

*Pacific  
Journal of  
Mathematics*

**PERIODS AND LEFSCHETZ ZETA FUNCTIONS**

JOSEFINA CASASAYAS, JAUME LLIBRE AND ANA NUNES

Volume 165 No. 1

September 1994

## PERIODS AND LEFSCHETZ ZETA FUNCTIONS

JOSEFINA CASASAYAS, JAUME LLIBRE AND ANA NUNES

**The goal of this paper is to obtain information on the set of periods for a transversal self-map of a compact manifold from the associated Lefschetz zeta function in the case when all its zeros and poles are roots of unity.**

**1. Introduction and statement of the results.** One of the most useful theorems for proving the existence of fixed points or, more generally, periodic points of a transversal self-map  $f$  of a compact manifold is the Lefschetz fixed point theorem. When studying the periodic points of  $f$ , i.e., the set

$$\text{Per}(f) = \{m \in \mathbb{N} : f \text{ has a periodic orbit of minimal period } m\},$$

it is convenient to use the Lefschetz zeta function of  $f$ ,  $Z_f(t)$ , which is a generating function for the Lefschetz numbers of all iterates of  $f$ . The function  $Z_f(t)$  is rational in  $t$  and can be computed from the homological invariants of  $f$  (see §3).

We shall study  $C^1$  self-maps  $f$  of a compact manifold which have only transversal periodic points, so called because the graph of  $f^m$  is transverse to the diagonal for all  $m > 0$ . The main contribution of this paper is the study of the periodic orbits of  $f$  when its Lefschetz zeta function has a finite factorization into terms of the form  $(1 \pm t^n)^{\pm 1}$ . A key point is the introduction of the notion of irreducible factor (see §3 for a precise definition). Our main result is the following.

**THEOREM A.** *Let  $f: M \rightarrow M$  be a transversal map of a compact manifold. Suppose that all the zeros and poles of its Lefschetz zeta function  $Z_f(t)$  are roots of unity, and that  $Z_f(t)$  has an irreducible factor of the form  $(1 \pm t^n)^{\pm 1}$ .*

- (a) *If  $n$  is odd then  $n \in \text{Per}(f)$ .*
- (b) *If  $n$  is even then  $\{\frac{n}{2}, n\} \cap \text{Per}(f) \neq \emptyset$ .*

The proof of this theorem will be given in §3. From Theorem A it follows that each irreducible factor of the form  $(1 \pm t^n)^{\pm 1}$  of the

Lefschetz zeta function forces at least one period ( $n$  if  $n$  is odd,  $n/2$  or  $n$  if  $n$  is even).

The set of periods obtained in this way will be called the *forced set of periods* of  $f$  and will be denoted by  $\text{FSP}(f)$ .

As an application of Theorem A and the algebraic results derived in §2, we obtain an upper bound for the cardinal and for the maximum period of the forced set of periods (see Corollaries 3.3 and 3.4).

Our main basic assumption throughout this work is that all the zeros and poles of the Lefschetz zeta function associated to  $f: M \rightarrow M$  are roots of unity (for different results under similar assumptions see Franks [F1], [F3], Fried [Fr], Matsuoka [Mt] and [CLN]). There are three interesting classes of transversal maps which satisfy our basic assumption. First, the set of maps whose set of periods  $\text{Per}(f)$  is finite (see Theorem 6 of [Fr]). Second, the self-maps of compact connected surfaces with  $\text{Per}(f)$  finite or  $h(f) = 0$ , see Corollaries 4.3 and 4.4. Finally, the self-maps of the  $n$ -dimensional torus with  $\text{Per}(f)$  finite or  $h(f) = 0$ , see Corollaries 5.1 and 5.2.

**2. Cyclotomic polynomials.** As usual, we shall use the notation  $c_n(t)$  for the  $n$ th cyclotomic polynomial given by

$$c_n(t) = \frac{1 - t^n}{\prod_{d|n, d < n} c_d(t)}$$

for  $n \in \mathbb{N} \setminus \{1\}$  and  $c_1(t) = 1 - t$ .

Notice that all the zeros of  $c_n(t)$  are roots of unity.

A proof of the next proposition may be found in [L].

**PROPOSITION 2.1.** *Let  $\xi$  be a primitive  $n$ th root of unity and  $P(t)$  a polynomial with rational coefficients. If  $P(\xi) = 0$  then  $c_n(t) | P(t)$ .*

Clearly, the degree  $\varphi(n)$  of  $c_n(t)$  verifies

$$n = \sum_{d|n} \varphi(d)$$

and so  $\varphi(n)$  is the Euler function, which may be computed through

$$\varphi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

Hence, if  $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$  is the prime decomposition of  $n$ , then

$$(2.1) \quad \varphi(n) = \prod_{i=1}^k p_i^{\alpha_i - 1} (p_i - 1).$$

TABLE 1

$c_1(t) = 1 - t$	$c_2(t) = 1 + t$	$c_3(t) = \frac{1 - t^3}{1 - t}$
$c_4(t) = 1 + t^2$	$c_5(t) = \frac{1 - t^5}{1 - t}$	$c_6(t) = \frac{1 + t^3}{1 + t}$
$c_7(t) = \frac{1 - t^7}{1 - t}$	$c_8(t) = 1 + t^4$	$c_9(t) = \frac{1 - t^9}{1 - t^3}$
$c_{10}(t) = \frac{1 + t^5}{1 + t}$	$c_{11}(t) = \frac{1 - t^{11}}{1 - t}$	$c_{12}(t) = \frac{1 + t^6}{1 + t^2}$
$c_{13}(t) = \frac{1 - t^{13}}{1 - t}$	$c_{14}(t) = \frac{1 + t^7}{1 + t}$	$c_{15}(t) = \frac{(1 - t^{15})(1 - t)}{(1 - t^3)(1 - t^5)}$
$c_{16}(t) = 1 + t^8$	$c_{17}(t) = \frac{1 - t^{17}}{1 - t}$	$c_{18}(t) = \frac{1 + t^9}{1 + t^3}$
$c_{19}(t) = \frac{1 - t^{19}}{1 - t}$	$c_{20}(t) = \frac{1 + t^{10}}{1 + t^2}$	$c_{21}(t) = \frac{(1 - t^{21})(1 - t)}{(1 - t^3)(1 - t^7)}$
$c_{22}(t) = \frac{1 + t^{11}}{1 + t}$	$c_{23}(t) = \frac{1 - t^{23}}{1 - t}$	$c_{24}(t) = \frac{1 + t^{12}}{1 + t^4}$
$c_{25}(t) = \frac{1 - t^{25}}{1 - t^5}$	$c_{26}(t) = \frac{1 + t^{13}}{1 + t}$	$c_{27}(t) = \frac{1 - t^{27}}{1 - t^9}$
$c_{28}(t) = \frac{1 + t^{14}}{1 + t^2}$	$c_{29}(t) = \frac{1 - t^{29}}{1 - t}$	$c_{30}(t) = \frac{(1 + t^{15})(1 + t)}{(1 + t^3)(1 + t^5)}$

In Table 1 we present a list of the first 30 cyclotomic polynomials and their degrees. The following rules follow easily from the definition of cyclotomic polynomials and their properties (see [L]).

$$(2.2) \quad p \text{ prime} \Rightarrow c_p(t) = \frac{1 - t^p}{1 - t},$$

$$(2.3) \quad p = 2^n \Rightarrow c_p(t) = 1 + t^{2^{n-1}},$$

$$(2.4) \quad p = 2r, r \text{ odd} \Rightarrow c_p(t) = c_r(-t),$$

$$(2.5) \quad p = 2^n r, r \text{ odd}, n > 1 \Rightarrow c_p(t) = c_{2r}(t^{2^{n-1}}),$$

$$(2.6) \quad p = p_1 p_2, p_1, p_2 \text{ prime} \Rightarrow c_p(t) = \frac{c_{p_1}(t^{p_2})}{c_{p_1}(t)} = \frac{c_{p_2}(t^{p_1})}{c_{p_2}(t)},$$

$$(2.7) \quad p = p_1^\alpha, p_1 \text{ prime} \Rightarrow c_p(t) = c_{p_1}(t^{p_1^{\alpha-1}}) = \frac{1 - t^{p_1^\alpha}}{1 - t^{p_1^{\alpha-1}}},$$

$$(2.8) \quad c_{p_1^{\alpha_1} \dots p_k^{\alpha_k}}(t) = c_{p_1 \dots p_k}(t^{p_1^{\alpha_1-1} \dots p_k^{\alpha_k-1}}),$$

$$(2.9) \quad p \text{ prime, } p \nmid r \Rightarrow c_{pr}(t) = \frac{c_r(t^p)}{c_r(t)}.$$

LEMMA 2.2. Let  $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$  be the prime decomposition of  $n \in \mathbb{N}$ . Then

$$(2.10) \quad c_{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}(t) = c_{p_k}(t^{p_k^{\alpha_k-1}})^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{1 \leq i_1 < \cdots < i_j \leq k-1} c_{p_k}((t^{p_{i_1} \cdots p_{i_j}})^{p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1}})^{(-1)^{k-1-j}}.$$

*Proof.* Using property (2.8), it is enough to show that

$$(2.11) \quad c_{p_1 \cdots p_k}(t) = c_{p_k}(t)^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{1 \leq i_1 < \cdots < i_j \leq k-1} c_{p_k}(t^{p_{i_1} \cdots p_{i_j}})^{(-1)^{k-1-j}},$$

and we shall prove (2.11) by induction with respect to  $k \in \mathbb{N}$ . For  $k = 1$  it holds trivially. Suppose  $k = 2$ . By property (2.6) we have

$$c_{p_1 p_2}(t) = c_{p_2}(t)^{-1} c_{p_2}(t^{p_1}),$$

and so (2.11) holds for  $k = 2$ . Suppose now that (2.11) holds for some  $k \in \mathbb{N}$ ,  $k \geq 2$ , and consider  $c_{p_1 \cdots p_{k+1}}(t)$  with  $p_1 < \cdots < p_{k+1}$ . Then, applying successively (2.9) and the induction hypothesis,

$$(2.12) \quad c_{p_1 \cdots p_{k+1}}(t) = \frac{c_{p_2 \cdots p_{k+1}}(t^{p_1})}{c_{p_2 \cdots p_{k+1}}(t)} = \frac{c_{p_{k+1}}(t^{p_1})^{(-1)^{k-1}}}{c_{p_{k+1}}(t)^{(-1)^{k-1}}} \cdot \prod_{j=1}^{k-1} \prod_{2 \leq i_1 < \cdots < i_j \leq k} \left[ \frac{c_{p_{k+1}}(t^{p_{i_1} p_{i_2} \cdots p_{i_j}})}{c_{p_{k+1}}(t^{p_{i_1} \cdots p_{i_j}})} \right]^{(-1)^{k-1-j}} = c_{p_{k+1}}(t)^{(-1)^k} c_{p_{k+1}}(t^{p_1})^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{2 \leq i_1 < \cdots < i_j \leq k} c_{p_{k+1}}(t^{p_{i_1} p_{i_2} \cdots p_{i_j}})^{(-1)^{k-j-1}} \cdot c_{p_{k+1}}(t^{p_{i_1} \cdots p_{i_j}})^{(-1)^{k-j}}.$$

Now it is easy to check that (2.12) is equal to

$$c_{p_{k+1}}(t)^{(-1)^k} \prod_{j=1}^{k-1} \prod_{1 \leq i_1 < \dots < i_j \leq k} c_{p_{k+1}}(t^{p_{i_1} \dots p_{i_j}})^{(-1)^{k-j}}.$$

Hence, (2.11) holds for  $k + 1$  and the lemma is proved.  $\square$

**PROPOSITION 2.3.** *Let  $c_n(t)$  be of degree  $\varphi(n) > 2$ . Then  $c_n(t)$  can be written as*

$$(2.13) \quad c_n(t) = \prod_{i=1}^m (1 - \sigma_1(i)t^{q_i})^{\sigma_2(i)},$$

where  $m \leq \varphi(n)/2$ ,  $q_i \in \mathbb{N}$  and  $\sigma_1(i), \sigma_2(i) \in \{-1, 1\}$  for  $i = 1, \dots, m$ . Moreover,  $q = \max_i q_i$  is smaller than or equal to  $n/2$  if  $n$  is even or  $n$  if  $n$  is odd.

*Proof.* Let  $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$  be the prime decomposition of  $n$ . The case  $k = 1$  is obvious from (2.7) and (2.3). Suppose first that  $k \geq 2$  and that none of the  $p_i$  is 2. From Lemma 2.2 and (2.2),  $c_n(t)$  factorizes in the form (2.13) with

$$m = 2 \sum_{j=0}^{k-1} \binom{k-1}{j} = 2^k.$$

From (2.1)

$$\varphi(n) = \prod_{i=1}^k p_i^{\alpha_i-1} (p_i - 1) \geq 2 \cdot 4 \cdot 2^{k-2} = 2^{k+1},$$

and the result follows with  $\sigma_1(i) = +1$  for  $i = 1, \dots, m$ .

Suppose now that  $k \geq 2$  and that one of the  $p_i$  is 2. Applying Lemma 2.2 with  $p_k = 2$  and (2.3),  $c_n(t)$  factorizes in the form (2.13) with  $\sigma_1(i) = -1$  for  $i = 1, \dots, m$  and

$$m = \sum_{j=0}^{k-1} \binom{k-1}{j} = 2^{k-1}.$$

But for  $k \geq 3$

$$\varphi(n) = \prod_{i=1}^k p_i^{\alpha_i-1} (p_i - 1) \geq 2 \cdot 8 \cdot 2^{k-3} = 2^{k+1},$$

TABLE 2

$d$	$n_0(d)$	$d$	$n_0(d)$	$d$	$n_0(d)$	$d$	$n_0(d)$	$d$	$n_0(d)$
1	2	2	6	3	9	4	12	5	15
6	18	7	21	8	30	9	33	10	37
11	41	12	45	13	48	14	52	15	56
16	60	17	63	18	67	19	71	20	75
21	78	22	82	23	86	24	90	25	93
26	97	27	101	28	105	29	108	30	112
31	116	32	120	33	123	34	127	35	131
36	135	37	138	38	142	39	146	40	150
41	153	42	157	43	161	44	165	45	168
46	172	47	176	48	210	49	214	50	218
51	223	52	227	53	231	54	236	55	240
56	245	57	249	58	253	59	258	60	262
61	266	62	271	63	275	64	280	65	284
66	288	67	293	68	297	69	301	70	306
71	310	72	315	73	319	74	323	75	328
76	332	77	336	78	341	79	345	80	350
81	354	82	358	83	363	84	367	85	371
86	376	87	380	88	385	89	389	90	393
91	398	92	402	93	406	94	411	95	415
96	420	97	424	98	428	99	433	100	437

and the result follows as before. If  $k = 2$ ,  $m = 2$ . But  $\varphi(n)$  is even and hence by hypothesis  $\varphi(n) \geq 4$ . So, again,  $m \leq \varphi(n)/2$ .

From the definition of  $c_n(t)$  it is clear that  $q \leq n$ , and from (2.4) and (2.5) it follows that  $q \leq n/2$  if  $n$  is even.  $\square$

Given  $d \in \mathbb{N}$ , let  $n_0(d)$  be defined by

$$(2.14) \quad n_0(d) = \left[ d \prod_{i=1}^k \frac{p_i}{p_i - 1} \right]$$

where  $[\cdot]$  denotes the integer part function and  $k \in \mathbb{N}$  is the greatest number of consecutive primes  $p_i$  such that  $p_1 = 2$  and  $(p_1 - 1) \cdots (p_k - 1) \leq d$ . See Table 2 for the first one hundred values of  $n_0(d)$ .

**PROPOSITION 2.4.** *Given  $d \in \mathbb{N}$ , there exists  $n_0 \in \mathbb{N}$  such that for  $n > n_0$ ,  $\varphi(n) > d$ . Moreover,  $n_0(d)$  is the best possible lower bound for  $n_0$ .*

*Proof.* For  $d = 8$  we obtain  $n_0(d) = 30$  and  $\varphi(30) = 8$ . So, (2.14) gives the best possible lower bound.

Let  $d \in \mathbb{N}$  and  $n_0 \in \mathbb{N}$  be given by (2.14). Let  $n$  be greater than  $n_0$  so that

$$n > \left[ d \prod_{i=1}^k \frac{p_i}{p_i - 1} \right] \geq \prod_{i=1}^k p_i,$$

and let  $n = p_1^{\alpha_1} \cdots p_j^{\alpha_j}$  be the prime decomposition of  $n$ . In order to prove that  $\varphi(n) > d$  we shall consider two cases. Suppose first that  $j > k$ . Then, from (2.1)

$$\varphi(n) = \prod_{i=1}^j p_{n_i}^{\alpha_i - 1} (p_{n_i} - 1) \geq \prod_{i=1}^j (p_{n_i} - 1) \geq \prod_{i=1}^{k+1} (p_i - 1) > d.$$

Suppose now that  $j \leq k$ . Then

$$\begin{aligned} \varphi(n) &= \prod_{i=1}^j p_{n_i}^{\alpha_i - 1} (p_{n_i} - 1) \geq n \prod_{i=1}^j \frac{p_{n_i} - 1}{p_{n_i}} \\ &> d \prod_{i=1}^k \frac{p_i}{p_i - 1} \prod_{i=1}^j \frac{p_{n_i} - 1}{p_{n_i}} \geq d, \end{aligned}$$

because the  $p_i$  are consecutive primes and so, for  $i = 1, \dots, j$

$$\frac{p_i}{p_i - 1} \cdot \frac{p_{n_i} - 1}{p_{n_i}} \geq 1. \quad \square$$

**3. Periods forced by the Lefschetz zeta function.** Let  $M$  be a compact manifold. A map  $f: M \rightarrow M$  is called *transversal* if

- (1)  $f$  is of class  $C^1$ ,
- (2)  $f(M) \subset \text{Int}(M)$ ,
- (3) for every periodic orbit  $\gamma$  of period  $p(\gamma)$ ,  $x \in \gamma$  and  $m \in \mathbb{N}$

$$\det(Df^{mp(\gamma)}(x) - I) \neq 0.$$

Let  $f: M \rightarrow M$  be a transversal map. Denote by PO the *set of periodic orbits* of  $f$  and, given  $\gamma \in \text{PO}$ , by  $p(\gamma)$  the *minimal period* of  $\gamma$ . Following the notation introduced in [CLN] we define  $u_+(\gamma)$  (resp.  $u_-(\gamma)$ ) as the number of real eigenvalues of  $Df^{p(\gamma)}(x)$ ,  $x \in \gamma$ , which are strictly greater than one (resp. strictly smaller than one). We also define the following subsets of PO:

$$\begin{aligned} \text{EE} &= \{\gamma \in \text{PO}: u_+(\gamma) \text{ and } u_-(\gamma) \text{ are even}\}, \\ \text{EO} &= \{\gamma \in \text{PO}: u_+(\gamma) \text{ is even and } u_-(\gamma) \text{ is odd}\}, \\ \text{OE} &= \{\gamma \in \text{PO}: u_+(\gamma) \text{ is odd and } u_-(\gamma) \text{ is even}\}, \\ \text{OO} &= \{\gamma \in \text{PO}: u_+(\gamma) \text{ and } u_-(\gamma) \text{ are odd}\}. \end{aligned}$$

Also, for each odd  $r \geq 1$  and  $n \geq 0$  let  $\text{PO}(2^n r)$  be the set

$$\text{PO}(2^n r) = \{\gamma \in \text{PO}: p(\gamma) = 2^n r\},$$

and denote by  $\text{EE}_{2^n r}$  (resp.  $\text{EO}_{2^n r}$ ,  $\text{OE}_{2^n r}$ ,  $\text{OO}_{2^n r}$ ) the cardinal of  $\text{EE} \cap \text{PO}(2^n r)$  (resp.  $\text{EO} \cap \text{PO}(2^n r)$ ,  $\text{OE} \cap \text{PO}(2^n r)$ ,  $\text{OO} \cap \text{PO}(2^n r)$ ).

Denote by  $\mu: \mathbb{N} \rightarrow \{-1, 0, 1\}$  the *Möbius function* defined by  $\mu(1) = 1$  and by the following rule: if  $n = p_1^{k_1} \cdots p_j^{k_j}$  is the prime decomposition of  $n$ ,  $\mu(n) = 0$  if  $k_i > 1$  for some  $i \in \{1, \dots, j\}$ , and  $\mu(n) = (-1)^j$  otherwise. Denote by  $\{\alpha_n\}_{n \in \mathbb{N}}$  the sequence of nonnegative integers defined by

$$(3.1) \quad \alpha_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right),$$

and, for each positive odd  $r$  and each nonnegative  $m$ , let

$$(3.2) \quad \beta_{2^m r} = \sum_{k=0}^m \alpha_{2^k r}.$$

Given a continuous self-map of a compact manifold  $M$  of dimension  $n$ , its *Lefschetz number* is defined as

$$L(f) = \sum_{k=0}^n (-1)^k \text{tr}(f_{*k}),$$

where  $f_{*k}: H_k(M; \mathbb{Q}) \rightarrow H_k(M; \mathbb{Q})$  is the endomorphism induced by  $f$  on the  $k$ th rational homology group of  $M$ . The Lefschetz fixed point theorem says that if  $L(f) \neq 0$  then  $f$  has a fixed point. For the purpose of studying the set  $\text{Per}(f)$ , it is useful to consider the *Lefschetz zeta function*

$$Z_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right),$$

which is a generating function for the Lefschetz numbers of all iterates of  $f$  and can be computed from the homological endomorphisms  $f_{*k}$

of  $f$  as follows:

$$(3.3) \quad Z_f(t) = \prod_{k=0}^n \det(I_{j_k} - t f_{*k})^{(-1)^{k+1}},$$

where  $j_k = \dim_{\mathbb{Q}} H_k(M; \mathbb{Q})$ , see [F2].

If  $f: M \rightarrow M$  is transversal, then the Lefschetz numbers of the iterates of  $f$  are related in a simple way to the periodic points of  $f$ , see [F1] for more details.

With this notation, we have the following theorem, which will be used in subsequent proofs.

**THEOREM 3.1 ([CLN]).** *Let  $f: M \rightarrow M$  be a transversal map such that its Lefschetz zeta function is of the form*

$$(3.4) \quad Z_f(t) = \prod_{p \in P} \prod_{i=1}^{N_p} (1 - \sigma_1(i, p) t^p)^{\sigma_2(i, p)},$$

where  $P$  is a finite subset of  $\mathbb{N}$ , the  $N_p$  are natural numbers and  $\sigma_1(i, p), \sigma_2(i, p) \in \{-1, 1\}$ . Then, for each odd integer  $r \geq 1$  and  $m = 0, 1, 2, \dots$  we have

$$(3.5) \quad \sum_{k=0}^m \text{EE}_{2^k r} + \text{EO}_{2^m r} + \gamma(r, m) = \text{OO}_{2^m r} + \sum_{k=0}^m \text{OE}_{2^k r},$$

where

$$(3.6) \quad \begin{aligned} \gamma(r, m) = & \sum_{\substack{q|r \\ 2^m q \in P}} \sum_{i=1}^{N_{2^m q}} \sigma_2(i, 2^m q) \sigma_1(i, 2^m q) \beta_{r/q} \\ & + \sum_{j=0}^{m-1} \sum_{\substack{q|r \\ 2^j q \in P}} \sum_{i=1}^{N_{2^j q}} \sigma_2(i, 2^j q) \left( \beta_{2^{m-j} r/q} + \frac{\sigma_2(i, 2^j q) - 1}{2} \beta_{r/q} \right), \end{aligned}$$

and we take the second summand equal zero when  $m = 0$ .

Consider a product of the form

$$(3.7) \quad \prod_{p \in P} \prod_{i=1}^{N_p} (1 - \sigma_1(i, p) t^p)^{\sigma_2(i, p)},$$

where  $P$  is a finite subset of  $\mathbb{N}$ ,  $N_p$  is a natural number for each  $p \in P$ , and  $\sigma_1(i, p), \sigma_2(i, p) \in \{-1, 1\}$ . A factor in (3.7) of the form  $(1 - \sigma_1(i, p)t^p)^{\sigma_2(i, p)}$  will be called *irreducible* if it remains after performing the following reductions:

$$\begin{aligned}
 \text{(R1)} \quad & (1 + t^p)(1 + t^p)^{-1} = 1, \\
 \text{(R2)} \quad & (1 - t^p)(1 - t^p)^{-1} = 1, \\
 \text{(R3)} \quad & (1 + t^p)(1 - t^p) = 1 - t^{2p}, \\
 \text{(R4)} \quad & (1 + t^p)^{-1}(1 - t^p)^{-1} = (1 - t^{2p})^{-1}, \\
 \text{(R5)} \quad & (1 + t^p)^{-1}(1 - t^{2p}) = 1 - t^p, \\
 \text{(R6)} \quad & (1 + t^p)(1 - t^{2p})^{-1} = (1 - t^p)^{-1}, \\
 \text{(R7)} \quad & (1 + t^p)(1 + t^{2p}) = (1 - t^p)^{-1}(1 - t^{4p}), \\
 \text{(R8)} \quad & (1 + t^p)^{-1}(1 + t^{2p})^{-1} = (1 - t^p)(1 - t^{4p})^{-1}.
 \end{aligned}$$

We remark that the reductions (R1) to (R8) should be performed in correlative order and consequently the whole process of reduction is finite.

**THEOREM 3.2.** *Let  $f: M \rightarrow M$  be a transversal map of a compact manifold. Suppose that all the zeros and poles of its Lefschetz zeta function  $Z_f(t)$  are roots of unity, and that  $Z_f(t)$  has an irreducible factor of the form  $(1 \pm t^n)^{\pm 1}$ .*

- (a) *If  $n$  is odd then  $n \in \text{Per}(f)$ .*
- (b) *If  $n$  is even then  $\{\frac{n}{2}, n\} \cap \text{Per}(f) \neq \emptyset$ .*

*Proof.* From (3.3) and Proposition 2.1, the Lefschetz zeta function of  $f$  is of the form

$$Z_f(t) = \frac{\prod_{i=1}^N c_{n_i}(t)}{\prod_{i=1}^Q c_{q_i}(t)}.$$

Moreover, applying Proposition 2.3 to each  $c_{n_i}(t)$ ,  $c_{q_i}(t)$ ,  $f$  is in the hypothesis of Theorem 3.1 and so, given  $r$  odd, equation (3.5) holds for  $m = 0$ . Now, if we have an irreducible factor of the form  $(1 - \sigma_1 t^r)^{\sigma_2}$ , then all the other possible irreducible factors associated to the same power of  $t$  are  $(1 - \sigma_1 t^r)^{\sigma_2}$  or  $(1 + \sigma_1 t^r)^{-\sigma_2}$  (see reduction rules (R1) to (R4)). Hence,

$$\left| \sum_{i=1}^{N_r} \sigma_2(i, r) \sigma_1(i, r) \right| \neq 0$$

because all the terms in the above sum are equal. So,  $r \in \text{Per}(f)$  and statement (a) is proved.

Now we shall prove statement (b). From the definition (3.6) of  $\gamma(r, m)$  for  $m = 1, 2, \dots$  we have

$$\begin{aligned} \gamma(r, m) - \gamma(r, m-1) &= \sum_{i=1}^{N_{2^m r}} \sigma_2(i, 2^m r) \sigma_1(i, 2^m r) \\ &\quad + \sum_{i=1}^{N_{2^{m-1} r}} \sigma_2(i, 2^{m-1} r) \left( \frac{1 - \sigma_1(i, 2^{m-1} r)}{2} \right), \end{aligned}$$

and subtracting (3.5) for  $m$  and  $m-1$  we obtain

$$\begin{aligned} (3.8) \quad & EE_{2^m r} + EO_{2^m r} - EO_{2^{m-1} r} + \gamma(r, m) - \gamma(r, m-1) \\ &= OE_{2^m r} + OO_{2^m r} - OO_{2^{m-1} r}. \end{aligned}$$

Suppose  $n = 2^m r$ ,  $m \geq 1, r$  odd. Consider an irreducible factor of the form  $1 - t^{2^m r}$ . The proof for the other possible irreducible factors  $(1 - t^{2^m r})^{-1}$ ,  $1 + t^{2^m r}$  and  $(1 + t^{2^m r})^{-1}$  is analogous. For this factor we have  $\sigma_1(\cdot, 2^m r) \sigma_2(\cdot, 2^m r) = 1$  and so all the factors of the form

$$(1 - \sigma_1(i, 2^m r) t^{2^m r}) \sigma_2(i, 2^m r)$$

that persist after reductions of the type (R1)–(R4) verify  $\sigma_1(i, 2^m r) \cdot \sigma_2(i, 2^m r) = 1$ . Moreover, the factors of the form

$$(1 - \sigma_1(i, 2^{m-1} r) t^{2^{m-1} r}) \sigma_2(i, 2^{m-1} r)$$

that can coexist with  $1 - t^{2^m r}$  satisfy

$$\sigma_2(i, 2^{m-1} r) \left( \frac{1 - \sigma_1(i, 2^{m-1} r)}{2} \right) \in \{0, 1\}$$

due to the reduction (R5). Then  $|\gamma(r, m) - \gamma(r, m-1)| > 0$  and equation (3.8) implies that if  $2^{m-1} r \notin \text{Per}(f)$  then  $2^m r \in \text{Per}(f)$ .

We remark that the proof for the irreducible factors  $(1 - t^{2^m r})^{-1}$ ,  $1 + t^{2^m r}$  and  $(1 + t^{2^m r})^{-1}$  uses the reductions (R6), (R7) and (R8), respectively, and of course also (R1) to (R4).

Let  $P(t)/Q(t)$  be a rational function. We define the *order* of  $P(t)/Q(t)$ , denoted by  $\text{order}(P(t)/Q(t))$ , as the number

$$\left\lceil \frac{\text{degree } P(t) + \text{degree } Q(t)}{2} \right\rceil + 1.$$

**COROLLARY 3.3.** *Let  $f: M \rightarrow M$  be a transversal map of a compact manifold. Suppose that all the zeros and poles of  $Z_f(t)$  are roots of unity. Then*

$$\text{Card}(\text{FSP}(f)) \leq \text{order}(Z_f(t)).$$

*Proof.* Since the Lefschetz zeta function of  $f$ ,  $Z_f(t)$ , is rational, we may write  $Z_f(t) = P(t)/Q(t)$  with  $P(t)$  and  $Q(t)$  polynomials.

By Proposition 2.1,  $P(t)$  and  $Q(t)$  factorize as the product of cyclotomic polynomials. We shall split  $P(t)$  (resp.  $Q(t)$ ) as a product  $P(t) = P_1(t)P_2(t)$  (resp.  $Q(t) = Q_1(t)Q_2(t)$ ), where  $P_1(t)$  (resp.  $Q_1(t)$ ) factorizes as a product of cyclotomic polynomials of degree strictly greater than 2 and  $c_4(t)$ , and  $P_2(t)$  (resp.  $Q_2(t)$ ) contains the remaining factors, i.e. all the  $c_i(t)$  for  $i \in \{1, 2, 3, 6\}$ . Since the reduction process given by rules (R1)–(R8) does not increase the number of factors of the form  $(1 \pm t^n)^{\pm 1}$ , from Proposition 2.3 and Theorem 3.2 it follows that the cardinal of the forced set of periods associated to the factors  $P_1(t)$  and  $Q_1(t)$  is smaller than or equal to  $(\text{degree } P_1(t) + \text{degree } Q_1(t))/2$ . Notice that this upper bound is an integer number.

To conclude the proof, it is enough to show that the contribution of the remaining factors ( $P_2(t)$  and  $Q_2(t)$ ) to the cardinal of the forced set of periods is smaller than or equal to

$$C = \left\lceil \frac{\text{degree } P_2(t) + \text{degree } Q_2(t)}{2} \right\rceil + 1.$$

Notice that  $P_2(t)/Q_2(t)$  is a rational function of the form  $c_1(t)^{a_1} \cdot c_2(t)^{a_2} c_3(t)^{a_3} c_6(t)^{a_6}$ , where the  $a_i \in \mathbb{Z}$ . The forced set of periods  $F$  associated to a product of this form is contained in  $\{1, 3\}$ . To prove that  $\text{Card}(F) \leq C$  it is enough to consider the case  $\text{Card}(F) = 2$  because  $C \geq 1$ . But if  $3 \in F$  then  $|a_3| + |a_6| \neq 0$ , and hence  $C \geq 2$  (see Table 1).  $\square$

Let  $n \in \mathbb{N}$ . We denote by  $S(n)$  the set  $\{1, 2, \dots, n\}$ .

**COROLLARY 3.4.** *Assume that we are in the hypotheses of Corollary 3.3. Let  $d$  be the maximum of the degrees of  $P(t)$  and  $Q(t)$ , where  $P(t)/Q(t) = Z_f(t)$ . Then*

$$(a) \quad \text{FSP}(f) \cap \{n \in \mathbb{N}: n \text{ is odd}\} \subset \begin{cases} S(n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(n_0(d) - 1) & \text{if } n_0(d) \text{ is even.} \end{cases}$$

$$(b) \quad \text{FSP}(f) \cap \{n \in \mathbb{N} : n \text{ is even}\} \\ \subset \begin{cases} S(2n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(2n_0(d) - 2) & \text{if } n_0(d) \text{ is even.} \end{cases}$$

*Proof.* By Proposition 2.1,  $P(t)$  and  $Q(t)$  factorize as products of cyclotomic polynomials. From Proposition 2.4 it follows that if the cyclotomic polynomial  $c_n(t)$  appears in the factorization of  $P(t)$  or  $Q(t)$ , then  $n \leq n_0(d)$ . Now, the maximum power of  $t$  in the decomposition (2.13) of  $c_n(t)$  is smaller than or equal to  $n/2$  (resp.  $n$ ) if  $n$  is even (resp. odd). Consequently, before applying the reduction procedure given by rules (R1) to (R8), the maximum power of  $t$  which appears in the factors  $(1 \pm t^j)^{\pm 1}$  of  $P(t)/Q(t)$  is  $n_0(d)$  (resp.  $n_0(d) - 1$ ) if  $n$  is odd (resp. even). Hence, taking into account the reduction rules, the corollary follows.  $\square$

**4. Transversal surface maps.** Throughout this section,  $M$  will be a compact connected surface of genus  $g$  and  $f: M \rightarrow M$  a transversal map.

Recall that  $H_0(M; \mathbb{Q}) \approx \mathbb{Q}$  and that  $H_1(M; \mathbb{Q}) \approx \mathbb{Q}^{2g}$ ,  $H_2(M; \mathbb{Q}) \approx \mathbb{Q}$  if  $M$  is orientable, and  $H_1(M; \mathbb{Q}) \approx \mathbb{Q}^{g-1}$ ,  $H_2(M; \mathbb{Q}) \approx \{0\}$  if  $M$  is non-orientable.

**PROPOSITION 4.1.** *If  $h(f) = 0$  then all the eigenvalues of  $f_{*0}$ ,  $f_{*1}$  and  $f_{*2}$  are either 0 or roots of unity.*

*Proof.* Since  $f_{*0} = \text{id}$ , 1 is the only eigenvalue of  $f_{*0}$ .

Let us consider now  $f_{*2}$ . If  $M$  is non-orientable, then 0 is the only eigenvalue of  $f_{*2}$ . If  $M$  is orientable,  $f_{*2}(1)$  is the degree  $D$  of  $f$ . From [MP] we know that if  $|D| > 1$  then  $h(f) \geq \log |D|$ . Hence, if  $h(f) = 0$ ,  $|D| \leq 1$  and so the only possible eigenvalues for  $f_{*2}$  are  $-1$ , 0 and 1.

Finally, consider  $f_{*1}$ . By Theorem 2 of [Mn], if  $h(f) = 0$  then all the eigenvalues  $\lambda$  of  $f_{*1}$  satisfy  $|\lambda| \leq 1$ . We claim that every nonzero eigenvalue  $\lambda$  of  $f_{*1}$  has modulus 1. Let  $\lambda_1, \dots, \lambda_k$  be the nonzero eigenvalues of  $f_{*1}$ . Then

$$(4.1) \quad \det(I - tf_{*1}) = \det(-t(f_{*1} - t^{-1}I)) = (-1)^k \prod_{i=1}^k (\lambda_i t - 1).$$

Moreover (4.1) must be a polynomial with integer coefficients, because  $f_{*1}$  is an integral matrix. Hence, in particular,  $\prod_{i=1}^k \lambda_i$  must

belong to  $\mathbb{Z}$ . Therefore,  $\prod_{i=1}^k |\lambda_i| \geq 1$ , and the claim follows because  $|\lambda_i| \leq 1$ .

So (4.1) is a polynomial with integer coefficients, constant term 1 and all its roots have modulus 1. By a standard result in algebra (see Lemma 1.6 of [W]) the proposition follows.  $\square$

**COROLLARY 4.2.** *If  $h(f) = 0$  or  $\text{Per}(f)$  is finite, then  $Z_f(t)$  is of the form*

$$Z_f(t) = \frac{\prod_{i=1}^N c_{n_i}(t)}{(1-t)\theta(t)},$$

where  $\theta(t)$  is either 1,  $1-t$  or  $1+t$  and  $c_{n_i}(t)$  is the  $n_i$ -cyclotomic polynomial.

*Proof.* The case  $h(f) = 0$  follows directly from the definition of the Lefschetz zeta function, Proposition 4.1 and Proposition 2.1. If  $\text{Per}(f)$  is finite, the corollary follows from Theorem 6 of [Fr].  $\square$

The following results improve slightly the statements of Corollaries 3.3 and 3.4 for transversal surface maps.

**COROLLARY 4.3.** *If  $h(f) = 0$  or  $\text{Per}(f)$  is finite, then*

$$\text{Card}(\text{FSP}(f)) \leq \begin{cases} g+1 & \text{if } M \text{ is orientable,} \\ \left\lfloor \frac{g-1}{2} \right\rfloor + 1 & \text{if } M \text{ is non-orientable.} \end{cases}$$

*Proof.* From Corollary 4.2, repeating the arguments of the proof of Corollary 3.3 and taking into account that the contribution of  $Q(t) = Q_2(t)$  to  $\text{FSP}(f)$  is at most period 1, the result follows.  $\square$

The following result is just a restatement of Corollary 3.4.

**COROLLARY 4.4.** *If  $h(f) = 0$  or  $\text{Per}(f)$  is finite, then*

- (a)  $\text{FSP}(f) \cap \{n \in \mathbb{N} : n \text{ is odd}\}$   
 $\subset \begin{cases} S(n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(n_0(d) - 1) & \text{if } n_0(d) \text{ is even.} \end{cases}$
- (b)  $\text{FSP}(f) \cap \{n \in \mathbb{N} : n \text{ is even}\}$   
 $\subset \begin{cases} S(2n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(2n_0(d) - 2) & \text{if } n_0(d) \text{ is even,} \end{cases}$

where  $d = 2g$  if  $M$  is orientable and  $d = g-1$  if  $M$  is non-orientable.

**5. Transversal  $N$ -torus-maps.** In this section we shall derive consequences of the results obtained in Section 3 for transversal self-maps

$f$  on  $n$ -dimensional manifolds such that for  $k = 2, \dots, n$  all the eigenvalues of  $f_{*k}$  can be obtained as products of the eigenvalues of  $f_{*1}$ . We remark that these conditions hold for continuous maps of the  $n$ -torus  $T^n$  and also for continuous maps of many Eilenberg-Mac Lane spaces.

The following corollary is easily obtained from Theorem 3.2 and Corollaries 3.3 and 3.4 repeating the arguments of the proof of Proposition 4.1.

**COROLLARY 5.1.** *Let  $f: M \rightarrow M$  be a transversal map on a compact manifold  $M$  of dimension  $m$  with  $h(f) = 0$ . Suppose that for  $k = 2, \dots, m$  all the eigenvalues of  $f_{*k}$  can be obtained as products of the eigenvalues of  $f_{*1}$ .*

(a) *If  $Z_f(t)$  has an irreducible factor of the form  $(1 \pm t^n)^{\pm 1}$  with  $n$  odd, then  $n \in \text{Per}(f)$ .*

(b) *If  $Z_f(t)$  has an irreducible factor of the form  $(1 \pm t^n)^{\pm 1}$  with  $n$  even, then  $\{n/2, n\} \cap \text{Per}(f) \neq \emptyset$ .*

(c)  $\text{Card}(\text{FSP}(f)) \leq \text{order}(Z_f(t))$ .

(d)

$$\begin{aligned} & \text{FSP}(f) \cap \{n \in \mathbb{N}: n \text{ is odd}\} \\ & \subset \begin{cases} S(n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(n_0(d) - 1) & \text{if } n_0(d) \text{ is even.} \end{cases} \end{aligned}$$

(e)

$$\begin{aligned} & \text{FSP}(f) \cap \{n \in \mathbb{N}: n \text{ is even}\} \\ & \subset \begin{cases} S(2n_0(d)) & \text{if } n_0(d) \text{ is odd,} \\ S(2n_0(d) - 2) & \text{if } n_0(d) \text{ is even.} \end{cases} \end{aligned}$$

Similarly, using Theorem 3.2, Corollaries 3.3 and 3.4 and Theorem 4.1 of [ABLSS] we have the following result.

**COROLLARY 5.2.** *Let  $f: T^m \rightarrow T^m$  be a transversal map and suppose that  $\text{Per}(f)$  is finite. Then, (a) to (e) of Corollary 5.1 hold.*

#### REFERENCES

- [ABLSS] Li. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, *Minimal sets of periods for torus maps via Nielsen numbers*, preprint of the Centre de Recerca Matemàtica (1991).
- [CLN] J. Casasayas, J. Llibre and A. Nunes, *Periodic orbits for transversal maps*, preprint of the Centre de Recerca Matemàtica (1991).

- [F1] J. Franks, *Some smooth maps with infinitely many hyperbolic periodic points*, Trans. Amer. Math. Soc., **226** (1977), 175–179.
- [F2] J. M. Franks, *Homology and Dynamical Systems*, CBMS Regional Conf. Series, Vol. 49, Amer. Math. Soc., 1982.
- [F3] J. Franks, *Period doubling and the Lefschetz formula*, Trans. Amer. Math. Soc., **287** (1985), 275–283.
- [Fr] D. Fried, *Periodic Points and Twisted Coefficients*, in Lecture Notes in Math., vol. 1007, Springer-Verlag (1983), 175–179.
- [L] S. Lang, *Algebra*, Addison-Wesley, 1971.
- [Mn] A. Manning, *Topological Entropy and the First Homology Group*, Lecture Notes in Math., vol. 468, Springer-Verlag (1975), 185–199.
- [MP] M. Misiurewicz and F. Przytycki, *Topological entropy and degree of smooth mappings*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Math., Astr. et Phys., **XXV** (1977), 573–578.
- [Mt] T. Matsuoka, *The number of periodic points of smooth maps*, Ergodic Theory & Dynamical Systems, **9** (1989), 153–163.
- [W] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer-Verlag, 1982.

Received December 16, 1991.

DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI  
 UNIVERSITAT DE BARCELONA  
 GRAN VIA 585, 08071 BARCELONA, SPAIN

DEPARTAMENT DE MATEMÀTIQUES  
 UNIVERSITAT AUTÒNOMA DE BARCELONA  
 BELLATERRA, 08193 BARCELONA, SPAIN

AND

CENTRE DE RECERCA MATEMÀTICA  
 UNIVERSITAT AUTÒNOMA DE BARCELONA  
 BELLATERRA, 08193 BARCELONA, SPAIN  
 ON LEAVE FROM DEPARTAMENTO DE FÍSICA, UNIVERSIDADE DE LISBOA  
 CAMPO GRANDE, ED C1, PISO 4, 1700 LISBOA, PORTUGAL

# PACIFIC JOURNAL OF MATHEMATICS

Founded by

E. F. BECKENBACH (1906–1982) F. WOLF (1904–1989)

## EDITORS

SUN-YUNG A. CHANG  
(Managing Editor)  
University of California  
Los Angeles, CA 90024-1555  
chang@math.ucla.edu

F. MICHAEL CHRIST  
University of California  
Los Angeles, CA 90024-1555  
christ@math.ucla.edu

HERBERT CLEMENS  
University of Utah  
Salt Lake City, UT 84112  
clemens@math.utah.edu

THOMAS ENRIGHT  
University of California, San Diego  
La Jolla, CA 92093  
tenright@ucsd.edu

NICHOLAS ERCOLANI  
University of Arizona  
Tucson, AZ 85721  
ercolani@math.arizona.edu

R. FINN  
Stanford University  
Stanford, CA 94305  
finn@gauss.stanford.edu

VAUGHAN F. R. JONES  
University of California  
Berkeley, CA 94720  
vfr@math.berkeley.edu

STEVEN KERCKHOFF  
Stanford University  
Stanford, CA 94305  
spk@gauss.stanford.edu

MARTIN SCHARLEMANN  
University of California  
Santa Barbara, CA 93106  
mgscharl@math.ucsb.edu

HAROLD STARK  
University of California, San Diego  
La Jolla, CA 92093

V. S. VARADARAJAN  
University of California  
Los Angeles, CA 90024-1555  
vsv@math.ucla.edu

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA  
UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
UNIVERSITY OF MONTANA  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Julie Honig, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 75 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$215.00 a year (10 issues). Special rate: \$108.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

---

The Pacific Journal of Mathematics at University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,  
Berkeley, CA 94720, A NON-PROFIT CORPORATION

This publication was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\text{T}\mathcal{E}\mathcal{X}$ ,  
the American Mathematical Society's  $\text{T}\mathcal{E}\mathcal{X}$  macro system.

Copyright © 1994 by Pacific Journal of Mathematics

# PACIFIC JOURNAL OF MATHEMATICS

Volume 165 No. 1 September 1994

---

The effect of dimension on certain geometric problems of irregularities of distribution	1
RALPH ALEXANDER	
The structure of $\mathfrak{sl}(2, 1)$ -supersymmetry: irreducible representations and primitive ideals	17
DIDIER ARNAL, HÉDI BENAMOR and GEORGES PINCZON	
Periods and Lefschetz zeta functions	51
JOSEFINA CASASAYAS, JAUME LLIBRE and ANA NUNES	
On closed hypersurfaces of constant scalar curvatures and mean curvatures in $S^{n+1}$	67
SHAOPING CHANG	
$R$ -groups and elliptic representations for $SL_n$	77
DAVID GOLDBERG	
The boundary distortion of a quasiconformal mapping	93
JUHA HEINONEN and PEKKA KOSKELA	
Strongly approximately transitive group actions, the Choquet-Deny theorem, and polynomial growth	115
WOJCIECH JAWORSKI	
$q$ -canonical commutation relations and stability of the Cuntz algebra	131
PALLE E. T. JORGENSEN, L. M. SCHMITT and REINHARD FRANK WERNER	
Complete open manifolds of non-negative radial curvature	153
YOSHIROH MACHIGASHIRA	
Perturbations of certain reflexive algebras	161
DAVID RYDER PITTS	
Interpolation submanifolds of the unitary group	181
YEREN XU	



0030-8730(1994)165:1;1-H