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**UNIT INDICES OF SOME IMAGINARY COMPOSITE
QUADRATIC FIELDS**

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Let K be an imaginary abelian number field of type $(2, 2, 2, 2)$ not containing the 8th cyclotomic field. Using the fundamental units of real quadratic subfields of K , we give a necessary and sufficient condition for the unit index Q_K of K to be equal to 2.

1. Introduction and results. Let K be an imaginary abelian number field and K_0 the maximal real subfield of K . Let E and E_0 be the groups of units of K and K_0 , respectively, and let W be the group of roots of unity in K . Then we call the group index

$$Q_K = [E : WE_0]$$

the unit index of K .

Using the character group of K , H. Hasse [2] gave sufficient conditions for Q_K to be equal to 1 or 2, by which we can determine Q_K for some types of fields K . However by his method we cannot always determine Q_K for arbitrary K , even if K is an imaginary composite quadratic field. (We call a field K a composite quadratic field if K is a composite of quadratic fields.) K. Yoshino and the author [3, 4] gave criteria to determine Q_K of K with Galois group $\text{Gal}(K/\mathbf{Q})$ of type $(2, 2)$ and $(2, 2, 2)$.

In this paper we extend our previous results [3, 4] to the case that K has Galois group $\text{Gal}(K/\mathbf{Q})$ of type $(2, 2, 2, 2)$ and does not contain the 8th cyclotomic field, and then, we give a necessary and sufficient condition for the unit index Q_K to be equal to 2.

NOTATION. $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$: the sets of natural numbers, rational integers and rational numbers, respectively,

$\stackrel{=}{2}$: the equality except rational quadratic factors,

$d_0, d_1, d_2, \dots, d_7$: square-free positive integers such that $d_4 \stackrel{=}{2} d_2 d_3$, $d_5 \stackrel{=}{2} d_3 d_1$, $d_6 \stackrel{=}{2} d_1 d_2$, $d_7 \stackrel{=}{2} d_1 d_2 d_3$ and $d_0 \neq d_i$ ($i = 1, 2, \dots, 7$),

$K = \mathbf{Q}(\sqrt{-d_0}, \sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$: an imaginary composite quadratic field of degree 16,

$K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$,
 E_0^+ : the group of totally positive units of K_0 ,
 E_0 : the group of units η of E_0^+ such that $K_0(\sqrt{\eta})$ is a composite quadratic field,

$$\begin{aligned}
 K_1 &= \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3}), & K_2 &= \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1}), \\
 K_3 &= \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}), & K_4 &= \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2 d_3}), \\
 K_5 &= \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3 d_1}), & K_6 &= \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1 d_2}), \\
 K_7 &= \mathbf{Q}(\sqrt{d_2 d_3}, \sqrt{d_3 d_1}), \\
 k_i &= \mathbf{Q}(\sqrt{d_i}) \quad (i = 1, 2, \dots, 7), \\
 \langle \sigma_i \rangle &= \text{Gal}(K_0/K_i) \quad (i = 1, 2, \dots, 7),
 \end{aligned}$$

$N(x)$, $\text{Sp}(x)$: the absolute norm and the absolute trace of x , respectively,

$$A = A(e_1, e_2, e_3) = \begin{cases} 2d_1^{e_1} d_2^{e_2} d_3^{e_3} & \text{if } d_0 = 1, \\ d_0 d_1^{e_1} d_2^{e_2} d_3^{e_3} & \text{otherwise,} \end{cases}$$

ε_i : the fundamental unit of $\mathbf{Q}(\sqrt{d_i})$, $\varepsilon_i > 1$ ($i = 1, 2, \dots, 7$).

When $N(\varepsilon_i) = +1$, we denote by Δ_i, Δ_i^* the square-free parts of $\text{Sp}(\varepsilon_i + 1)$, $\text{Sp}(\varepsilon_i - 1)$, respectively, and by m_i, n_i the natural numbers such that $\text{Sp}(\varepsilon_i + 1) = \Delta_i m_i^2$, $\text{Sp}(\varepsilon_i - 1) = \Delta_i^* n_i^2$. Then we have

$$(1) \quad \sqrt{\varepsilon_i} = \frac{1}{2}(m_i \sqrt{\Delta_i} + n_i \sqrt{\Delta_i^*}).$$

When $d_i d_j \stackrel{2}{=} d_k$ with $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1$, we denote by $\Delta_{ij} = \Delta_{ji}$ the square-free integer such that

$$\Delta_{ij} \stackrel{2}{=} \text{Sp}_{\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})/\mathbf{Q}}(\varepsilon_i \varepsilon_j \varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k).$$

(We take $(i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 5), (3, 6)$ and $(4, 5)$.)

When $d_i d_j d_k \stackrel{2}{=} d_l$ with $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = N(\varepsilon_l) = -1$ and when $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j}, \sqrt{d_k}) = K_0$, we denote by Δ_{ijk} the square-free integer such that

$$\Delta_{ijk} \stackrel{2}{=} \text{Sp}_{K_0/\mathbf{Q}} \left(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l + 1 - \sum_{\alpha < \beta} \varepsilon_\alpha \varepsilon_\beta \right)$$

where α, β run through i, j, k and l .

For a totally positive unit η of K_0 let

$$(2) \quad \xi^*(\eta) = \eta + \eta^{\sigma_1} + 2(-1)^{s_1} \sqrt{\eta \eta^{\sigma_1}},$$

$$(3) \quad \theta^*(\eta) = \xi^*(\eta) + \xi^*(\eta)^{\sigma_2} + 2(-1)^{s_2} \sqrt{\xi^*(\eta) \xi^*(\eta)^{\sigma_2}},$$

$$(4) \quad d^*(\eta) = \theta^*(\eta) + \theta^*(\eta)^{\sigma_3} + 2(-1)^{s_3} \sqrt{\theta^*(\eta) \theta^*(\eta)^{\sigma_3}} \quad (s_i = 0 \text{ or } 1)$$

under the condition that

$$(5) \quad \sqrt{\eta \eta^{\sigma_1}} \in K_1, \quad \sqrt{\xi^*(\eta) \xi^*(\eta)^{\sigma_2}} \in k_3 \quad \text{and} \quad \sqrt{\theta^*(\eta) \theta^*(\eta)^{\sigma_3}} \in \mathbf{Q}.$$

We remark that for a totally positive unit η of K_0 this condition (5) is satisfied if and only if η is contained in \bar{E}_0 . This remark can be proved by Lemmas 4 and 5 (cf. proof of Theorem 4).

Throughout this paper we assume that K does not contain the 8th cyclotomic field $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$. Our result is the following

MAIN THEOREM. *Under the above notation and assumption we have that $Q_K = 2$ if and only if*

$$\prod_i \Delta_i^{a_i} \cdot \prod_{i,j} \Delta_{ij}^{b_{ij}} \cdot \prod_{i,j,k} \Delta_{ijk}^{c_{ijk}} \cdot d^*(\eta_0)^f = \frac{A(e_1, e_2, e_3)}{2}$$

for some $a_i, b_{ij}, c_{ijk}, f, e_i = 0, 1$ and $\eta_0 \in \bar{E}_0$ represented in the form

$$\eta_0 = \sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i} \cdot \prod_{N(\varepsilon_i)=-1} \varepsilon_i^{v_i}},$$

where $u_i, v_i = 0$ or 1 . The number of i 's for which $u_i = 1$ is neither 1 nor 2.

More precisely we have the following Theorems 1–6.

THEOREM 1. *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7) = -1$, we have*

$$Q_K = 2 \Leftrightarrow \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \Delta_{123}^c = \frac{A(e_1, e_2, e_3)}{2}$$

for some $b_i, c, e_i = 0, 1$. Especially, if $\sqrt{\Delta_{ij}}$ is contained in $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for every (i, j) , then $Q_K = 1$.

THEOREM 2. *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_6) = -1$ and $N(\varepsilon_7) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \Delta_7^a \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} = \frac{A(e_1, e_2, e_3)}{2}$$

for some $a, b_i, e_i = 0, 1$.

THEOREM 3. *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_5) = -1$ and $N(\varepsilon_6) = N(\varepsilon_7) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \frac{1}{2} = A(e_1, e_2, e_3)$$

for some $a_i, b_i, e_i = 0, 1$.

THEOREM 4. (1) *In the case that $N(\varepsilon_1) = \cdots = N(\varepsilon_4) = -1$ and $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^b d^*(\eta_0)^f \frac{1}{2} = A(e_1, e_2, e_3)$$

for some $a_i, b, f, e_i = 0, 1$ and $\eta_0 \in \overline{E}_0$ such that

$$\eta_0 = \sqrt{\varepsilon_5 \varepsilon_6 \varepsilon_7} \prod_{i=1}^4 \varepsilon_i^{v_i} \quad (v_i = 0 \text{ or } 1).$$

(2) *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ and $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \Delta_4^{a_4} \Delta_5^{a_5} \Delta_6^{a_6} \Delta_{123}^c \frac{1}{2} = A(e_1, e_2, e_3)$$

for some $a_i, c, e_i = 0, 1$.

THEOREM 5. (1) *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ and $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \prod_{i=4}^7 \Delta_i^{a_i} \cdot d^*(\eta_0)^f \frac{1}{2} = A(e_1, e_2, e_3)$$

for some $a_i, f, e_i = 0, 1$ and $\eta_0 \in \overline{E}_0$ such that

$$\frac{\eta_0}{\prod_{i=1}^3 \varepsilon_i^{v_i}} = \sqrt{\varepsilon_4 \varepsilon_5 \varepsilon_7}, \sqrt{\varepsilon_5 \varepsilon_6 \varepsilon_7} \text{ or } \sqrt{\varepsilon_6 \varepsilon_4 \varepsilon_7} \quad (v_i = 0 \text{ or } 1).$$

(2) *In the case that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ and the others $N(\varepsilon_i) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \prod_{N(\varepsilon_i)=+1} \Delta_i^{a_i} \cdot \Delta_{12}^b \cdot d^*(\eta_0)^f \frac{1}{2} = A(e_1, e_2, e_3)$$

for some $a_i, b, f, e_i = 0, 1$ and $\eta_0 \in \overline{E}_0$ such that

$$\frac{\eta_0}{\prod_{N(\varepsilon_i)=-1} \varepsilon_i^{v_i}} = \sqrt{\varepsilon_3 \varepsilon_4 \varepsilon_5 \varepsilon_7}, \sqrt{\varepsilon_3 \varepsilon_4 \varepsilon_5}, \sqrt{\varepsilon_3 \varepsilon_4 \varepsilon_7}, \\ \sqrt{\varepsilon_3 \varepsilon_5 \varepsilon_7} \text{ or } \sqrt{\varepsilon_4 \varepsilon_5 \varepsilon_7} \quad (v_i = 0 \text{ or } 1).$$

THEOREM 6. *In the case that $N(\varepsilon_3) = N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$, we have*

$$Q_K = 2 \Leftrightarrow \prod_{N(\varepsilon_i)=+1} \Delta_i^{a_i} \cdot d^*(\eta_0)^f = \frac{A(e_1, e_2, e_3)}{2}$$

for some $a_i, f, e_i = 0, 1$ and $\eta_0 \in \overline{E}_0$ such that

$$\frac{\eta_0}{\sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i}}} = \varepsilon_1^{v_1} \varepsilon_2^{v_2}, \varepsilon_1^{v_1} \text{ or } 1 \quad (u_i, v_i = 0 \text{ or } 1)$$

according as $N(\varepsilon_1) = N(\varepsilon_2) = -1$; $N(\varepsilon_1) = -1$ and $N(\varepsilon_2) = +1$; or $N(\varepsilon_1) = N(\varepsilon_2) = +1$. The number of i 's for which $u_i = 1$ is neither 1 nor 2.

REMARK 1. In Main Theorem η_0 is not represented in the form

$$\eta_0 = \sqrt{\varepsilon_i \varepsilon_j \varepsilon_k} \cdot \prod_{N(\varepsilon_l)=-1} \varepsilon_l^{v_l}$$

where $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = +1$ and $d_i d_j = \frac{d_k}{2}$ (cf. proof of Case (2) of Theorem 4).

REMARK 2. For some $\eta_0 \in \overline{E}_0$ we can actually calculate the rational integers $d^*(\eta_0)$ defined by (4). For example, we can obtain the following: *Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = +1$ and that $\eta_0 = \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ is totally positive. Then $\eta_0 \in \overline{E}_0$ if and only if*

$$(6) \quad \Delta_1 = \frac{d_2 d_3}{2}, \quad \Delta_2 = \frac{d_3 d_1}{2}, \quad \Delta_3 = \frac{d_1 d_2}{2}.$$

If this condition (6) is satisfied, we have

$$\begin{aligned} d^*(\eta_0) &= m_1 m_2 m_3 \sqrt{\Delta_1 \Delta_2 \Delta_3} \\ &\quad + 2\Delta_1^* \{(-1)^{s_1} n_2 n_3 + (-1)^{s_2} n_3 n_1 + (-1)^{s_3} n_1 n_2\} \\ &\quad - 8(-1)^{s_1+s_2+s_3} \quad (s_i = 0 \text{ or } 1) \end{aligned}$$

where $\Delta_i, \Delta_i^*, m_i, n_i$ and s_i are as in the notation.

2. Properties of \overline{E}_0 and lemmas on (2, 2)-extensions. In this section we give a proposition and some lemmas which will be used in the proofs of theorems.

Let $\langle x, y, \dots \rangle$ be a group generated by x, y, \dots . Let E_0^* be the subgroup of E_0 generated by the units of $\mathbf{Q}(\sqrt{d_i})$ for $i = 1, 2, \dots, 7$. Let $(E_0^*)^+$ be the subgroup of E_0 generated by totally positive units of E_0^* , i.e., $(E_0^*)^+ = E_0^* \cap E_0^+$.

PROPOSITION 1. (1) *If $N(\varepsilon_1) = \cdots = N(\varepsilon_7) = -1$, then*

$$(E_0^*)^+ = \langle \varepsilon_2 \varepsilon_3 \varepsilon_4, \varepsilon_3 \varepsilon_1 \varepsilon_5, \varepsilon_1 \varepsilon_2 \varepsilon_6, \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7 \rangle E_0^{*2}.$$

(2) *If $N(\varepsilon_1) = \cdots = N(\varepsilon_6) = -1$ and $N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_2 \varepsilon_3 \varepsilon_4, \varepsilon_3 \varepsilon_1 \varepsilon_5, \varepsilon_1 \varepsilon_2 \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(3) *If $N(\varepsilon_1) = \cdots = N(\varepsilon_5) = -1$ and $N(\varepsilon_6) = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_2 \varepsilon_3 \varepsilon_4, \varepsilon_3 \varepsilon_1 \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(4₁) *If $N(\varepsilon_1) = \cdots = N(\varepsilon_4) = -1$ and $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_2 \varepsilon_3 \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(4₂) *If $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ and $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7, \varepsilon_4, \varepsilon_5, \varepsilon_6 \rangle E_0^{*2}.$$

(5₁) *If $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ and $N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(5₂) *If $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ and the others $N(\varepsilon_i) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_1 \varepsilon_2 \varepsilon_6, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_7 \rangle E_0^{*2}.$$

(6) *If $N(\varepsilon_1) = N(\varepsilon_2) = -1$ and $N(\varepsilon_3) = \cdots = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(7) *If $N(\varepsilon_1) = -1$ and $N(\varepsilon_2) = \cdots = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_2, \varepsilon_3, \dots, \varepsilon_7 \rangle E_0^{*2}.$$

(8) *If $N(\varepsilon_1) = \cdots = N(\varepsilon_7) = +1$, then*

$$(E_0^*)^+ = \langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_7 \rangle E_0^{*2}.$$

Proof. We only prove the case (1), because the other cases are proved in the same way.

For an element $\alpha \neq 0$ of K we define $s(\alpha) = 0$ or 1 by $(-1)^{s(\alpha)} = \alpha/|\alpha|$.

For $\eta \in (E_0^*)^+$, putting $\eta = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7}$ ($x_i \in \mathbf{Z}$), we have a system of simultaneous linear equations

$$\begin{cases} s(\varepsilon_1)x_1 + s(\varepsilon_2)x_2 + \cdots + s(\varepsilon_7)x_7 \equiv 0 \\ s(\varepsilon_1^{\sigma_1})x_1 + s(\varepsilon_2^{\sigma_1})x_2 + \cdots + s(\varepsilon_7^{\sigma_1})x_7 \equiv 0 \\ \quad \quad \quad \dots \\ s(\varepsilon_1^{\sigma_7})x_1 + s(\varepsilon_2^{\sigma_7})x_2 + \cdots + s(\varepsilon_7^{\sigma_7})x_7 \equiv 0. \end{cases} \quad (\text{mod } 2)$$

By Gauss-Jordan elimination (see, for example, H. Anton, *Elementary Linear Algebra*, John Wiley & Sons (1973), pp. 18–20) we see that this system has the following four linearly independent solutions:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

To these solutions correspond units $\varepsilon_2\varepsilon_3\varepsilon_4$, $\varepsilon_3\varepsilon_1\varepsilon_5$, $\varepsilon_1\varepsilon_2\varepsilon_6$, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7$ respectively. Thus we have

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_2\varepsilon_6, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7 \rangle E_0^{*2}. \quad \square$$

In general, let K/k be a $(2, 2)$ -extension with Galois group $\text{Gal}(K/k) = \langle \sigma, \tau \rangle$. Then, as used by H. Wada [6], we have

$$\alpha^2 = \frac{\alpha^{1+\sigma}\alpha^{1+\tau}}{(\alpha^\sigma)^{1+\sigma\tau}}$$

for $\alpha \in K$, $\alpha \neq 0$. By this simple formula we see that $E_0^4 \subseteq E_0^*$. Moreover, we have $\overline{E}_0^2 \subseteq E_0^*$ by the following

LEMMA 1. *Let $\eta \in \overline{E}_0$ and put $\eta^4 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ ($x_i \in \mathbf{Z}$). Then, every x_i is even.*

Proof. Since $K_0(\sqrt{\eta}) = K_0(\sqrt{d})$ for some $d \in \mathbf{N}$, we can put $\eta = d\alpha_0^2$ ($\alpha_0 \in K_0$). Taking the norm N_{K_0/k_i} of $\varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} = d^4\alpha_0^8$, we have $\varepsilon_i^{4x_i} = d^{16}N_{K_0/k_i}(\alpha_0)^8$. This implies that x_i is even. \square

LEMMA 2. *Let $\eta \in \overline{E}_0$ and put*

$$(7) \quad \eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

Then, all x_i are even or at least three x_i 's are odd.

Proof. For the simplicity we denote by N_i the norm N_{K_0/k_i} for each i .

First, for example, we assume that $x_1 \equiv 1$, $x_i \equiv 0 \pmod{2}$ ($i = 2, 3, \dots, 7$). Taking the norm N_3 of the equation (7), we have $N_3(\eta) = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\varepsilon_6^{x_6} \in K_3$. On the other hand, putting $\eta = d\alpha_0^2$ ($d \in \mathbf{N}$,

$\alpha_0 \in K_0$), we have $N_3(\eta) = d^2 N_3(\alpha_0)^2$. Therefore, $\sqrt{\varepsilon_1}$ is contained in $K_3 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$. In the same way, taking the norm N_2 of (7), we see that $\sqrt{\varepsilon_1}$ is contained in $K_2 = \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1})$. Thus $\sqrt{\varepsilon_1}$ is contained in $K_2 \cap K_3 = \mathbf{Q}(\sqrt{d_1})$, which is impossible.

Secondly, for example, we assume that $x_1 \equiv x_2 \equiv 1$, $x_i \equiv 0 \pmod{2}$ ($i = 3, 4, \dots, 7$). Taking the norms N_2, N_4 of (7), we see that $\sqrt{\varepsilon_1}$ is contained in $\mathbf{Q}(\sqrt{d_1})$, which is also impossible.

Thus there is no case that exactly one or two of x_i are odd. \square

LEMMA 3. Let $\eta \in \overline{E}_0$ and put

$$(8) \quad \eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

(1) If there exists an even x_i , then $N(\varepsilon_j) = +1$ for each odd x_j .

(2) If there exists “ i ” for which $x_i \equiv 0 \pmod{2}$ or $N(\varepsilon_i) = +1$, then x_j is even when $N(\varepsilon_j) = -1$.

(3) If $x_1 \equiv x_2 \equiv \cdots \equiv x_7 \equiv 1 \pmod{2}$, then $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7)$.

Proof. (1) Suppose that $x_1 \equiv 1$, $x_2 \equiv 0 \pmod{2}$. Taking the norm N_3 of (8), we have $N_3(\eta) = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \varepsilon_6^{x_6}$. Again, taking the norms N_1, N_2 of this equation, we have by $\eta \gg 0$ that

$$\begin{aligned} N_1(N_3(\eta)) &= N(\varepsilon_1)^{x_1} \varepsilon_2^{2x_2} N(\varepsilon_6)^{x_6} > 0, \\ N_2(N_3(\eta)) &= \varepsilon_1^{2x_1} N(\varepsilon_2)^{x_2} N(\varepsilon_6)^{x_6} > 0. \end{aligned}$$

Hence $N(\varepsilon_6)^{x_6} = +1$ and then $N(\varepsilon_1) = +1$.

(2) We suppose that $x_1 \equiv 0 \pmod{2}$ or $N(\varepsilon_1) = +1$ and that $N(\varepsilon_2) = -1$.

Taking the norm N_3 of (8), we have $N_3(\eta) = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \varepsilon_6^{x_6}$. Again, taking the norm N_6 of this equation, we have

$$N_6(N_3(\eta)) = N(\varepsilon_1)^{x_1} N(\varepsilon_2)^{x_2} \varepsilon_6^{2x_6} > 0,$$

and so $x_2 \equiv 0 \pmod{2}$.

(3) Taking the norm N_1 of (8), we have $N_1(\eta) = \varepsilon_2^{x_2} \varepsilon_3^{x_3} \varepsilon_4^{x_4}$. Moreover, taking the norms N_2, N_3 of this equation, we have

$$\begin{aligned} N_2(N_1(\eta)) &= N(\varepsilon_2)^{x_2} \varepsilon_3^{2x_3} N(\varepsilon_4)^{x_4} > 0, \\ N_3(N_1(\eta)) &= \varepsilon_2^{2x_2} N(\varepsilon_3)^{x_3} N(\varepsilon_4)^{x_4} > 0. \end{aligned}$$

Then $N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_4)$.

In the same way, taking the norms N_2, N_3, N_6 of (8), we obtain $N(\varepsilon_3) = N(\varepsilon_1) = N(\varepsilon_5), N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6), N(\varepsilon_3) = N(\varepsilon_6) = N(\varepsilon_7)$. \square

For a field k we denote by “ $\stackrel{=}{2}$ in k ” the equality except a square of a number of k .

LEMMA 4 (*F. Halter-Koch [1, Satz 1]*). *Let K_1 be a field with $\text{char}(K_1) \neq 2$. Let K_0 be a quadratic extension of K_1 and $K_0(\sqrt{\eta_0})$ ($\eta_0 \in K_0$) a biquadratic (quartic) extension of K_1 . Then $K_0(\sqrt{\eta_0})/K_1$ is bicyclic if and only if $N_{K_0/K_1}(\eta_0) \stackrel{=}{2} 1$ in K_1 .*

By this Lemma 4 we can easily obtain

LEMMA 5. *Let K_1 be an algebraic number field and K_0 a quadratic extension of K_1 . Let $K_0(\sqrt{\eta_0})$ ($\eta_0 \in K_0, \eta_0 \notin K_1$) be a biquadratic bicyclic extension of K_1 with $\text{Gal}(K_0(\sqrt{\eta_0})/K_1) = \langle \sigma, \tau \rangle$ and $\text{Gal}(K_0(\sqrt{\eta_0})/K_0) = \langle \tau \rangle$. Let F be the intermediate field of $K_0(\sqrt{\eta_0})/K_1$ fixed by σ . Then we have*

$$F = K_1(\sqrt{\eta_0} + \sqrt{\eta_0}^\sigma).$$

3. Proof of theorems. For the proof of Main Theorem, it is enough to prove Theorems 1–6, because the cases of Proposition 1 cover all the possible cases of the combination of $N(\varepsilon_i) = \pm 1$.

Let K' be the quadratic extension of K generated by a primitive 2^{n+1} th root of unity, $2^n \parallel \#W$, and let K'_0 be the maximal real subfield of K' .

When $d_i d_j \stackrel{=}{2} d_k$ and $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1$, let

$$\eta_{ij} = \varepsilon_i \varepsilon_j \varepsilon_k, \quad \xi_{ij} = \varepsilon_i \varepsilon_j \varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k.$$

Then it follows from T. Kubota [5, §5] that

$$(9) \quad \eta_{ij} \text{Sp}(\xi_{ij}) = \xi_{ij}^2.$$

For the multi-quadratic field $K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$, we can prove:

LEMMA 6. *Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$. Let*

$$\eta = \eta_{123} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7,$$

$$\xi = \xi_{123} = \eta + 1 - (\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1 + \varepsilon_1 \varepsilon_7 + \varepsilon_2 \varepsilon_7 + \varepsilon_3 \varepsilon_7).$$

Then we have

$$(10) \quad \eta \operatorname{Sp}(\xi) = \xi^2.$$

Proof. Since

$$\xi^{\sigma_1} = \varepsilon'_1 \varepsilon_2 \varepsilon_3 \varepsilon'_7 + 1 - \varepsilon'_1 \varepsilon_2 - \varepsilon_2 \varepsilon_3 - \varepsilon_3 \varepsilon'_1 - \varepsilon'_1 \varepsilon'_7 - \varepsilon_2 \varepsilon'_7 - \varepsilon_3 \varepsilon'_7,$$

it holds that $\varepsilon_1 \varepsilon_7 \xi^{\sigma_1} = -\xi$, where ε' is the conjugate of ε with respect to \mathbf{Q} . In the same way we have

$$\begin{aligned} \varepsilon_2 \varepsilon_7 \xi^{\sigma_2} &= \varepsilon_3 \varepsilon_7 \xi^{\sigma_3} = \varepsilon_2 \varepsilon_3 \xi^{\sigma_4} = \varepsilon_3 \varepsilon_1 \xi^{\sigma_5} = \varepsilon_1 \varepsilon_2 \xi^{\sigma_6} = -\xi, \\ \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7 \xi^{\sigma_7} &= \xi. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Sp}_{K_0/\mathbf{Q}}(\xi) &= \xi + \xi^{\sigma_1} + \cdots + \xi^{\sigma_7} \\ &= \xi \left(1 - \sum_{i < j} \frac{1}{\varepsilon_i \varepsilon_j} + \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7} \right) \end{aligned}$$

where i, j run through 1, 2, 3 and 7. Thus we have $\eta \operatorname{Sp}_{K_0/\mathbf{Q}}(\xi) = \xi^2$. \square

LEMMA 7. *Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7) = -1$ and that $\sqrt{\Delta_{ij}} \notin \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for some (i, j) . Then we have $\overline{E}_0 = (E_0^*)^+ E_0^2$.*

Proof. Let $\eta \in \overline{E}_0$. By Lemma 1 we have

$$(11) \quad \eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

Assume that every x_i is odd. Taking the norm N_1 of (11), we have by Lemma 4 that $\varepsilon_2^{x_2} \varepsilon_3^{x_3} \varepsilon_4^{x_4} = 1$ in K_1 , because $K_0(\sqrt{\eta})/K_1$ is a $(2, 2)$ -extension or $\sqrt{\eta}$ is contained in K_0 . Therefore $\sqrt{\varepsilon_2 \varepsilon_3 \varepsilon_4} \in K_1$, and then by (9) we have $\sqrt{\Delta_{23}} \in K_1 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3})$. Similarly, taking the norms N_2, N_3, N_4, N_5, N_6 and N_7 of (11), we have $\sqrt{\Delta_{ij}} \in \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for every (i, j) . This contradicts the assumption. Hence there is an even integer among x_i 's, and it follows from (2) of Lemma 3 that every x_i is even. Therefore, $\eta \in (E_0^*)^+ E_0^2$. Thus we have $\overline{E}_0 \subseteq (E_0^*)^+ E_0^2$.

The inverse inclusion $(E_0^*)^+ E_0^2 \subseteq \overline{E}_0$ is shown by the equations

$$(12) \quad \sqrt{\eta} \sqrt{\operatorname{Sp}(\xi)} = \xi$$

for $(\eta, \xi) = (\eta_{ij}, \xi_{ij})$ and (η_{ijk}, ξ_{ijk}) , since $(E_0^*)^+ E_0^2 / E_0^2$ is represented by $\eta_{12}, \eta_{23}, \eta_{31}$ and η_{123} .

Proof of Theorem 1. First we assume that $\sqrt{\Delta_{ij}} \notin \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for some (i, j) .

Suppose that $Q_K = 2$. Then there exists a unit $\eta \in \bar{E}_0$ such that $K_0(\sqrt{\eta}) = K'_0$ (Hasse [2, Satz 15]). By Lemma 7 we have $\eta = \varepsilon_1^{a_1} \varepsilon_2^{a_2} \cdots \varepsilon_7^{a_7} \varepsilon_0^2$ ($a_i \in \mathbf{Z}$, $\varepsilon_0 \in E_0$) such that $\varepsilon_1^{a_1} \varepsilon_2^{a_2} \cdots \varepsilon_7^{a_7}$ is totally positive, and by (1) of Proposition 1 $\eta = \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3} \eta_{123}^c \varepsilon^2$ ($b_i, c \in \mathbf{Z}$, $\varepsilon \in E_0$). Therefore it follows from (12) that

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \Delta_{123}^c}).$$

Since $K'_0 = K_0(\sqrt{2})$ or $K_0(\sqrt{d_0})$ according as $d_0 = 1$ or not, we have $K'_0 = K_0(\sqrt{A'})$ for some $A' = A(e'_1, e'_2, e'_3)$. Therefore

$$K_0(\sqrt{\Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \Delta_{123}^c}) = K_0(\sqrt{A'}).$$

Thus we have

$$(13) \quad \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \Delta_{123}^c \equiv_2 A(e_1, e_2, e_3)$$

for some $e_i = 0, 1$. Because, if $K_0(\sqrt{m}) = K_0(\sqrt{A'})$ for a rational integer m and $A' = A(e'_1, e'_2, e'_3)$, then $\mathbf{Q}(\sqrt{m/A'})$ is equal to \mathbf{Q} or $\mathbf{Q}(\sqrt{m/A'})$ is a quadratic subfield of K_0 , and so

$$m = A' d_1^{e''_1} d_2^{e''_2} d_3^{e''_3} r^2$$

for some $e''_1, e''_2, e''_3 = 0, 1$ and some $r \in \mathbf{Q}$. Therefore, putting $e_i \equiv e'_i + e''_i \pmod{2}$ ($i = 1, 2, 3$), we have

$$m \equiv_2 A(e_1, e_2, e_3).$$

Conversely, if this equation (13) holds, then the square root of $\eta := \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3} \eta_{123}^c$ generates K'_0 over K_0 , i.e., $K_0(\sqrt{\eta}) = K'_0$. Thus, by H. Hasse [2, Satz 15] we have $Q_K = 2$.

Secondly, we assume that $\sqrt{\Delta_{ij}} \in \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for every (i, j) . Then it does not hold that

$$\Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \Delta_{123}^c \equiv_2 A(e_1, e_2, e_3)$$

for any $b_i, c, e_i = 0, 1$.

In fact, by the assumption and by $\eta_{123} = \eta_{12}\eta_{36}\varepsilon_6^{-2}$ we have $K_0(\sqrt{\Delta_{ij}}) = K_0$ for every (i, j) and $K_0(\sqrt{\Delta_{123}}) = K_0(\sqrt{\Delta_{12}\Delta_{36}}) = K_0$. Consequently, we have

$$\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^c = \frac{d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3}}{2} \neq A(e_1, e_2, e_3),$$

where $\alpha_i = 0$ or 1 .

In this case we can show that $Q_K = 1$ as follows:

Assume that $Q_K = 2$. Then there is a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K'_0$. By Lemma 1 we have $\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ ($x_i \in \mathbf{Z}$). It follows from (2) of Lemma 3 that all x_i are even or odd.

If all x_i are even, then $\eta \in (E_0^*)^+$ and we have $\eta = \eta_{12}^{b_1}\eta_{23}^{b_2}\eta_{31}^{b_3}\eta_{123}^c\varepsilon_0^2$ for some $b_i, c \in \mathbf{Z}$ and $\varepsilon_0 \in E_0^*$. Since $\eta_{123} = \eta_{12}\eta_{36}\varepsilon_6^{-2}$, we obtain by the assumption that $\sqrt{\eta} \in K_0$, which contradicts that $K_0(\sqrt{\eta})$ is a quadratic extension over K_0 . Therefore, all x_i are odd. Then $\eta = \sqrt{\varepsilon_1\varepsilon_1\cdots\varepsilon_7}\prod_{i=1}^7\varepsilon_i^{y_i}$ for some $y_i \in \mathbf{Z}$. Since $\varepsilon_1\varepsilon_2\cdots\varepsilon_7 = \eta_{13}\eta_{23}\eta_{36}\varepsilon_3^{-2}$, we have

$$\eta = \sqrt{\eta_{13}}\sqrt{\eta_{23}}\sqrt{\eta_{36}}\varepsilon_3^{-1}\prod_{i=1}^7\varepsilon_i^{y_i}.$$

By (9) we have $\sqrt{\eta_{13}}r_{13}\sqrt{\Delta_{13}} = \xi_{13}$ for some $r_{13} \in \mathbf{N}$. And by the assumption we have $\Delta_{13} = \frac{d_1^{a_1} d_3^{a_3}}{2}$ for some $a_1, a_3 = 0, 1$. Hence $\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\Delta_{13}}$ is totally positive. Moreover, from $\xi_{13}^{\sigma_1} < 0$, $\xi_{13}^{\sigma_2} > 0$, $\xi_{13}^{\sigma_3} < 0$ it follows that $\varepsilon_1\varepsilon_3\xi_{13}$ is totally positive. Therefore

$$\varepsilon_1\varepsilon_3\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\eta_{13}} = \frac{1}{r_{13}} \cdot \frac{\varepsilon_1^{a_1}\varepsilon_3^{a_3}}{\sqrt{\Delta_{13}}} \cdot \varepsilon_1\varepsilon_3\xi_{13}$$

is totally positive, and then this unit is square in $K_2 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_3})$ (M. Hirabayashi and K. Yoshino [4, Proposition 2, IV]). So we can put

$$\varepsilon_1\varepsilon_3\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\eta_{13}} = \varepsilon_{13}^2$$

where ε_{13} is a unit of K_2 . In the same way we obtain

$$\varepsilon_2\varepsilon_3\varepsilon_2^{b_2}\varepsilon_3^{b_3}\sqrt{\eta_{23}} = \varepsilon_{23}^2, \quad \varepsilon_3\varepsilon_6\varepsilon_3^{c_3}\varepsilon_6^{c_6}\sqrt{\eta_{36}} = \varepsilon_{36}^2 \quad (b_i, c_j = 0, 1)$$

where ε_{23} and ε_{36} are units of K_1 and K_6 , respectively. Therefore we have

$$\eta = \varepsilon_{13}^2\varepsilon_{23}^2\varepsilon_{36}^2\prod_{i=1}^7\varepsilon_i^{z_i} \quad (z_i \in \mathbf{Z}).$$

Since $\prod_{i=1}^7 \varepsilon_i^{z_i}$ is totally positive, we have, as before,

$$\prod_{i=1}^7 \varepsilon_i^{z_i} = \eta_{12}^{\alpha_1} \eta_{23}^{\alpha_2} \eta_{31}^{\alpha_3} (\eta_{12} \eta_{36})^{\alpha_4} \varepsilon_0^2$$

for some $\alpha_i \in \mathbf{Z}$ and $\varepsilon_0 \in E_0^*$. By the assumption each η_{ij} is square in $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ and so is η in K_0 , which is also contradiction. \square

LEMMA 8. *If exactly one or two of $N(\varepsilon_i)$ ($i = 1, 2, \dots, 7$) are $+1$, then we have $\overline{E}_0 = (E_0^*)^+ E_0^2$.*

Proof. It is enough to prove the following two Cases (1) and (2).

Case (1): $N(\varepsilon_1) = \dots = N(\varepsilon_5) = -1$ and $N(\varepsilon_6) = N(\varepsilon_7) = +1$.

Let $\eta \in \overline{E}_0$ and let $\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \dots \varepsilon_7^{x_7}$ ($x_i \in \mathbf{Z}$). By (2) of Lemma 3 we see that x_1, x_2, \dots, x_5 are even. Then it follows from Lemma 4 that

$$\eta \eta^{\sigma_4} = \varepsilon_1^{x_1} \varepsilon_4^{x_4} \varepsilon_7^{x_7} \frac{1}{2} = 1 \quad \text{in } K_4,$$

$$\eta \eta^{\sigma_5} = \varepsilon_2^{x_2} \varepsilon_5^{x_5} \varepsilon_7^{x_7} \frac{1}{2} = 1 \quad \text{in } K_5.$$

Now, we assume that x_7 is odd. Then $\varepsilon_7 \frac{1}{2} = 1$ in $K_4 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_4})$ and in $K_5 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_5})$. Therefore, $\Delta_7 \frac{1}{2} = d_1^{e_1} d_4^{e_4}$, $\Delta_7 \frac{1}{2} = d_2^{e_2} d_5^{e_5}$ for some $e_1, e_2, e_4, e_5 = 0, 1$. These equations lead that $\Delta_7 \frac{1}{2} = (d_1 d_2 d_3)^{e_1} \frac{1}{2} = d_7^{e_1}$, which is impossible (Kubota [5, Hilfssatz 9]). Thus x_7 is even. Similarly, by the equations

$$\eta \eta^{\sigma_3} = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \varepsilon_6^{x_6} \frac{1}{2} = 1 \quad \text{in } K_3,$$

$$\eta \eta^{\sigma_6} = \varepsilon_3^{x_3} \varepsilon_6^{x_6} \varepsilon_7^{x_7} \frac{1}{2} = 1 \quad \text{in } K_6,$$

we see that x_6 is even. Therefore all x_i are even and so $\eta \in E_0^*$. Thus $\overline{E}_0 \subseteq (E_0^*)^+ E_0^2$.

The inverse inclusion $(E_0^*)^+ E_0^2 \subseteq \overline{E}_0$ is shown by the equations (1) and (12).

Case (2): $N(\varepsilon_1) = N(\varepsilon_2) = \dots = N(\varepsilon_6) = -1$ and $N(\varepsilon_7) = +1$.

Let $\eta \in \overline{E}_0$ and let $\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \dots \varepsilon_7^{x_7}$ ($x_i \in \mathbf{Z}$). Then, by (2) of Lemma 3 we see that x_1, x_2, \dots, x_6 are even. In the same way as in the proof of Case (1) we can show that x_7 is even and that $\overline{E}_0 = (E_0^*)^+ E_0^2$. \square

Proof of Theorems 2 and 3. We only prove Theorem 2, because we prove Theorem 3 in a similar way.

Suppose that $Q_K = 2$. Then there exists a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K'_0 = K_0(\sqrt{A})$ where $A = A(e_1, e_2, e_3)$. By Lemma 8 and (2) of Proposition 1 we can put $\eta = \varepsilon_7^a \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3} \varepsilon^2$ ($a, b_i \in \mathbf{Z}$, $\varepsilon \in E_0$) and we have

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_7^a \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3}}).$$

Consequently,

$$(14) \quad \Delta_7^a \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} = A(e_1, e_2, e_3).$$

Conversely, if this equation (14) holds, then a square root of $\eta := \varepsilon_7^a \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3}$ generates K'_0 over K_0 , i.e., $K'_0 = K_0(\sqrt{\eta})$. Therefore we have $Q_K = 2$. \square

Proof of Theorem 4.

Case (1): $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_4) = -1$ and $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$.

Suppose that $Q_K = 2$. Then there is a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K'_0$. By Lemma 1 and (4₁) of Proposition 1 we have

$$\eta^2 = \eta_{23}^{x_2} \varepsilon_5^{x_3} \varepsilon_6^{x_6} \varepsilon_7^{x_7} \prod_{i=1}^7 \varepsilon_i^{2y_i}$$

where $x_i, y_i \in \mathbf{Z}$. From (2) of Lemma 3 it follows that $x_2 \equiv 0 \pmod{2}$. Hence by Lemma 2 we see that $x_5 \equiv x_6 \equiv x_7 \pmod{2}$.

In the case that $x_5 \equiv x_6 \equiv x_7 \equiv 0 \pmod{2}$, we have

$$\eta = \varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7} \eta_{23}^b \varepsilon_0^2$$

for some $a_i, b = 0, 1$ and $\varepsilon_0 \in E_0^*$. Therefore,

$$K'_0 = K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^b})$$

and then

$$(15) \quad \Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^b = A(e_1, e_2, e_3)$$

for some $e_i = 0, 1$.

In the case that $x_5 \equiv x_6 \equiv x_7 \equiv 1 \pmod{2}$, let

$$\eta_0 := \sqrt{\varepsilon_5 \varepsilon_6 \varepsilon_7} \prod_{i=1}^4 \varepsilon_i^{v_i} \quad (v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive. Then we have $\eta = \varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7} \eta_{23}^b \eta_0 \varepsilon_0^2$ where $a_i, b = 0, 1$ and $\varepsilon_0 \in E_0^*$. Since $\varepsilon_5, \varepsilon_6, \varepsilon_7, \eta_{23}, \eta \in \overline{E}_0$, we see $\eta_0 \in \overline{E}_0$. Then it follows from Lemma 5 that

$$K_0(\sqrt{\eta_0}) = K_0(\sqrt{\xi^*(\eta_0)}) = K_0(\sqrt{\theta^*(\eta_0)}) = K_0(\sqrt{d^*(\eta_0)})$$

where $\xi^*(\eta_0)$, $\theta^*(\eta_0)$ and $d^*(\eta_0)$ is defined by (2), (3) and (4), respectively. Here we take $s_i = 0$ or 1 ($i = 1, 2, 3$) in accordance with

$$\begin{aligned} \xi^*(\eta_0) &= (\sqrt{\eta_0} + \sqrt{\eta_0} \sigma_1)^2, & \theta^*(\eta_0) &= (\sqrt{\xi^*(\eta_0)} + \sqrt{\xi^*(\eta_0)} \sigma_2)^2, \\ d^*(\eta_0) &= (\sqrt{\theta^*(\eta_0)} + \sqrt{\theta^*(\eta_0)} \sigma_3)^2, \end{aligned}$$

respectively. Therefore

$$K'_0 = K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^b d^*(\eta_0)})$$

and then we have

$$(16) \quad \Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^b d^*(\eta_0) \underset{2}{=} A(e_1, e_2, e_3)$$

for some $e_i = 0, 1$.

Conversely, if the equation (15) or (16) holds, the square root of $\eta := \varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7} \eta_{23}^b$ or $\varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7} \eta_{23}^b \eta_0$ generates K'_0 over K_0 , respectively, i.e., $K'_0 = K_0(\sqrt{\eta})$. Then we have $Q_K = 2$.

Case (2): $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ and $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$.

Suppose that $Q_K = 2$. Then by Lemma 1 and (4₂) of Proposition 1 we have

$$(17) \quad \eta^2 = \varepsilon_4^{x_4} \varepsilon_5^{x_5} \varepsilon_6^{x_6} \eta_{123}^z \prod_{i=1}^7 \varepsilon_i^{2y_i}$$

where $x_i, y_i, z \in \mathbf{Z}$. Then it follows from (2) of Lemma 3 that $z \equiv 0 \pmod{2}$, and from Lemma 2 that $x_4 \equiv x_5 \equiv x_6 \pmod{2}$.

If $x_4 \equiv x_5 \equiv x_6 \equiv 0 \pmod{2}$, then $\eta \in (E_0^*)^+$. By (4₂) of Proposition 1 we have $\eta = \varepsilon_4^{a_4} \varepsilon_5^{a_5} \varepsilon_6^{a_6} \eta_{123}^c \varepsilon_0^2$ for some $a_i, c = 0, 1$ and $\varepsilon_0 \in E_0^*$. Therefore,

$$(18) \quad K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_4^{a_4} \Delta_5^{b_5} \Delta_6^{b_6} \Delta_{123}^c}).$$

If $x_4 \equiv x_5 \equiv x_6 \equiv 1 \pmod{2}$, taking norms N_1 and N_4 of the equation (17), we have by Lemma 4 that

$$\begin{aligned} \eta^{1+\sigma_1} &= \varepsilon_4^{x_4} \varepsilon_2^{2y_2} \varepsilon_3^{2y_3} \varepsilon_4^{2y_4} \underset{2}{=} 1 \quad \text{in } K_1, \\ \eta^{1+\sigma_4} &= \varepsilon_4^{x_4} \varepsilon_1^{2y_1} \varepsilon_7^{2y_7} \varepsilon_4^{2y_4} \underset{2}{=} 1 \quad \text{in } K_4. \end{aligned}$$

Then $\sqrt{\Delta_4}$ is contained in $K_1 \cap K_4 = \mathbf{Q}(\sqrt{d_2 d_3})$, and then $\Delta_4 \equiv \frac{1}{2}$ or $d_2 d_3$, which is impossible (T. Kubota [5, Hilfssatz 9]).

Thus, if $Q_K = 2$ we have the equation (18) and hence

$$(19) \quad \Delta_4^{a_4} \Delta_5^{a_5} \Delta_6^{a_6} \Delta_{123}^c \equiv \frac{1}{2} A(e_1, e_2, e_3)$$

for some $e_i = 0, 1$.

Conversely, when the equation (19) holds, we can show, as before, that $Q_K = 2$. \square

Proof of Theorem 5. (1) Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ and that $N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$. By Lemma 1 and (5₁) of Proposition 1 we have

$$(20) \quad \eta^2 = \varepsilon_4^{x_4} \varepsilon_5^{x_5} \varepsilon_6^{x_6} \varepsilon_7^{x_7} \prod_{i=1}^7 \varepsilon_i^{2y_i}$$

for any $\eta \in \overline{E}_0$ where $x_i, y_i \in \mathbf{Z}$. Then by Lemma 2 we have the following three cases:

- (i) $x_4 \equiv x_5 \equiv x_6 \equiv x_7 \equiv 0 \pmod{2}$;
- (ii) Among x_4, x_5, x_6 and x_7 , exactly one x_i is even;
- (iii) $x_4 \equiv x_5 \equiv x_6 \equiv x_7 \equiv 1 \pmod{2}$.

Case (i). We have $\eta \in (E_0^*)^+$ and we may put $\eta = \varepsilon_4^{a_4} \varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7}$ ($a_i \in \mathbf{Z}$). Then we obtain, as before,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_4^{a_4} \Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7}}).$$

Case (ii). We first consider the case that $x_4 \equiv x_5 \equiv x_6 \equiv 1, x_7 \equiv 0 \pmod{2}$. Taking norms N_1 and N_4 of (20), we have

$$\eta^{1+\sigma_1} = \varepsilon_4^{x_4} \varepsilon_2^{2y_2} \varepsilon_3^{2y_3} \equiv 1 \quad \text{in } K_1 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3}),$$

$$\eta^{1+\sigma_4} = \varepsilon_4^{x_4} \varepsilon_1^{2y_1} \varepsilon_7^{2y_7} \equiv 1 \quad \text{in } K_4 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_4}).$$

Then, as before, $\sqrt{\Delta_4}$ is contained in $\mathbf{Q}(\sqrt{d_4})$, which is impossible.

Next we consider the other cases, for example, $x_4 \equiv x_5 \equiv x_7 \equiv 1, x_6 \equiv 0 \pmod{2}$. Let

$$\eta_0 := \sqrt{\varepsilon_4 \varepsilon_5 \varepsilon_7} \prod_{i=1}^3 \varepsilon_i^{v_i} \quad (v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive. Then we can prove the assertion in the same way as in the proof of Case (1) of Theorem 4.

Case (iii). As before, taking norms N_1, N_2, N_3 and N_7 of (20), we obtain

$$\begin{aligned}\Delta_4 \equiv_2 d_2 \text{ or } d_3; \quad \Delta_5 \equiv_2 d_3 \text{ or } d_1; \quad \Delta_6 \equiv_2 d_1 \text{ or } d_2; \\ \Delta_4 \Delta_5 \Delta_6 \equiv_2 d_2 d_3, d_3 d_1 \text{ or } d_1 d_2,\end{aligned}$$

which is impossible.

(2) Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ and the others $N(\varepsilon_i) = +1$. We have by (5₂) of Proposition 1

$$\eta^2 = \varepsilon_3^{x_3} \varepsilon_4^{x_4} \varepsilon_5^{x_5} \varepsilon_7^{x_7} \eta_{12}^{x_1} \prod_{i=1}^7 \varepsilon_i^{2y_i}$$

for any $\eta \in \overline{E}_0$ where $x_i, y_i \in \mathbf{Z}$. By (2) of Lemma 3 we have $x_1 \equiv 0 \pmod{2}$. Therefore we obtain, as before, the following cases:

- (i) $x_3 \equiv x_4 \equiv x_5 \equiv x_7 \equiv 0 \pmod{2}$;
- (ii) Among x_3, x_4, x_5 and x_7 , exactly one x_i is even;
- (iii) $x_3 \equiv x_4 \equiv x_5 \equiv x_7 \equiv 1 \pmod{2}$.

By the same argument in (1) of this proof we can prove the assertion for each case. \square

Proof of Theorem 6. In the following we only consider the first case: $N(\varepsilon_1) = N(\varepsilon_2) = -1$, since the other cases are proved in the same way.

Let

$$\eta_0 := \sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i} \cdot \prod_{N(\varepsilon_i)=-1} \varepsilon_i^{v_i}} \quad (u_i, v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive.

For any $\eta \in \overline{E}_0$ we may put $\eta = \varepsilon_3^{a_3} \cdots \varepsilon_7^{a_7} \cdot \eta_0^f$ where $a_i, f = 0$ or 1 . Then we have, as before,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_3^{a_3} \cdots \Delta_7^{a_7} d^*(\eta_0)^f}).$$

Thus we obtain that $Q_K = 2$ if and only if

$$\Delta_3^{a_3} \cdots \Delta_7^{a_7} d^*(\eta_0)^f \equiv_2 A(e_1, e_2, e_3),$$

as desired. \square

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