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We define the Dirichlet space \mathcal{D} on the unit polydisc \mathbb{U}^n of \mathbb{C}^n . \mathcal{D} is a semi-Hilbert space of holomorphic functions, contains the holomorphic polynomials densely, is invariant under compositions with the biholomorphic automorphisms of \mathbb{U}^n , and its semi-norm is preserved under such compositions. We show that \mathcal{D} is unique with these properties. We also prove \mathcal{D} is unique if we assume that the semi-norm of a function in \mathcal{D} composed with an automorphism is only equivalent in the metric sense to the semi-norm of the original function. Members of a subclass of \mathcal{D} given by a norm can be written as potentials of L^2 -functions on the n -torus \mathbb{T}^n . We prove that the functions in this subclass satisfy strong-type inequalities and have tangential limits almost everywhere on $\partial\mathbb{U}^n$. We also make capacity estimates on the size of the exceptional sets on $\partial\mathbb{U}^n$.

1. Introduction. Möbius-invariant spaces. Let \mathbb{U} be the open unit disc in \mathbb{C} and \mathbb{T} be the unit circle bounding it. The open unit polydisc \mathbb{U}^n and the torus \mathbb{T}^n in \mathbb{C}^n are the cartesian products of n unit discs and n unit circles, respectively. \mathbb{T}^n is the distinguished boundary of \mathbb{U}^n and forms only a small part of the topological boundary $\partial\mathbb{U}^n$ of \mathbb{U}^n . We denote by \mathcal{M} the group of all biholomorphic automorphisms of \mathbb{U}^n (the Möbius group). The subgroup of linear automorphisms in \mathcal{M} is denoted by \mathcal{U} . The space of holomorphic functions with domain \mathbb{U}^n will be called $\mathcal{H}(\mathbb{U}^n)$ and will carry the topology of uniform convergence on compact subsets of \mathbb{U}^n .

A semi-inner product on a complex vector space \mathcal{H} is a sesquilinear functional on $\mathcal{H} \times \mathcal{H}$ with all the properties of an inner product except that it is possible to have $\langle\langle a, a \rangle\rangle = 0$ when $a \neq 0$. $\|a\| = \sqrt{\langle\langle a, a \rangle\rangle}$ is the associated semi-norm. We assume $\langle\langle \cdot, \cdot \rangle\rangle$ is not identically zero.

DEFINITION 1.1. \mathcal{H} is called a Hilbert space of holomorphic functions on \mathbb{U}^n if

- (i) \mathcal{H} is a linear subspace of $\mathcal{H}(\mathbb{U}^n)$,
- (ii) the semi-inner product $\langle\langle \cdot, \cdot \rangle\rangle$ of \mathcal{H} is complete,
- (iii) \mathcal{H} contains all (holomorphic) polynomials,
- (iv) polynomials are dense in \mathcal{H} in the topology of the semi-norm $\|\cdot\|$ of \mathcal{H} .

A space \mathcal{H} of functions on \mathbb{U}^n is \mathcal{M} -invariant if $f \circ \Psi \in \mathcal{H}$ whenever $f \in \mathcal{H}$ and $\Psi \in \mathcal{M}$. An \mathcal{M} -invariant Hilbert space \mathcal{H} of holomorphic functions on \mathbb{U}^n will be called an \mathcal{M} -space for brevity. \mathcal{U} -invariance and \mathcal{U} -space have similar definitions.

\mathbb{N} , \mathbb{Z}_+ , \mathbb{Z} , \mathbb{R} denote the set of nonnegative integers, positive integers, integers, and real numbers, respectively. A multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ is a point in \mathbb{N}^n . \sum_{α} indicates a summation with α running over all the points in \mathbb{N}^n , and $\sum'_{\alpha \in \mathbb{I}}$ is a summation where we consider only those α in the index set \mathbb{I} with all positive components. Let also $D_j = \partial/\partial z_j$ and $\bar{D}_j = \partial/\partial \bar{z}_j$. The following abbreviated notations will be used:

$$\begin{aligned} |\alpha| &= \alpha_1 + \dots + \alpha_n, & z^\alpha &= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \\ \alpha! &= \alpha_1! \dots \alpha_n!, & D^\alpha &= D_1^{\alpha_1} \dots D_n^{\alpha_n}. \end{aligned}$$

The Dirichlet space $\mathcal{D}(\mathbb{U}^n)$ is the class of $f(z) = \sum_{\alpha} f_{\alpha} z^{\alpha} \in \mathcal{H}(\mathbb{U}^n)$ with

$$(1.1) \quad \|f\|_{\mathcal{D}}^2 = \sum_{\alpha} \alpha_1 \dots \alpha_n |f_{\alpha}|^2 = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \alpha_1 \dots \alpha_n |f_{\alpha}|^2 < \infty.$$

Equivalently, $\mathcal{D}(\mathbb{U}^n)$ is the class of those $f \in \mathcal{H}(\mathbb{U}^n)$ with

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{U}^n} |D_1 \dots D_n f|^2 d\mu_n < \infty,$$

where μ_n is the Lebesgue measure on \mathbb{U}^n normalized so that $\mu_n(\mathbb{U}^n) = 1$. The semi-norm $\|\cdot\|_{\mathcal{D}}$ is obtained from the semi-inner product

$$\langle\langle f, g \rangle\rangle_{\mathcal{D}} = \sum_{\alpha} \alpha_1 \dots \alpha_n f_{\alpha} \bar{g}_{\alpha} = \int_{\mathbb{U}^n} (D_1 \dots D_n f) \overline{(D_1 \dots D_n g)} d\mu_n.$$

Main results. In this work, we first prove two theorems which show that the Dirichlet space is unique among \mathcal{M} -spaces that have certain properties.

THEOREM A. *Let \mathcal{H} be an \mathcal{M} -space and suppose that*

$$(1.2) \quad \|f\| = \|f \circ \Psi\| \quad (f \in \mathcal{H}, \Psi \in \mathcal{M}).$$

Then

$$\|f\| = C \|f\|_{\mathcal{D}} \quad (f \in \mathcal{H}),$$

where $C = \|z_1 \dots z_n\|^2$. Thus \mathcal{H} is $\mathcal{D}(\mathbb{U}^n)$.

Note that the assumption on the semi-norm is equivalent to $\langle\langle f \circ \Psi, g \circ \Psi \rangle\rangle = \langle\langle f, g \rangle\rangle$, and the conclusion implies that $\langle\langle f, g \rangle\rangle = C \langle\langle f, g \rangle\rangle_{\mathcal{D}}$, for all $f, g \in \mathcal{H}$ and $\Psi \in \mathcal{M}$.

For the second theorem, we need a strengthening of condition (ii) of Definition 1.1. To derive it, we write the Taylor series expansion at 0 of an $f \in \mathcal{H}$ by separating the higher and lower dimensional terms. For example, when $n = 2$, using (z, w) for (z_1, z_2) and (k, l) for (α_1, α_2) , we write

$$f(z, w) = f_{00} + \sum_{k=1}^{\infty} f_{k0} z^k + \sum_{l=1}^{\infty} f_{0l} w^l + \sum_{k,l=1}^{\infty} f_{kl} z^k w^l.$$

Since we assume conditions (iii) and (iv) of Definition 1.1, we can define a norm on \mathcal{H} by

$$\| \| f \| \|^2 = |f_{00}|^2 + \sum_{k=1}^{\infty} |f_{k0}|^2 \|z^k\|_1^2 + \sum_{l=1}^{\infty} |f_{0l}|^2 \|w^l\|_1^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 \|z^k w^l\|^2,$$

where $\| \cdot \|$ is still the semi-norm of $\mathcal{H} \subset \mathcal{H}(\mathbb{U}^n)$ and $\| \cdot \|_1$ denotes the semi-norm of a similar $\mathcal{H}' \subset \mathcal{H}(\mathbb{U})$. We already know (see [1]) that $\| \cdot \|_1$ is equivalent to $\| \cdot \|_{\mathcal{D}}$ in \mathbb{U} . Since our proof of Theorem B is by induction on the dimension of the polydisc, the above definition of $\| \| \cdot \| \|$ makes sense. Now we make the alternate assumption

(ii)' \mathcal{H} is complete in the norm $\| \| \cdot \| \|$.

A similar condition was assumed in [1], whereas [4] assumes (ii).

THEOREM B. *Let \mathcal{H} be an \mathcal{M} -space in the sense modified by (ii)' and assume that there is a positive constant $\delta < 1$ such that*

$$(1.3) \quad \delta \| f \| \leq \| f \circ \Psi \| \leq \frac{1}{\delta} \| f \| \quad (f \in \mathcal{H}, \Psi \in \mathcal{M}).$$

Then there exists positive constants K_1 and K_2 such that

$$K_1 \| \| f \| \|_{\mathcal{D}} \leq \| f \| \leq K_2 \| f \|_{\mathcal{D}} \quad (f \in \mathcal{H}).$$

Thus $\mathcal{D}(\mathbb{U}^n)$ is unique again.

The proofs of these theorems will be presented in §2.

Next, we consider a subspace of the Dirichlet space, one that is defined by a genuine norm similar to $\| \| \cdot \| \|$. This space is not \mathcal{M} -invariant any more, but the stronger conditions on it allow us to prove

that it has *tangential* limits as we approach $\partial\mathbb{U}^n$. In fact, tangential limits exist for a wider class of functions which are potentials of certain functions in $\mathcal{L}^2(\mathbb{T}^n)$. The precise definitions and theorems are stated in §3. Theorems C and D at the end of that section are the major results in this direction.

In earlier work, Arazy and Fisher [1] proved, under slightly different hypotheses, the analogs of Theorem A and Theorem B in \mathbb{U} . Zhu [5] found the equivalent of Theorem A for the unit ball in \mathbb{C}^n when $n \geq 2$. Nagel, Rudin and Shapiro [3] obtained the unit-disc versions of Theorems C and D.

After the submission of the manuscript, we were informed by a referee that in the preprint *Invariant Hilbert Spaces of Analytic Functions on Bounded Symmetric Domains* by J. Arazy and S. D. Fisher, results analogous to Theorems A and B were established for all irreducible bounded symmetric domains.

NOTATION. λ_n is the Lebesgue measure on \mathbb{T}^n both normalized to have mass 1; i.e., it is the Haar measure on the compact abelian group \mathbb{T}^n . If $p \in [1, \infty)$, its *conjugate* is $q = p/(p - 1)$. The \mathcal{L}^p - and ℓ^p -spaces will have their usual meaning. z_j will usually be an element of \mathbb{U} and ζ_j of \mathbb{T} . Apart from the usual big \mathcal{O} notation, we will use $u \sim v$ to mean both $u = \mathcal{O}(v)$ and $v = \mathcal{O}(u)$, and $u \approx v$ to mean u/v has a finite positive limit.

The *Poisson integral* of an $f \in \mathcal{L}^1(\mathbb{T}^n)$ is

$$P[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^n \frac{1 - |z_j|^2}{|1 - z_j \bar{\zeta}_j|^2} d\lambda_n(\zeta) \quad (z \in \mathbb{U}^n),$$

and its *Cauchy integral* is

$$C[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^n \frac{1}{1 - z_j \bar{\zeta}_j} d\lambda_n(\zeta) \quad (z \in \mathbb{U}^n),$$

where the products are called the *Poisson kernel* $P(z, \zeta)$ and the *Cauchy kernel* $C(z, \zeta)$ for \mathbb{U}^n , respectively. These transforms have the following invariance properties: If $f \in \mathcal{L}^1(\lambda_n)$, $\Psi \in \mathcal{M}$, and $U \in \mathcal{U}$, then

$$P[f \circ \Psi] = P[f] \circ \Psi \quad \text{and} \quad C[f \circ U] = C[f] \circ U.$$

The automorphisms of \mathbb{U}^n for $n \geq 2$ are generated by the following three subgroups: *rotations* in each variable separately

$$R_\theta(z) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n),$$

Möbius transformations in each variable separately

$$\Phi_w(z) = (\phi_{w_1}(z_1), \dots, \phi_{w_n}(z_n)),$$

and the coordinate permutations. Here $\theta \in [-\pi, \pi]^n$ and $w \in \mathbb{U}^n$ are fixed, Möbius transformations are in the form

$$(1.4) \quad \phi_w(z) = \frac{w - z}{1 - \bar{w}z} \quad (w \in \mathbb{U}, z \in \bar{\mathbb{U}}),$$

and the coordinate permutations are nothing but the $n!$ members of the symmetric group \mathcal{S}_n on n objects. Thus an arbitrary $\Psi \in \mathcal{M}$ can be written in the form

$$\Psi(z) = (e^{i\theta_1} \phi_{w_1}(z_{\sigma(1)}), \dots, e^{i\theta_n} \phi_{w_n}(z_{\sigma(n)})),$$

for some $w \in \mathbb{U}^n$ and $\theta \in [-\pi, \pi]^n$, and $\sigma \in \mathcal{S}_n$. \mathcal{U} is generated by $\sigma \in \mathcal{S}_n$ and the rotations R_θ . Each Möbius transformation Φ_w is an involution (its inverse is itself) exchanging 0 and w . \mathcal{M} acts transitively on \mathbb{U}^n : if $a, b \in \mathbb{U}^n$, then $\Phi_b \circ \Phi_a \in \mathcal{M}$ moves a to b (and b to a). Finally, \mathcal{M}^* denotes the component of the identity in \mathcal{M} ; i.e., \mathcal{M}^* is \mathcal{M} without the action of \mathcal{S}_n .

2. Uniqueness of the Dirichlet space. We start by showing that $\mathcal{D}(\mathbb{U}^n)$ has all the properties of a Hilbert space in the sense of Definition 1.1. Clearly the polynomials are in $\mathcal{D}(\mathbb{U}^n)$ and $\|z^\alpha\|^2 = \alpha_1 \cdots \alpha_n$ for all $\alpha \in \mathbb{N}^n$. A quick look at (1.1) shows that the polynomials are dense in $\mathcal{D}(\mathbb{U}^n)$ with respect to $\|\cdot\|_{\mathcal{D}}$. Again from (1.1), identifying g by $\{g_\alpha\}$, we see that $\mathcal{D}(\mathbb{U}^n)$ is a weighted ℓ^2 -space, hence every Cauchy sequence $\{f_j\}$ in $\mathcal{D}(\mathbb{U}^n)$ converges in $\|\cdot\|_{\mathcal{D}}$ to some $f \in \mathcal{D}(\mathbb{U}^n)$ represented by $\{f_\alpha\}$ for $\alpha \in \mathbb{Z}_+^n$. To show that f is holomorphic, let $f_m(z) = \sum_{k=1}^m \sum'_{|\alpha|=k} f_\alpha z^\alpha$ and pick $\varepsilon > 0$. For any $0 < r < 1$ and positive integers $m > l > n$,

$$\begin{aligned} \sup_{z \in r\bar{\mathbb{U}}^n} |(f_m - f_l)(z)| &= \sup_{z \in r\bar{\mathbb{U}}^n} \left| \sum_{k=l+1}^m \sum'_{|\alpha|=k} f_\alpha z^\alpha \right| \leq \sum_{k=l+1}^m \sum'_{|\alpha|=k} |f_\alpha| r^{|\alpha|} \\ &\leq \left(\sum_{k=l+1}^m \sum'_{|\alpha|=k} |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m \sum'_{|\alpha|=k} r^{2|\alpha|} \right)^{1/2} \\ &< \left(\sum_{k=l+1}^m \sum_{|\alpha|=k} \alpha_1 \cdots \alpha_n |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m k^n r^{2k} \right)^{1/2}. \end{aligned}$$

The first factor is less than ε when l and m are large enough because $f \in \mathcal{D}(\mathbb{U}^n)$, and the second factor is bounded as $l, m \rightarrow \infty$. Hence

$f(z) = \sum_{\alpha} f_{\alpha} z^{\alpha}$ is uniformly convergent on compact subsets of \mathbb{U}^n , and this proves $f \in \mathcal{H}(\mathbb{U}^n)$. Note that we need not know f_{α} if $\alpha \in \mathbb{N}^n \setminus \mathbb{Z}_+^n$. These coefficients of f can be taken arbitrarily as long as f remains holomorphic.

LEMMA 2.1. \mathcal{M} is generated by \mathcal{S}_n , rotations $R_{\omega}(z) = (e^{i\omega} z_1, z_2, \dots, z_n)$ with $\omega \in [-\pi, \pi]$, and Möbius transformations of the form $\Phi_t(z) = (\phi_t(z_1), z_2, \dots, z_n)$ with $0 \leq t < 1$.

PROPOSITION 2.2. $\mathcal{D}(\mathbb{U}^n)$ is \mathcal{M} -invariant.

Proof. The integral form of the Dirichlet semi-norm uses the measure μ_n which is invariant under rotations and permutations. Thus $\mathcal{D}(\mathbb{U}^n)$ is \mathcal{U} -invariant. To prove invariance under Möbius transformations, in view of Lemma 2.1, it suffices to consider

$$w = \Phi_r(z) = \left(\frac{r - z_1}{1 - rz_1}, z_2, \dots, z_n \right).$$

Then $D_1^z(f \circ \Phi_r) = (D_1^w f)dw_1/dz_1$ and

$$\begin{aligned} |D_1^z D_2 \cdots D_n(f \circ \Phi_r)|^2 &= |D_1^w D_2 \cdots D_n f|^2 \frac{(r^2 - 1)^2}{|1 - rz_1|^4} \\ &= |D_1^w D_2 \cdots D_n f|^2 J_{\Re} \Phi_r(z), \end{aligned}$$

since $dw_1/dz_1 = (r^2 - 1)/(1 - rz_1)^2$, where $J_{\Re} \Phi_r$ is the real Jacobian of Φ_r . Therefore

$$\begin{aligned} \|f \circ \Phi_r\|_{\mathcal{D}}^2 &= \int_{\mathbb{U}^n} |D_1^z D_2 \cdots D_n(f \circ \Phi_r)(z)|^2 d\mu_n(z) \\ &= \int_{\mathbb{U}^n} |D_1^w D_2 \cdots D_n f(w)|^2 \frac{(r^2 - 1)^2}{|1 - rz_1|^4} \frac{1}{J_{\Re} \Phi_r(z)} d\mu_n(w) \\ &= \int_{\mathbb{U}^n} |D_1^w D_2 \cdots D_n f(w)|^2 d\mu_n(w) = \|f\|_{\mathcal{D}}^2. \quad \square \end{aligned}$$

Note that when $n \geq 2$, $\mathcal{D}(\mathbb{U}^n)$ does not put any conditions on the infinitely many power series coefficients of f , those with at least one $\alpha_j = 0$, i.e., those in $\mathbb{N}^n \setminus \mathbb{Z}_+^n$. Thus if each term in the Taylor expansion of some $f \in \mathcal{H}(\mathbb{U}^n)$ depends on fewer than n variables, then $\|f\|_{\mathcal{D}} = 0$ and $f \in \mathcal{D}$. The Dirichlet space can also be thought

of as a quotient space of holomorphic functions satisfying (1.1) where the functions whose Taylor series differ by terms depending on at most $n - 1$ variables are identified. Trivially any holomorphic function f of fewer than n variables or any constant f has $\|f\|_{\mathcal{D}} = 0$ and is in $\mathcal{D}(\mathbb{U}^n)$. For comparison, when $n = 1$, only constants (a one-dimensional subspace) have zero Dirichlet semi-norm.

We can define a *modified* Dirichlet space $\tilde{\mathcal{D}}(\mathbb{U}^n)$ similar to $\mathcal{D}(\mathbb{U}^n)$ by considering a norm instead of a semi-norm. This requires some control on *all* the power series coefficients of $f \in \mathcal{H}(\mathbb{U}^n)$. For simplicity let's look at the case $n = 2$. With notation as before, let

$$\begin{aligned} \|f\|_{\tilde{\mathcal{D}}}^2 &= |f_{00}|^2 + \sum_{k=1}^{\infty} k|f_{k0}|^2 + \sum_{l=1}^{\infty} l|f_{0l}|^2 + \sum_{k,l=1}^{\infty} kl|f_{kl}|^2 \\ &= |f(0, 0)|^2 + \int_{\mathbb{U}} |D_1 f(z, 0)|^2 d\mu_1(z) \\ &\quad + \int_{\mathbb{U}} |D_2 f(0, w)|^2 d\mu_1(w) \\ &\quad + \int_{\mathbb{U}^2} |D_1 D_2 f(z, w)|^2 d\mu_2(z, w). \end{aligned}$$

This norm is \mathcal{U} -invariant, but not \mathcal{M} -invariant; in fact, none of its first three terms is preserved under compositions with Φ_r .

Proof of Theorem A. First, $\langle\langle z^\alpha, z^\beta \rangle\rangle = 0$ if $\alpha \neq \beta$. To see this, assume, without loss of generality, $\alpha_1 \neq \beta_1$. Let ω be an irrational multiple of π and consider the rotation $R_\omega(z) = (e^{i\omega} z_1, z_2, \dots, z_n)$. By the \mathcal{M} -invariance of $\langle\langle \cdot, \cdot \rangle\rangle$,

$$\begin{aligned} \langle\langle z^\alpha, z^\beta \rangle\rangle &= \langle\langle z^\alpha \circ R_\omega, z^\beta \circ R_\omega \rangle\rangle \\ &= \langle\langle e^{i\alpha_1 \omega} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}, e^{i\beta_1 \omega} z_1^{\beta_1} z_2^{\beta_2} \dots z_n^{\beta_n} \rangle\rangle \\ &= e^{i(\alpha_1 - \beta_1)\omega} \langle\langle z^\alpha, z^\beta \rangle\rangle, \end{aligned}$$

and the desired orthogonality result follows.

Put $C_\alpha = \langle\langle z^\alpha, z^\alpha \rangle\rangle$. Note that C_α is defined only for $\alpha \in \mathbb{N}^n$. If β is another multi-index and $\beta = \sigma(\alpha)$ for some $\sigma \in \mathcal{S}_n$, then by the \mathcal{M} -invariance of $\langle\langle \cdot, \cdot \rangle\rangle$ again, $C_\alpha = C_\beta$.

Now let $0 < r_j < 1$, $\Psi = (\phi_{r_1}, \dots, \phi_{r_n})$, and consider $f(z) = \prod_{j=1}^n (1 - r_j z_j) \in \mathcal{H}$. Then since z^α is orthogonal to z^β for $\alpha \neq \beta$,

we get

$$\begin{aligned} \langle\langle f, f \rangle\rangle &= \left\| \prod_{j=1}^n (1 - r_j z_j) \right\|^2 = \left\| 1 - \sum_{j=1}^n r_j z_j + \cdots + (-1)^m \prod_{j=1}^n r_j z_j \right\|^2 \\ &= C_{(0, \dots, 0)} + \left(\sum_{j=1}^n r_j^2 \right) C_{(1, 0, \dots, 0)} \\ &\quad + \left(\sum_{l>j=1}^n r_j^2 r_l^2 \right) C_{(1, 1, 0, \dots, 0)} + \cdots + \left(\prod_{j=1}^n r_j^2 \right) C_{(1, \dots, 1)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (f \circ \Psi)(z) &= \prod_{j=1}^n \left(1 - r_j \frac{r_j - z_j}{1 - r_j z_j} \right) = \prod_{j=1}^n \frac{1 - r_j^2}{1 - r_j z_j} \\ &= \prod_{j=1}^n \left[(1 - r_j^2) \sum_{\alpha_j=0}^{\infty} r_j^{\alpha_j} z_j^{\alpha_j} \right] = \left(\prod_{j=1}^n (1 - r_j^2) \right) \left(\sum_{\alpha} r^{\alpha} z^{\alpha} \right); \end{aligned}$$

now the density of the polynomials and the axioms of a Hilbert space imply

$$\langle\langle f \circ \Psi, f \circ \Psi \rangle\rangle = \left(\prod_{j=1}^n (1 - r_j^2)^2 \right) \sum_{\alpha} r^{2\alpha} C_{\alpha}.$$

Putting $x_j = r_j^2$ and using the \mathcal{M} -invariance of the semi-norm gives

$$\begin{aligned} (2.1) \quad C_{(0, \dots, 0)} &+ \left(\sum_{j=1}^n x_j \right) C_{(1, 0, \dots, 0)} + \left(\sum_{l>j=1}^n x_j x_l \right) C_{(1, 1, 0, \dots, 0)} \\ &+ \cdots + \left(\prod_{j=1}^n x_j \right) C_{(1, \dots, 1)} = \left(\prod_{j=1}^n (1 - x_j)^2 \right) \sum_{\alpha} x^{\alpha} C_{\alpha}. \end{aligned}$$

The constant terms $(C_{(0, \dots, 0)})$ cancel, and if we set the coefficients of $x_1, x_1 x_2, \dots,$ and $x_1 x_2 \cdots x_n$ on either side equal to each other, we obtain, respectively,

$$\begin{aligned} C_{(1, 0, \dots, 0)} &= C_{(1, 0, \dots, 0)} - 2C_{(0, \dots, 0)}, \\ C_{(1, 1, 0, \dots, 0)} &= C_{(1, 1, 0, \dots, 0)} - 2^2 C_{(1, 0, \dots, 0)}, \\ &\vdots \\ C_{(1, \dots, 1)} &= C_{(1, \dots, 1)} - 2^n C_{(1, \dots, 1, 0)}. \end{aligned}$$

These imply $C_{(0, \dots, 0)} = 0$, $C_{(1, 0, \dots, 0)} = 0$, \dots , $C_{(1, \dots, 1, 0)} = 0$. After the elimination of the terms that are zero, (2.1) simplifies to

$$\begin{aligned} \sum_{\alpha}' x^{\alpha} C_{\alpha} &= \frac{x_1 \cdots x_n C_{(1, \dots, 1)}}{(1 - x_1)^2 \cdots (1 - x_n)^2} \\ &= C_{(1, \dots, 1)} \left(\sum_{\alpha_1=1}^{\infty} \alpha_1 x_1^{\alpha_1} \right) \cdots \left(\sum_{\alpha_n=1}^{\infty} \alpha_n x_n^{\alpha_n} \right); \end{aligned}$$

and this implies

$$C_{\alpha} = \alpha_1 \cdots \alpha_n C_{(1, \dots, 1)} \quad (\alpha \in \mathbb{Z}_+^n).$$

Thus the norm of a monomial of fewer than n variables is zero. Since the polynomials are dense in \mathcal{H} , the same result is true for any $f \in \mathcal{H}$ whose Taylor expansion consists of monomials depending on fewer than n variables. But $C_{(1, \dots, 1)} \neq 0$, because otherwise, since the polynomials are dense in \mathcal{H} , $\langle \cdot, \cdot \rangle$ would be identically zero contrary to hypothesis. Then renaming $C_{(1, \dots, 1)} = C$ completes the proof. \square

Proof of Theorem B. We will only show how the two-variable case is obtained from the one-variable case. This then can be adapted to prove by induction the case for arbitrary \mathbb{U}^n . Unless explicitly stated, subscripted C 's will denote positive constants that are independent of any parameters.

Step 1. We begin by introducing two other semi-inner products on \mathcal{H} . For $f, g \in \mathcal{H}$, let

$$[f, g] = \int_{\mathbb{T}^2} \langle f \circ R_{\theta}, g \circ R_{\theta} \rangle d\lambda_2(\theta)$$

and

$$\langle f, g \rangle = m \langle f \circ \Phi, g \circ \Phi \rangle,$$

where m is an *invariant mean* (see [2]) on the abelian subgroup

$$\mathcal{N} = \{ \Phi = (\phi_s, \phi_t) : 0 \leq s, t < 1 \}$$

of \mathcal{M} . To actually make \mathcal{N} abelian, in this proof we change our definition of a Möbius transformation so that $\phi_w(z)$ is the negative of what is given in (1.4). The earlier definition was adopted to make ϕ_w an involution, since it simplified calculations involving ϕ_w^{-1} . The required boundedness condition for the existence of this nonunique

mean is furnished by (1.3). Rotations and \mathcal{N} , along with \mathcal{S}_2 , suffice to generate \mathcal{M} , by Lemma 2.1.

Now $[\cdot, \cdot]$ is rotation-invariant

$$(2.2) \quad [f, g] = [f \circ R_\theta, g \circ R_\theta] \quad (\theta \in [-\pi, \pi]^n),$$

and $\langle \cdot, \cdot \rangle$ is \mathcal{N} -invariant

$$(2.3) \quad \langle f, g \rangle = \langle f \circ \Phi, g \circ \Phi \rangle \quad (\Phi \in \mathcal{N}).$$

Moreover, (1.3) implies

$$(2.4) \quad \delta^2 \|f\|^2 \leq [f, f] \leq \frac{1}{\delta^2} \|f\|^2 \quad (f \in \mathcal{H}),$$

and

$$(2.5) \quad \delta^2 \|f\|^2 \leq \langle f, f \rangle \leq \frac{1}{\delta^2} \|f\|^2 \quad (f \in \mathcal{H});$$

and combining these two, we further obtain

$$(2.6) \quad \delta^4 [f, f] \leq \langle f, f \rangle \leq \frac{1}{\delta^4} [f, f] \quad (f \in \mathcal{H}).$$

(2.4) and (2.5) show that the semi-norms associated to $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ are both equivalent to $\|\cdot\|$.

As in the proof of Theorem A, the rotation-invariance of $[\cdot, \cdot]$ gives the orthogonality condition

$$(2.7) \quad [z^{k_1} w^{l_1}, z^{k_2} w^{l_2}] = 0 \quad ((k_1, l_1) \neq (k_2, l_2)),$$

which leads to

$$[f, f] = \sum_{k, l=0}^{\infty} |f_{kl}|^2 [z^k w^l, z^k w^l] \quad (f \in \mathcal{H}).$$

Therefore, to prove the theorem, it suffices to show that

$$K_1 kl \leq [z^k w^l, z^k w^l] \leq K_2 kl \quad ((k, l) \in \mathbb{N}^2)$$

or, equivalently,

$$K_3 kl \leq \langle z^k w^l, z^k w^l \rangle \leq K_4 kl \quad ((k, l) \in \mathbb{N}^2)$$

for some positive constants K_1, K_2, K_3 , and K_4 . Clearly $K_2 \geq K_1$ and $K_4 \geq K_3$.

Step 2. Claim:

$$(2.8) \quad \langle z^k w^l, z^k w^l \rangle \neq 0 \quad \text{if } k \geq 1 \text{ and } l \geq 1.$$

Suppose it is zero for some (N, M) ; then $[z^N w^M, z^N w^M] = 0$ also. Then for $0 \leq s, t < 1$, if we use (2.3), (2.6) and power series expansion

$$\begin{aligned} 0 &= \langle z^N w^M, z^N w^M \rangle \\ &= \left\langle \left(\frac{z-s}{1-sz} \right)^N \left(\frac{w-t}{1-wt} \right)^M, \left(\frac{z-s}{1-sz} \right)^N \left(\frac{w-t}{1-wt} \right)^M \right\rangle \\ &\geq \delta^8 \left[\left(\frac{z-s}{1-sz} \right)^N \left(\frac{w-t}{1-wt} \right)^M, \left(\frac{z-s}{1-sz} \right)^N \left(\frac{w-t}{1-wt} \right)^M \right] \\ &= \left[\sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l, \sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l \right] \\ &= \sum_{k,l=0}^{\infty} |c_{kN}(s)|^2 |c'_{lM}(t)|^2 [z^k w^l, z^k w^l]. \end{aligned}$$

A tedious computation shows that the coefficients $c_{kN}(s) \neq 0$ for any k, N , and s as given above; the same is obviously true for $c'_{lM}(t)$. Thus $[z^k w^l, z^k w^l] = 0$. This means that every element in \mathcal{H} has zero norm and contradicts our basic assumption that $\langle \cdot, \cdot \rangle$ is not identically zero. Hence the claim is proved.

The one-variable result can be stated as

$$(2.9) \quad C_1 k \leq \langle z^k, z^k \rangle_1 \leq C_2 k \quad (k \in \mathbb{N}).$$

It is a consequence of condition (ii)' of Definition 1.1 and of (2.8) that the subspace of \mathcal{H} consisting of functions whose Taylor series expansion at 0 depend only on z is closed. Then (2.9) implies that, for fixed $M \in \mathbb{N}$,

$$(2.10) \quad C_3 k \leq \langle z^k w^M, z^k w^M \rangle \leq C_4 k \quad (k \in \mathbb{N}),$$

and we have a similar equation when the power of z is held constant. Of course, the constants C_3 and C_4 are different for different M . It is our aim to find their explicit dependence on M . If we had only finitely many M , we could pick C_3 and C_4 independently of M and the proof would be over. In the sequel, whenever we have only finitely many n or M , we will use this fact without further reference.

Step 3 (upper bound). Let $M \in \mathbb{N}$ be fixed and $k, j \in \mathbb{N}$. Put

$$\begin{aligned} \alpha_{kM}^{jM} &= \langle z^k w^M, z^j w^M \rangle, & \beta_{kM} &= \langle z^k w^M, z^k w^M \rangle, \\ b_{kM} &= [z^k w^M, z^k w^M]. \end{aligned}$$

By (2.3), $\alpha_{kM}^{jM} = \langle \phi_s^k w^M, \phi_s^j w^M \rangle$ for any $s \in [0, 1)$. Differentiate both sides of this equality with respect to s and set $s = 0$. Then take $k = j + 1$ and add the resulting expressions from $j = 0$ to $j = N \geq 1$; and finally divide both sides by $N + 1$. The result is

$$(2.11) \quad \frac{\beta_{N+1,M}}{N+1} = \frac{2S_{NM}}{N(N+1)} - \frac{\alpha_{N+2,M}^{NM}}{N},$$

where $S_{NM} = \frac{1}{2}\beta_{0M} + \sum_{k=1}^N \beta_{kM}$. In particular, $\alpha_{N+2,M}^{NM}$ is real. Now using (2.3) and (2.5), and letting $s^2 = \frac{N}{N+1}$, we obtain

$$\begin{aligned} \frac{1}{\delta^4}\beta_{1M} &= \frac{1}{\delta^4}\langle \phi_s w^M, \phi_s w^M \rangle \geq [\phi_s w^M, \phi_s w^M] \\ &= s^2 b_{0M} + (1-s^2)^2 \sum_{k=0}^{\infty} s^{2k} b_{k+1,M} \\ &\geq \frac{\delta^4}{(N+1)^2} \sum_{k=0}^N \left(\frac{N}{N+1}\right)^k \beta_{k+1,M} \geq \frac{\delta^4}{(N+1)^2 e} \sum_{k=0}^N \beta_{k+1,M}, \end{aligned}$$

which implies

$$S_{NM} \leq \left(\frac{e\beta_{1M}}{\delta^8} + \frac{\beta_{0M}}{2}\right) N^2.$$

Using (2.10) twice on the right side gives

$$(2.12) \quad S_{NM} \leq C_5 M N^2.$$

It is this inequality and its pair (2.15) below that allow us to pass from one variable to several variables.

As a special case, when $M = 0$, we get $S_{N0} = 0$ for all $N \geq 1$. Symmetric nature of the calculation shows also $S_{0N} = 0$ for all $N \geq 1$. It follows that

$$(2.13) \quad \beta_{N0} = \langle z^N, z^N \rangle = 0 \quad \text{and} \quad \beta_{0N} = \langle w^N, w^N \rangle = 0 \quad (N \in \mathbb{Z}_+).$$

The inequality

$$\delta^8(\beta_{NM} + \beta_{N+2,M}) \leq \langle (z^N + z^{N+2})w^M, (z^N + z^{N+2})w^M \rangle$$

is a direct consequence of (2.6) and (2.7). Using this, after some routine calculation, (2.11) can be written as

$$\begin{aligned} 2\frac{\beta_{N+1,M}}{N+1} &\leq \frac{4S_{NM}}{N(N+1)} + \frac{2\beta_{N+2,M}}{N(N+2)} + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right) \\ &\leq 6C_5 M + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right), \end{aligned}$$

which is equivalent to $2\gamma_{N+1, M} \leq (1 - \delta^8)(\gamma_{NM} + \gamma_{N+2, M})$ if we let

$$\gamma_{NM} = \frac{\beta_{NM}}{N} - \frac{3C_5M}{\delta^8}.$$

A result in [1] shows $\gamma_{N+1, M} \leq |\gamma_{NM}|$ for positive N . Then using (2.8), we get

$$\frac{\beta_{NM}}{N} - \frac{3C_5M}{\delta^8} \leq \left| \frac{\beta_{1M}}{1} - \frac{3C_5M}{\delta^8} \right| \leq C_6M.$$

Multiplying both sides by M , we conclude

$$\beta_{NM} \leq \left(C_6 + \frac{3C_5}{\delta^8} \right) NM = K_4 NM,$$

which holds for $N \geq 1$ and $M \geq 0$, and for $M \geq 1$ and $N \geq 0$, by the symmetry of the computation.

Step 4 (lower bound). If we combine the result of Step 1 with (2.6), we also get

$$(2.14) \quad b_{kM} \leq \frac{K_4}{\delta^4} kM.$$

We use (2.6), (2.7), (2.3), (2.14), and take $s^2 = \frac{N}{N+1}$ to calculate

$$\begin{aligned} \delta^8 s^2 b_{1M} &\leq \delta^8 [(1 + sz)w^M, (1 + sz)w^M] \\ &= \delta^4 \langle (1 + s\phi_s)w^M, (1 + s\phi_s)w^M \rangle \\ &\leq (1 - s^2)^2 \left[\frac{w^M}{1 - sz}, \frac{w^M}{1 - sz} \right] = \left(\frac{1}{N + 1} \right)^2 \sum_{k=0}^{\infty} s^{2k} b_{kM} \\ &\leq \frac{1}{(N + 1)^2} \left(\sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M \sum_{k=mN+1}^{\infty} k \left(\frac{N}{N + 1} \right)^k \right), \end{aligned}$$

where m will be determined shortly. After approximating the second sum by an integral, we have

$$\delta^8 \left(\frac{N}{N + 1} \right) b_{1M} \leq \frac{1}{(N + 1)^2} \sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M(m + 1)e^{-m}.$$

Because of (2.10), M in the last term can be replaced with $C_8 b_{1M}$. Now choose m so large that $K_2 \delta^{-4} (m + 1) e^{-m} \leq \delta^8 / 3$. Using (2.10) once again and some simplification yields

$$(2.15) \quad \sum_{k=0}^N b_{kM} \geq C_9 MN^2,$$

which is the reverse inequality for (2.12). Combining (2.15) with (2.11), we obtain

$$(2.16) \quad \delta^4 C_9 M \leq \frac{\beta_{N+1, M}}{N+1} + \left(\frac{\beta_{NM}}{N} \right)^{1/2} \left(\frac{\beta_{N+2, M}}{N+2} \right)^{1/2}.$$

Now let

$$\phi_s^N(z) = \left(\frac{z-s}{1-sz} \right)^N = \sum_{k=0}^{\infty} c_{kN}(s) z^k$$

and consider

$$\begin{aligned} b_{NM} &\geq \delta^4 \beta_{NM} = \delta^4 \langle \phi_s^N w^M, \phi_s^N w^M \rangle \geq \delta^8 [\phi_s^N w^M, \phi_s^N w^M] \\ &= \delta^8 \sum_{k=1}^{\infty} |c_{kN}(s)|^2 b_{kM} \geq \delta^8 |c_{N+1, N}(s)|^2 b_{N+1, M}. \end{aligned}$$

A calculation in [1] shows that $|c_{N+1, N}(s)|^2 \geq 1/2$ for all $s \in [0, 1)$ and $N \geq 1$. Thus there is a constant C_{10} such that

$$(2.17) \quad \beta_{N+1, M} \leq C_{10} \beta_{NM}.$$

Now (2.16) and (2.17) together imply

$$\beta_{NM} \geq \left(\frac{\delta^4 C_9}{4C_{10}} \right) NM = K_3 NM,$$

for $N \geq 1$ and $M \geq 0$, and for $M \geq 1$ and $N \geq 0$.

Step 5. The only term we have not yet accounted for is $\beta_{00} = \langle 1, 1 \rangle$. Since it represents a one-dimensional subspace of \mathcal{H} , we now know $\mathcal{H} = \mathcal{D}(\mathbb{U}^2)$. To complete the proof, we will also show $\beta_{00} = b_{00} = 0$. To obtain a contradiction, suppose $b_{00} = [1, 1] \neq 0$. Let $f \in \mathcal{H}$, Φ be a Möbius transformation, and denote the power series coefficients of f and $f \circ \Phi$ by f_{kl} and f'_{kl} , respectively. Because of (2.13)

$$\|f\|^2 \geq \delta^2 [f, f] = \delta^2 \left(b_{00} |f(0, 0)|^2 + \sum_{k, l=1}^{\infty} |f_{kl}|^2 b_{kl} \right)$$

and

$$[f \circ \Phi, f \circ \Phi] = b_{00} |f(\Phi(0, 0))|^2 + \sum_{k, l=1}^{\infty} |f'_{kl}|^2 b_{kl}.$$

We have

$$\begin{aligned} \sum_{k, l=1}^{\infty} |f'_{kl}|^2 b_{kl} &\geq K_1 \sum_{k, l=1}^{\infty} |f_{kl}|^2 kl = K_1 \|f \circ \Phi\|_{\mathcal{D}}^2 = K_1 \|f\|_{\mathcal{D}}^2 \\ &= K_1 \sum_{k, l=1}^{\infty} |f_{kl}|^2 kl \geq \frac{K_1}{K_2} \sum_{k, l=1}^{\infty} |f_{kl}|^2 b_{kl}. \end{aligned}$$

Hence

$$\begin{aligned} b_{00}|f(0, 0)|^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} &= [f, f] \geq \delta^8 [f \circ \Phi, f \circ \Phi] \\ &= \delta^8 \left(b_{00} |f(\Phi(0, 0))|^2 + \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl} \right) \\ &= \delta^8 \left(b_{00} |f(\Phi(0, 0))|^2 + \frac{K_1}{K_2} \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} \right), \end{aligned}$$

from which we obtain

$$b_{00}|f(0, 0)|^2 + \left(1 - \delta^8 \frac{K_1}{K_2} \right) \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} \geq \delta^8 b_{00} |f(\Phi(0, 0))|^2.$$

The left hand side of this equation is finite since it is equivalent to $\|f\|^2$. Since $\Phi(0, 0)$ can be any point in \mathbb{U}^2 , it follows that every element of \mathcal{H} , i.e., of $\mathcal{D}(\mathbb{U}^2)$, is bounded. But a Dirichlet space contains unbounded elements. In \mathbb{U} , this is seen most easily by the Area Theorem; in \mathbb{U}^2 , we take an unbounded function depending only on one variable. Therefore $b_{00} = 0$ and we are done. \square

COROLLARY 2.3. *Theorem B is true even if (1.3) holds only for $\Psi \in \mathcal{M}^*$. Theorem A is true even if (1.2) holds only for $\Psi \in \mathcal{M}^*$.*

Proof. The proof of Theorem B uses coordinate permutations nowhere. Theorem A is a consequence of Theorem B. \square

3. Boundary behavior. Dirichlet-type spaces. This section requires some new notions that were studied in \mathbb{U} in [3] and [4]. For each $\delta = (\delta_1, \dots, \delta_n)$ with each $0 \leq \delta_j \leq 1$, we define the *Dirichlet-type spaces* $\mathcal{D}_\delta(\mathbb{U}^n)$ to consist of those $f(z) = \sum_{\alpha} f_{\alpha} z^{\alpha} \in \mathcal{H}(\mathbb{U}^n)$ that satisfy

$$\|f\|_{\mathcal{D}_\delta} = \sum_{\alpha} \alpha_1^{2\delta_1} \cdots \alpha_n^{2\delta_n} |f_{\alpha}|^2 < \infty.$$

This definition makes sense even if some $\delta_j = 0$ if we interpret $0^0 = 1$. In fact, if all the $\delta_j = 0$, then $\mathcal{D}_\delta(\mathbb{U}^n) = \mathcal{H}^2(\mathbb{U}^n)$. The space corresponding to $\delta_1 = \dots = \delta_n = 1$ consists of functions f with $D_1 \cdots D_n f \in \mathcal{H}^2(\mathbb{U}^n)$. When $\delta_1 = \dots = \delta_n = 1/2$, we have the Dirichlet space. For $n = 1$, all Dirichlet-type spaces are contained in $\mathcal{H}^2(\mathbb{U})$, but this is not true if $n > 1$.

Some subclasses of $\mathcal{D}_\delta(\mathbb{U}^n)$ have certain integral representations: If $F \in \mathcal{L}^2(\mathbb{T}^n)$, for $0 \leq \delta_j < 1$ and $z \in \mathbb{U}^n$, set

$$(3.1) \quad f(z) = \int_{\mathbb{T}^n} F(\zeta) \prod_{j=1}^n \frac{1}{(1 - z_j \bar{\zeta}_j)^{1-\delta_j}} d\lambda_n(\zeta).$$

The product is the Cauchy kernel each of whose factors is raised to a fractional power. Omitting j , each factor can be expanded as

$$(1 - z\bar{\zeta})^{\delta-1} = \sum_{\alpha=0}^{\infty} b_\alpha z^\alpha \bar{\zeta}^\alpha$$

where

$$b_\alpha = \frac{\Gamma(1 - \delta + \alpha)}{\Gamma(1 - \delta)\Gamma(1 + \alpha)} \sim \frac{1}{\alpha^\delta}.$$

In particular $b_0 = 1$. Let c_α be the $(\alpha_1, \dots, \alpha_n)$ th Fourier coefficient of F ; i.e.,

$$c_\alpha = \int_{\mathbb{T}^n} \bar{\zeta}^\alpha F(\zeta) d\lambda_n(\zeta).$$

Setting $f_\alpha = b_{\alpha_1} \cdots b_{\alpha_n} c_\alpha$, we get

$$\begin{aligned} f(z) &= \int_{\mathbb{T}^n} \prod_{j=1}^n \sum_{\alpha_j=0}^{\infty} b_{\alpha_j} z_j^{\alpha_j} \bar{\zeta}_j^{\alpha_j} F(\zeta) d\lambda_n(\zeta) \\ &= \sum_{\alpha} b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha \int_{\mathbb{T}^n} \bar{\zeta}^\alpha F(\zeta) d\lambda_n(\zeta) \\ &= \sum_{\alpha} b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha c_\alpha = \sum_{\alpha} f_\alpha z^\alpha. \end{aligned}$$

Now

$$\|f\|_{\mathcal{D}_\delta} = \sum_{\alpha} \alpha_1^{2\delta_1} \cdots \alpha_n^{2\delta_n} |b_{\alpha_1}|^2 \cdots |b_{\alpha_n}|^2 |c_\alpha|^2 \sim \sum_{\alpha} |c_\alpha|^2 < \infty.$$

Hence $f \in \mathcal{D}_\delta(\mathbb{U}^n)$; i.e., any f given by (3.1) is in a Dirichlet-type space.

But not all $f \in \mathcal{D}_\delta(\mathbb{U}^n)$ have integral representations as in (3.1), because a Dirichlet-type space does not control all the power series coefficients of its members. However, we can define a space $\tilde{\mathcal{D}}_\delta(\mathbb{U}^n)$ similar to $\tilde{\mathcal{D}}(\mathbb{U}^n)$ in which an integral representation is possible. Let's concentrate on the case $n = 2$ again for simplicity. With obvious

notation, $f \in \tilde{\mathcal{D}}_\delta(\mathbb{U}^2)$ if and only if

$$(3.2) \quad \|f\|_{\tilde{\mathcal{D}}_\delta}^2 = |f_{00}|^2 + \sum_{k=1}^\infty k^{2\delta_1} |f_{k0}|^2 + \sum_{l=1}^\infty l^{2\delta_2} |f_{0l}|^2 + \sum_{k,l=1}^\infty k^{2\delta_1} l^{2\delta_2} |f_{kl}|^2 < \infty.$$

Given $f(z) = \sum_{k,l} f_{kl} z^k w^l \in \tilde{\mathcal{D}}_\delta(\mathbb{U}^2)$, let $c_{kl} = f_{kl}/b_k b_l$ if $(k, l) \in \mathbb{N}^2$ (recall that $b_0 = 1$), and let $c_{kl} = 0$ otherwise. Then, using (3.2),

$$\sum_{(k,l) \in \mathbb{Z}^2} |c_{kl}|^2 = |c_{00}|^2 + \sum_{k=1}^\infty |c_{k0}|^2 + \sum_{l=1}^\infty |c_{0l}|^2 + \sum_{k,l=1}^\infty |c_{kl}|^2 \sim \|f\|_{\tilde{\mathcal{D}}_\delta}^2 < \infty.$$

Thus there is an $F \in \mathcal{L}^2(\mathbb{T}^2)$ such that $\hat{F}(k, l) = c_{kl}$. Therefore

$$\begin{aligned} f(z) &= \sum_{k,l=0}^\infty f_{kl} z^k w^l = \sum_{k,l=0}^\infty c_{kl} b_k b_l z^k w^l \\ &= \sum_{k,l=0}^\infty b_k b_l z^k w^l \int_{\mathbb{T}^2} \bar{\zeta}_1^k \bar{\zeta}_2^l F(\zeta) d\lambda_2(\zeta) \\ &= \int_{\mathbb{T}^2} \frac{F(\zeta) d\lambda_2(\zeta)}{(1 - \bar{\zeta}_1 z)^{1-\delta_1} (1 - \bar{\zeta}_2 w)^{1-\delta_2}}. \end{aligned}$$

Clearly F is not unique. In fact, c_{kl} can be defined arbitrarily for $(k, l) \notin \mathbb{N}^2$ as long as we retain $\sum_{(k,l) \in \mathbb{Z}^2} |c_{kl}|^2 < \infty$.

Kernels and potentials. From now on, we will also use $e^{i\theta_j}$ for $\zeta_j \in \mathbb{T}$, $e^{i\varphi_j}$ for $\eta_j \in \mathbb{T}$, and $r_j e^{i\theta_j} = r_j \zeta_j$ for $z_j \in \mathbb{U}$. The point $(1, \dots, 1) \in \mathbb{T}^n$ corresponding to $\theta_1 = \dots = \theta_n = 0$ will act like the origin in \mathbb{R}^n . Now the Poisson kernel takes the more familiar form

$$P_r(\theta) = \prod_{j=1}^n \frac{1 - r_j^2}{1 - 2r_j \cos \theta_j + r_j^2},$$

and it is considered as a function of θ indexed by r . So the \mathcal{L}^p -norm of a Poisson integral will be obtained by an integration on the θ -variable and will still depend on r .

A kernel K is a nonnegative \mathcal{L}^1 -function on \mathbb{T}^n which is even and decreasing in each $|\theta_j|$ when the other variables are kept fixed. We will also have $K(1, \dots, 1) = \infty$ and normalize as $\|K\|_1 = 1$. A potential is the convolution of an \mathcal{L}^p -function F on \mathbb{T}^n with a kernel. Thus (3.1) defines $f(z) \in \tilde{\mathcal{D}}_\delta(\mathbb{U}^n)$ as a potential. The Poisson integral is simply the convolution with the Poisson kernel.

Let's define the *Bessel kernels* on the torus. For $0 < \delta_j \leq 1$, let

$$\begin{aligned} G_\delta(\zeta) &= \prod_{j=1}^n g_{\delta_j}(\theta_j) = \prod_{j=1}^n \left(1 + \frac{1}{2} \sum_{\alpha_j \neq 0} |\alpha_j|^{-\delta_j} \zeta_j^{\alpha_j} \right) \\ &= \prod_{j=1}^n \left(1 + \sum_{\alpha_j=1}^{\infty} \alpha_j^{-\delta_j} \cos(\alpha_j \theta_j) \right), \end{aligned}$$

where each g_{δ_j} is a Bessel kernel on the unit circle. $g_\delta(0) = \infty$, g_δ is a decreasing function of $|\omega|$ for $\omega \neq 0$, $g_\delta(\omega) > 0$, and $g_\delta(-\omega) = g_\delta(\omega)$. Each $g_{\delta_j} \in \mathcal{L}^1(\mathbb{T})$, so $G_\delta \in \mathcal{L}^1(\mathbb{T}^n)$, and $\|G_\delta\|_1 = \prod_{j=1}^n \|g_{\delta_j}\|_1 = 1$. When $0 < \delta < 1$,

$$g_\delta(\omega) \approx \left| \sin \frac{\omega}{2} \right|^{\delta-1} \quad \text{as } \omega \rightarrow 0.$$

Also $g_1(\omega) = 1 - \log|2 \sin(\omega/2)|$. $P_r[g_\delta] = P_r * g_\delta$ is the harmonic extension of g_δ to \mathbb{U} . As $r \rightarrow 1$, it satisfies the following:

$$(3.3) \quad \|P_r * g_\delta\|_q \sim (1-r)^{\delta-1/p} \quad (\delta p < 1),$$

$$(3.4) \quad \|P_r * g_\delta\|_q \sim \left(\log \frac{1}{1-r} \right)^{1/q} \quad (\delta p = 1, p > 1),$$

$$\|P_r * g_1\|_\infty \sim \log \frac{1}{1-r} \quad (\delta = 1, p = 1).$$

$P_r[G_\delta]$ possesses these properties in each variable separately.

On the unit circle, for $0 < \delta < 1$, the *modified Bessel kernels* are

$$\tilde{g}_\delta(\omega) = (1 - e^{i\omega})^{\delta-1} \quad \text{and} \quad \tilde{g}_1(\omega) = \log \frac{1}{1 - e^{i\omega}}.$$

On the torus, let $\tilde{G}_\delta(\zeta) = \prod_{j=1}^n \tilde{g}_{\delta_j}(\theta_j)$. These functions are not positive, so they are not properly kernels, but they are dominated by the Bessel kernels: There are constants $C_\delta > 0$ such that $|\tilde{G}_\delta| \leq C_\delta G_\delta$. If each δ_j is less than 1,

$$P_r[\tilde{G}_\delta](\theta) = \prod_{j=1}^n P_{r_j}[\tilde{g}_{\delta_j}](\theta_j) = \prod_{j=1}^n (1 - z_j)^{\delta_j-1},$$

with a logarithmic term if some $\delta_k = 1$. For $F \in \mathcal{L}^p(\mathbb{T}^n)$, the map that takes F to $G_\delta * F$ is one-to-one, and the Cauchy integral of $\tilde{G}_\delta * F$ is the same as its Poisson integral:

$$\begin{aligned} P_r[\tilde{G}_\delta * F](\theta) &= (P_r * \tilde{G}_\delta * F)(\theta) \\ &= \int_{\mathbb{T}^n} F(\zeta) \prod_{j=1}^n \frac{1}{(1 - z_j \bar{\zeta}_j)^{1-\delta_j}} d\lambda_n(\zeta) = f(z). \end{aligned}$$

Thus we have obtained the integral in (3.1). From now on, F and f will always be related as in this equation. $\delta_j = 1$ does not give rise to a Cauchy-type integral; so we will not pay any attention to this case any more.

Tangential limits. Define the *tangential approach regions* to the unit circle:

$$(3.5) \quad A_{\gamma,c}(\varphi) = \left\{ re^{i\theta} : 1 - r > c \left| \sin \frac{\theta - \varphi}{2} \right|^\gamma \right\},$$

$$E_{\gamma,c}(\varphi) = \left\{ re^{i\theta} : 1 - r > \exp \left(-c \left| \sin \frac{\theta - \varphi}{2} \right|^{-\gamma} \right) \right\}.$$

$A_{\gamma,c}(\varphi)$ has (polynomial) order of contact γ , and $E_{\gamma,c}(\varphi)$ makes exponential contact, with \mathbb{T} . A function f defined in \mathbb{U} has $A_\gamma(E_\gamma)$ -limit L at $e^{i\varphi}$ if $f(z) \rightarrow L$ as $z \rightarrow e^{i\varphi}$ within $A_{\gamma,c}(E_{\gamma,c})$ for every $c > 0$. In [3], it was shown that Poisson integrals of the modified Bessel potentials have A_γ -limits a.e. on \mathbb{T} if $\delta p < 1$ for $\gamma = \frac{1}{1-\delta p}$, and E_γ -limits a.e. on \mathbb{T} if $\delta p = 1$ for $\gamma = q - 1$.

Let $Q = Q(\eta, s)$ be the cube centered at $\eta \in \mathbb{T}^n$ with sides $s = (s_1, \dots, s_n)$, where each s_j has the same order as $\max\{s_j : 1 \leq j \leq n\} \rightarrow 0$. Its volume is $\lambda_n(Q) = s_1 \cdots s_n$. If $F \in \mathcal{L}^p(\mathbb{T}^n)$, its Hardy-Littlewood maximal function is

$$(M_p F)(\eta) = \sup_{0 < s_1, \dots, s_n \leq 1} \left(\frac{1}{\lambda_n(Q)} \int_Q |f|^p d\lambda_n \right)^{1/p}.$$

M_1 is of weak type $(1, 1)$; and since $M_p F = (M_1 |F|^p)^{1/p}$, M_p is of weak type (p, p) . Thus there are C_p such that

$$\lambda_n(\{M_p F > t\}) \leq \frac{C_p}{t^p} \|F\|_p^p \quad (F \in \mathcal{L}^p(\mathbb{T}^n), t \in (0, \infty)).$$

The proofs of the following assertions are similar to the proofs given in [3] for $n = 1$ and will be omitted. Some of them are valid in more general situations. The first result is obtained using the straightforward inequality

$$\int_{\mathbb{T}^n} |F| G_\delta d\lambda_n \leq (M_1 F)(1, \dots, 1) \int_{\mathbb{T}^n} G_\delta d\lambda_n = (M_1 F)(1, \dots, 1),$$

which holds for any $F \in \mathcal{L}^1(\mathbb{T}^n)$, and whose proof is also in [3].

THEOREM 3.1. *There is a $C_p < \infty$ such that for $F \in \mathcal{L}^p(\mathbb{T}^n)$ and $\zeta, \eta \in \mathbb{T}^n$,*

$$|(G_\delta * F)(\zeta)| \leq C_p \left[(M_p F)(\eta) \left(\prod_{j=1}^n |\zeta_j - \eta_j|^{1/p} \right) \|G_\delta\|_q + (M_1 F)(\eta) \right].$$

Convolution of two kernels is a kernel; so Theorem 3.1 holds with $P_r * G_\delta = P_r[G_\delta]$ in place of G_δ , which has the desired properties $\|P_r * G_\delta\|_1 = \|P_r\|_1 \|G_\delta\|_1 = 1$ and $\|P_r * G_\delta\|_q = \prod_{j=1}^n \|P_{r_j} * g_{\delta_j}\|_q$. The Hölder inequality gives $M_1F \leq M_pF$. In addition, $|\zeta_j - \eta_j|$ can be replaced by $|\theta_j - \varphi_j|$, or even by $|\sin((\theta_j - \varphi_j)/2)|$, since they are all of the same order as $\zeta \rightarrow \eta$. Lastly, we can put \tilde{G}_δ in place of G_δ on the left side of the inequality since the latter dominates the former. Hence Theorem 3.1 yields

THEOREM 3.2. *If $F \in \mathcal{L}^p(\mathbb{T}^n)$, then, using $z = re^{i\theta}$, for all $e^{i\theta}$, $e^{i\varphi} \in \mathbb{T}^n$,*

$$(3.6) \quad |f(z)| \leq C_p (M_p F)(\varphi) \left[1 + \prod_{j=1}^n \left| \sin \frac{\theta_j - \varphi_j}{2} \right|^{1/p} \|P_{r_j} * g_{\delta_j}\|_q \right].$$

For given φ , any bound on the product on the right side gives a bound on $f(z)$. This leads us to the tangential approach regions to \mathbb{T}^n . So fix an $\eta \in \mathbb{T}^n$. As $z \rightarrow \zeta$, all $r_j \rightarrow 1$; and because of (3.3) and (3.4), $\|P_r * G_\delta\|_q \sim \prod_{j=1}^n b_j$, where $b_j = (1 - r_j)^{\delta_j - 1/p}$ or $b_j = (\log(1/(1 - r_j)))^{1/q}$ depending on whether $\delta_j p < 1$ or $\delta_j p = 1$, respectively. In other words, an approach region should be determined by

$$\prod_{j=1}^n b_j \left| \sin \frac{\theta_j - \varphi_j}{2} \right|^{1/p} < c.$$

So define $B_{\gamma,c}(\eta)$ by

$$(3.7) \quad B_{\gamma,c}(\eta) = B_{\gamma,c}(\varphi) = \left\{ z \in \mathbb{U}^n : \prod_{j=1}^n b_j^{-1/\gamma_j} \left| \sin \frac{\theta_j - \varphi_j}{2} \right| < c \right\}.$$

Each of the factors in the above product is related to one of the regions in (3.5). In particular, points in a cartesian product of one-dimensional approach regions such as $B_{\gamma_1,c_1}(\varphi_1) \times \cdots \times B_{\gamma_n,c_n}(\varphi_n)$, where each $B_{\gamma_j,c_j}(\varphi_j)$ is either $A_{\gamma_j,c_j}(\varphi_j)$ or $E_{\gamma_j,c_j}(\varphi_j)$, satisfy the criterion for being in $B_{\gamma,c}(\eta)$. Hence an approach region can make exponential contact with \mathbb{T}^n in one (complex) direction and polynomial contact in another. For $\eta \in \mathbb{T}^n$, the *maximal functions* associated to these approach regions are defined as

$$(M_{G_\delta,\gamma,c} f)(\eta) = \sup\{|f(z)| : z \in B_{\gamma,c}(\eta)\}.$$

A function f defined in \mathbb{U}^n is said to have B_γ -limit L at $\eta \in \mathbb{T}^n$ if $f(z) \rightarrow L$ as $z \rightarrow \eta$ within $B_{\gamma,c}(\eta)$ for every $c > 0$.

THEOREM 3.3. *If $F \in \mathcal{L}^p(\mathbb{T}^n)$ and $t \in (0, \infty)$, there is a $C = C(p, c)$ such that*

$$\lambda_n(\{M_{G_\delta, \gamma, c} f > t\}) < \frac{C}{t^p} \|F\|_p^p,$$

where

$$(3.8) \quad \gamma_j = \frac{1}{1 - \delta_j p} \quad \text{if } \delta_j p < 1 \quad \text{and} \quad \gamma_j = q - 1 \quad \text{if } \delta_j p = 1.$$

This follows from the weak-type- (p, p) estimate for M_p . The weak-type estimate gives rise to a convergence theorem via classical arguments. This is the content of part (i) of Theorem C. The case $p = 1$ of part (ii) of that theorem also follows from Theorem 3.3. Henceforth, p, δ , and γ will always be related by (3.8).

Capacities. For $E \subset \mathbb{T}^n$, let $T(G_\delta, p, E)$ be the set of all nonnegative $F \in \mathcal{L}^p(\mathbb{T}^n)$ such that $(G_\delta * F)(\zeta) \geq 1$ for all $\zeta \in E$. The p -capacity of E is

$$\Sigma_{G_\delta}(E) = \inf\{\|F\|_p^p : F \in T(G_\delta, p, E)\}.$$

$\Sigma_{G_\delta}(E) = 0$ implies $\lambda_n(E) = 0$. The functions $G_\delta * F$ are defined Σ_{G_δ} -almost everywhere. If $F \in T(P_r * G_\delta, p, E)$, then $\Sigma_{P_r}(E) \leq \Sigma_{P_r * G_\delta}(E)$. For $\eta \in \mathbb{T}^n$ and fixed $\rho > 1$, let

$$\Gamma(\eta) = \{z \in \mathbb{U}^n : |z_j - \eta_j| < \rho(1 - |z_j|), 1 \leq j \leq n\},$$

and set $S(E) = \mathbb{U}^n \setminus \bigcup_{\eta \notin E} \Gamma(\eta)$. $\Gamma(\eta)$ is the cartesian product of n sets each of which is asymptotic, as $z_j \rightarrow \eta_j$, to an angle-shaped approach region in \mathbb{U} with vertex at η_j . For $\eta \in \mathbb{T}^n$, the *nontangential maximal function* is

$$(Nf)(\eta) = \sup\{|f(z)| : z \in \Gamma(\eta)\}.$$

For $W \subset \mathbb{U}^n$, $J_c^\gamma(W)$ is the set of $\eta \in \mathbb{T}^n$ for which W intersects $B_{\gamma, c}(\eta)$.

LEMMA 3.4. *There exists a constant $b = b(n) > 0$ such that if $F \geq 0$ on \mathbb{T}^n , $F \geq 1$ on $E \in \mathbb{T}^n$, and $z \in S(E)$, then $P_r[F](z) > b$.*

We will use this lemma with $G_\delta * F$ in place of F . It leads to the following lower estimate for capacities.

PROPOSITION 3.5. *If $F \in \mathcal{L}^1(\mathbb{T}^n)$ and $0 < t < \infty$, then $\{M_{G_\delta, \gamma, cf} > t\}$ is contained in $J_c^\gamma(S(\{Nf > t\}))$. Thus, there is a $C = C(p, c)$ such that if $t \in (0, \infty)$ and $F \in \mathcal{L}^1(\mathbb{T}^n)$, we have*

$$\lambda_n(\{M_{G_\delta, \gamma, cf} > t\}) \leq C \Sigma_{G_\delta}(\{Nf > t\}).$$

THEOREM 3.6. *For $1 < p < \infty$, there is a constant $C_p < \infty$ such that if $F \in \mathcal{L}^p(\mathbb{T}^n)$ and $F \geq 0$, then*

$$\int_0^\infty \Sigma_{G_\delta}(\{G_\delta * F > t\}) d(t^n) \leq C_p \|F\|_p^p.$$

Combining Lemma 3.4, Proposition 3.5, and Theorem 3.6 with the fact that G_δ dominates \tilde{G}_δ and that $N(G_\delta * F) = G_\delta * NF$, we obtain the strong-type estimates in part (ii) of Theorem C.

THEOREM C. *Let $1 \leq p < \infty$, $F \in \mathcal{L}^p(\mathbb{T}^n)$, $0 < \delta_j < 1$, define f as in (3.1), pick γ_j as in (3.8), and for $\zeta \in \mathbb{T}^n$, construct $B_{\gamma, c}(\zeta)$ as in (3.7).*

- (i) *The B_γ -limit of f exists a.e. $[\lambda_n]$ on \mathbb{T}^n*
- (ii) *There are positive constants C_p such that*

$$\begin{aligned} \|M_{G_\delta, \gamma, cf}\|_p &\leq C_p \|F\|_p && (1 < p < \infty), \\ \lambda_n(\{M_{G_\delta, \gamma, cf} > t\}) &\leq \frac{C_1}{t} \|F\|_1 && (p = 1, 0 < t < \infty). \end{aligned}$$

If $\zeta \in \partial\mathbb{U}^n \setminus \mathbb{T}^n$, then only one component of ζ , say the n th, has $|\zeta_n| = 1$. Then the first $n - 1$ factors in the product in (3.6) are bounded as $z \rightarrow \zeta$. So in this case, it suffices to apply the one-variable result in the n th variable. The approach regions are restricted only in the n th component as in (3.5), and (z_1, \dots, z_{n-1}) can approach $(\zeta_1, \dots, \zeta_{n-1}) \in \partial\mathbb{U}^n$ in any manner whatsoever. Theorem C remains valid except that in part (ii), we would use one-dimensional norm and Lebesgue measure.

When $p = 2$ and all the $\delta_j = 1/2$, this theorem takes care of $\tilde{\mathcal{D}}(\mathbb{U}^n)$, but cannot deal with $\mathcal{D}(\mathbb{U}^n)$. Thus the functions in the modified Dirichlet space have tangential $(B_{(1, \dots, 1)})$ limits at almost every boundary point of the unit polydisc. When $n = 1$, since $\mathcal{H}^2(\mathbb{U})$ includes all Dirichlet-type spaces, elements of $\mathcal{D}(\mathbb{U})$ have nontangential limits a.e. on \mathbb{T} .

Now we will look at the size of the exceptional sets. From Lemma 3.4, part (ii) of Theorem C, and the first part of Proposition 3.5, we obtain

LEMMA 3.7. *If $1 < p < \infty$, then for some $C = C(p, c)$*

$$\Sigma_{P_r}(J_c^\gamma(S(E))) \leq C \Sigma_{P_r * G_\delta}(E).$$

Hence for $F \in \mathcal{L}^p(\mathbb{T}^n)$,

$$\Sigma_{P_r}(\{M_{G_\delta, \gamma, cf} > t\}) \leq C \Sigma_{P_r * G_\delta}(\{Nf > t\}).$$

THEOREM 3.8. *If $1 < p < \infty$ and $F \in \mathcal{L}^p(\mathbb{T}^n)$, then*

$$\int_0^\infty \Sigma_{P_r}(\{M_{G_\delta, \gamma, cf} > t\}) d(t^n) \leq C \|F\|_p^p,$$

and thus

$$\Sigma_{P_r}(\{M_{G_\delta, \gamma, cf} > t\}) \leq C \frac{\|F\|_p^p}{t^p}.$$

This theorem is an analog of Theorem C in the language of capacities and proved similarly.

THEOREM D. *Let $1 < p < \infty$, $F \in \mathcal{L}^p(\mathbb{T}^n)$, and f be as in (3.1).*

(i) *There is a set $E_1 \subset \mathbb{T}^n$ with $\Sigma_{P_r * G_\delta}(E_1) = 0$ such that the nontangential limit of f exists at every point of $\mathbb{T}^n \setminus E_1$.*

(ii) *There is a set $E_2 \subset \mathbb{T}^n$ with $\Sigma_{P_r}(E_2) = 0$ such that the B_γ -limit of f exists at every point of $\mathbb{T}^n \setminus E_2$.*

This result is a consequence of the basic properties of capacities and Theorem 3.10. For points on $\partial U^n \setminus \mathbb{T}^n$, the one-variable result can again be used to reach a similar conclusion. Hence if $p = 2$ and all the $\delta_j = 1/2$, the points on ∂U^n where the modified Dirichlet space does not have nontangential limits have zero capacity in some sense.

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