, where $\tilde{u} = \pi^*u = u \circ \pi$ and now g_K denotes the standard metric on the two dimensional space form S_K with curvature K . Let $\tau: \tilde{D}^* \rightarrow \tilde{D}^*$ be the map $\tau(r, \theta) = (r, \theta + 2\pi)$. Then

$$\tau^*\tilde{h}^*g_K = \tau^*(e^{\tilde{u}}\tilde{g}) = e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K.$$

Therefore, $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ is a local isometry of S_K . By a result in differential geometry (Corollary 6.4, p. 256 in [5]), $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ can be extended uniquely to a global isometry of S_K . Locally

$$\tilde{h} \circ \tau = \rho \circ \tilde{h}, \quad \rho \in \text{Isom}(S_K).$$

Since \tilde{D}^* is simply connected, this holds globally. Moreover, ρ is analytic since \tilde{h} and τ are analytic. Therefore, there exists a locally

univalent multi-valued meromorphic function $h(z) = \tilde{h}(\pi^{-1}z)$ satisfying $h(ze^{2\pi i}) = \rho(h(z))$, $\rho \in \text{Isom}(S_K)$, ρ analytic, in D^* such that

$$u = \log \frac{|h'|^2}{(1 + (K/4)|h|^2)^2}.$$

Here $h(ze^{2\pi i})$ denotes the value of h after a turn along the circle centered at the origin with radius $|z|$.

By a change of coordinates, we only need to prove the theorem for $K = 4$, $K = 0$, and $K = -4$, where now ρ is an analytic isometry of the standard unit sphere, the Euclidean plane, and the Poincaré disc respectively.

For $K = 4$, ρ is given by

$$\frac{w - a}{1 + \bar{a}w} = e^{i\theta} \frac{z - a}{1 + \bar{a}z}$$

and

$$\frac{w - a}{1 + \bar{a}w} = e^{i\theta} \frac{1 + \bar{a}z}{z - a}$$

for some $a \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. In the first case, let

$$f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)}.$$

Then f satisfies

$$f(ze^{2\pi i}) = e^{i\theta} f(z), \quad \forall z \in D^*.$$

Consider the function

$$g(z) = f(z)z^{-\alpha}$$

on D^* , where $\alpha = \theta/2\pi$. We have

$$\begin{aligned} g(ze^{2\pi i}) &= f(ze^{2\pi i})(ze^{2\pi i})^{-\alpha} \\ &= f(z)e^{i\theta} z^{-\alpha} e^{-2\pi\alpha i} = g(z) \end{aligned}$$

for all $z \in D^*$. Hence $g(z)$ is a single-valued function and therefore analytic in D^* . So $f(z)$ takes the form $g(z)z^\alpha$. Using the fact that $w = (z - a)/(1 + \bar{a}z)$ is an isometry of the standard unit sphere,

$$u = \log \frac{|h'|^2}{(1 + |h|^2)^2} = \log \frac{|f'|^2}{(1 + |f|^2)^2},$$

which proves the first case.

In the second case, letting

$$f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)},$$

we have $f(ze^{4\pi i}) = f(z)$. Hence there exists a single-valued analytic function φ in the punctured disc satisfying $f(z) = e^{i\theta/2}\varphi(\sqrt{z})$. The condition $f(ze^{2\pi i})f(z) = e^{i\theta}$ implies $\varphi(z)\varphi(-z) = 1$. The proof of the positive case is completed.

For $K = 0$, we notice that analytic isometries of the Euclidean plane are of the form $w = e^{i\theta}z + c$, which can be represented by $w - a = e^{i\theta}(z - a)$ or $w = z + c$. Similar argument as in the positive case gives us the desired result.

Finally, for $K = -4$, analytic isometries of the Poincaré disc are in one of the following forms:

$$\begin{aligned} \frac{w - a}{1 - \bar{a}w} &= e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad \text{with } |a| < 1, \\ \frac{w - e^{i\theta_1}}{w - e^{i\theta_2}} &= k \frac{z - e^{i\theta_1}}{z - e^{i\theta_2}}, \quad \text{with } k > 1, \theta_1 \neq \theta_2 \in \mathbb{R}, \\ \frac{w - e^{i\theta}}{w + e^{i\theta}} &= \frac{z - e^{i\theta}}{z + e^{i\theta}} + c, \quad \text{with } \theta \in \mathbb{R}, c \in \mathbb{C}. \end{aligned}$$

Using the same argument as above one can show that f assumes one of the following forms:

- (i) $g(z)z^\alpha$,
- (ii) $e^{i\theta_1}(e^{i\theta_2} - g(z)z^{i\alpha})/(e^{-i\theta_2} - g(z)z^{i\alpha})$, and
- (iii) $e^{i\alpha}(1 + g(z) + \alpha \log z)/(1 - g(z) - \alpha \log z)$,

where g is analytic in D^* , and $\alpha, \theta_1, \theta_2, \theta \in \mathbb{R}$. Observe that in (5) ($K = -4$) u becomes singular at $|f| = 1$. Hence, by the analyticity of f and the regularity of u , the image of f lies either inside or outside D . Replacing f by $1/f$ if $|f| > 1$, we may assume $f(D^*)$ is contained in D . This immediately implies that the expression in (i) can be rewritten as $h(z)z^\beta$ where $h(0) \neq 0$ and $\beta \geq 0$.

In the following let h stand for an analytic function in D with $h(0) \neq 0$. We shall show that in (ii) and (iii) $\alpha = 0$ and $g(z) = h(z)$, and consequently they are special cases of (i). To see this first observe that in case (ii) the image of D^* under the map $g(z)z^{i\alpha}$ lies in a half plane, which, modulo a rotation, may be taken to be the upper half plane. We have

$$0 < \arg(g(z)z^{i\alpha}) = \arg g(z) + \alpha \log|z| < \pi \pmod{2\pi}.$$

Applying the maximum principle to $\text{Im } g(z)$ in the annulus $r_j < |z| < r_{j_0}$, $r_j = e^{-2j\pi/|\alpha|}$, $j > j_0$, j_0 large, we conclude that $\text{Im } g(z) > 0$ for

all z in a deleted neighborhood of 0. Hence 0 cannot be an essential singularity of g . Now we can write $g(z) = h(z)z^k$, $k \in \mathbb{Z}$. Then the inequality

$$0 < \arg(g(z)z^{i\alpha}) = \arg h(z) + \alpha \log|z| + k \arg z < \pi \pmod{2\pi}$$

implies $\alpha = k = 0$. Similarly one can show that in (iii) $\alpha = 0$ and $g(z) = h(z)$. This completes our proof of the theorem. \square

Now we can deduce an asymptotic radial symmetry result for equation (4) from Theorem 3. First we need a lemma from complex analysis.

LEMMA 4. *Suppose that $g(z)$ is a holomorphic function in D^* which has an essential singularity at the origin. Then the multi-valued function $f(z) = z^\alpha g(z)$, $\alpha \in \mathbb{R}$, takes all values infinitely many times except at most one value.*

Proof. Consider the single-valued function $\phi(z) = z^{k-\alpha} f(z) = z^k g(z)$, where k is an integer such that $k > \alpha$. Since g has an essential singularity at the origin, so has ϕ . The sequence

$$\phi_n(z) = \phi\left(\frac{z}{2^n}\right)$$

is not a normal sequence on some annulus $\Gamma: r/4 < |z| < 2r$. In particular, the sequence is not a normal sequence on intersection Ω of Γ with any sector: $|\arg z - \arg z_0| < \varepsilon$, in the unit disc. Therefore the sequence

$$f_n(z) = f\left(\frac{z}{2^n}\right)$$

cannot be normal on Ω . Now, applying the Montel theorem [7], we see that for any $a \in \mathbb{C}$, except at most one point, there exist infinitely many n such that f_n takes the value a in Ω . This implies that f takes the value a infinitely many times in the sector. \square

THEOREM 5. *Let u be a smooth real solution of the equation (4) with $K > 0$ in the punctured disc D^* . Then u is asymptotically radial, more precisely,*

$$u(z) = \alpha \log|z| + O(1) \quad \text{as } |z| \rightarrow 0, \alpha > -2,$$

if and only if

$$\int_{D^*} e^u < +\infty.$$

Proof. By Theorem 3, the metric $e^u|dz|^2$ is the pull-back of the spherical metric with curvature K via the holomorphic map f . Moreover f is a covering map on $D \setminus \{z < 0\}$ since $f' \neq 0$ for all $z \in D^*$. If g takes the value ∞ infinitely many times, then so does f . This implies $e^u|dz|^2$ has infinite volume, i.e. $\int_{D^*} e^u = +\infty$. So we may assume g takes ∞ for finitely many times. Then g is holomorphic near the essential singularity and we can apply Lemma 4 (in case $f(z) = g(z)z^\alpha$) to conclude that f covers the image of f in the sphere infinitely many times. Thus $\int_{D^*} e^u = +\infty$. Therefore, the integrability condition implies that g at most has a pole at the origin. Simple calculation now establishes the asymptotic radial symmetry of the solution u . \square

REMARK. Theorem 5 no longer holds for $K = 0$. In fact, it is straightforward to show that $\int e^u|dz|^2 < \infty$ for some deleted neighborhood of 0 if and only if $f(z) = h(z)z^\alpha$, $h(0) \neq 0$ and $\alpha > 0$. In particular, all radially symmetric solutions corresponding to $f(z) = h(z)z^k + c \log z$, $k \in \mathbb{Z}$, $c \neq 0$, satisfy $\int e^u|dz|^2 = \infty$ in any deleted neighborhood of 0.

On the other hand, Theorem 5 holds for $K < 0$. In fact, all solutions are asymptotic radially symmetric and satisfy $\int e^u|dz|^2 < \infty$.

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