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ON CLOSED HYPERSURFACES OF CONSTANT SCALAR  
CURVATURES AND MEAN CURVATURES IN  $S^{n+1}$

SHAOPING CHANG

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ON CLOSED HYPERSURFACES  
OF CONSTANT SCALAR CURVATURES  
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We consider in this note the following question: given a closed Riemann  $n$ -manifold of constant scalar curvature, how can it be minimally immersed in the round  $(n+1)$ -sphere? Our main result states that the immersion has to be isoparametric if the number of its distinct principal curvatures is three identically. This provides another piece of supporting evidence to a conjecture of Chern.

**0. Introduction.** Consider  $\mathcal{M}_{\text{closed}}^n$  the set of all the closed minimal hypersurfaces of constant scalar curvatures  $R$  in the unit round  $(n+1)$ -sphere  $S^{n+1}$ . Let  $\mathcal{R}_n \subset \mathbf{R}$  be the collection of all the possible values of such  $R$ 's. Chern [12] posed the following:

*Chern Conjecture.* For any  $n \geq 3$ ,  $\mathcal{R}_n$  is a discrete subset of the real numbers.

This is a very interesting conjecture in the theory of minimal submanifolds in spheres. To attack this problem, it will be most helpful if one has a good guess on what  $\mathcal{M}_{\text{closed}}^n$  is for each  $n$ . When  $n = 3$ , from his work on the exterior differential systems R. Bryant [1] proposed the following:

*Bryant Conjecture.* A piece of minimal hypersurface of constant scalar curvature in  $S^4$  is isoparametric of type  $g \leq 3$ .

Here a hypersurface (not necessarily compact)  $M^n$  in  $S^{n+1}$  is said to be *isoparametric of type  $g$*  if it has constant principal curvatures  $\lambda_1 < \dots < \lambda_g$  with respective constant multiplicities  $m_1, \dots, m_g$ . Such hypersurfaces with  $g \leq 3$  are classified due to Cartan's work [2] in 1939.

Note that the Bryant conjecture is very strong because  $M^3$  is not assumed to be closed. Nevertheless, there is good evidence that it may be true. In [3], together with the works of Simons [11] and Peng-Terng [10], the author was able to establish the Chern Conjecture when  $n = 3$  by showing that each  $M^3 \in \mathcal{M}_{\text{closed}}^3$  is an isoparametric hypersurface. Hence,  $\mathcal{R}_3 = \{0, 3, 6\}$ . Also, the Bryant Conjecture was verified

when  $M^3$  has multiple principal curvatures somewhere.

Therefore, we would like to pursue such a point of view for the study of  $\mathcal{M}_{\text{closed}}^n$  in higher dimensions. Suppose that  $M^n$  also satisfies the following:

*Condition (g)*: The number  $g$  of distinct principal curvatures is constant.

Recall that there is one minimal hypersurface among each family of isoparametric hypersurfaces (cf. [9]). All the closed minimal isoparametric hypersurfaces by definition are members of  $\mathcal{M}_{\text{closed}}^n$  and satisfy Condition (g). Conversely, it is straightforward to check that any  $M^n \in \mathcal{M}_{\text{closed}}^n$  satisfying Condition (g) with  $g \leq 2$  has constant principal curvatures and thus is isoparametric. When  $g = 3$ , as a consequence of the main result of the present paper, one has the following:

**THEOREM.** *If  $M^n \in \mathcal{M}_{\text{closed}}^n$  satisfies Condition (g) with  $g = 3$ , then  $M^n$  is either an equator  $S^n$ , a product of spheres  $S^p \times S^q$  or a Cartan minimal hypersurface.*

**REMARK.** The Bryant conjecture will be established if one can exhibit such a theorem without assuming  $M^n$  to be compact.

We now state the following:

**MAIN THEOREM.** *A closed hypersurface  $M^n$  of constant scalar curvature  $R$  and constant mean curvature  $H$  in  $S^{n+1}$  is isoparametric provided it has 3 distinct principal curvatures everywhere.*

**REMARK.** When the principal curvatures are all non-simple, R. Miyaoka [7] exhibited that  $M^n$  is isoparametric even without assuming the scalar curvature is constant.

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**1. Notations and the reduction of the proof.** Throughout the paper, we use  $A, B, C, \dots$ , for indices ranging from 1 to  $n$  and denote by  $\delta_{AB}$  the Kronecker symbols.

For each point  $x \in M^n$ , let  $\lambda(x)$ ,  $\mu(x)$  and  $\sigma(x)$  be the three distinct principal curvatures of multiplicities  $p(x)$ ,  $q(x)$  and  $r(x)$ , respectively, at  $x$ .

In order to establish the Main Theorem, we need to show that all the three continuous functions  $\lambda$ ,  $\mu$  and  $\sigma$  on  $M^n$  are indeed constant functions.

We first observe that all the three integer-valued functions  $p$ ,  $q$  and  $r$  are constant integers.

Indeed, consider the following system of linear equations with  $p$ ,  $q$  and  $r$  as unknowns:

$$\begin{aligned}
 (*) \quad & p + q + r = n, \\
 & p\lambda + q\mu + r\sigma = H, \\
 & p\lambda^2 + q\mu^2 + r\sigma^2 = S,
 \end{aligned}$$

where  $S$  is the square length of the second fundamental form.

Since  $\lambda$ ,  $\mu$  and  $\sigma$  are distinct everywhere, we can solve for  $p$ ,  $q$  and  $r$  in terms of  $\lambda$ ,  $\mu$ ,  $\sigma$  and  $S$ , which are all continuous on  $M^n$ .

This shows that  $p$ ,  $q$  and  $r$  are constant as desired since they need to be integers.

**REMARK.** By the same argument, one can see that for  $\forall g$ , Condition (g) always yields the constancy of the multiplicities.

Therefore, we can choose a local frame  $\{e_i, e_\alpha, e_a\}$  where the indices  $i$ ,  $\alpha$  and  $a$  range from 1 to  $p$ ,  $p + 1$  to  $p + q$  and  $p + q + 1$  to  $p + q + r (= n)$ , respectively, such that the second fundamental form  $h = \sum_{A,B} h_{AB} \omega_A \omega_B$  is given by

$$(h) \quad (h_{AB}) = \begin{pmatrix} \lambda I_p & & \\ & \mu I_q & \\ & & \sigma I_r \end{pmatrix}$$

where for each integer  $s$ , we denote by  $I_s$  the identity matrix of rank  $s$ , and  $\{\omega_i, \omega_\alpha, \omega_a\}$  is the dual co-frame of  $\{e_i, e_\alpha, e_a\}$ .

Recall that the structure equations of  $M^n$  are given by the following:

$$\begin{aligned}
 d\omega_A &= \sum_B \omega_{AB} \wedge \omega_B, \\
 d\omega_{AB} &= \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} R_{ABCD} \omega_C \wedge \omega_D,
 \end{aligned}$$

where  $\omega_{AB}$ 's denote the connection forms of  $M^n$  and  $R_{ABCD}$  the curvature tensor.

Define  $\nabla h = \sum_{A,B,C} h_{ABC} \omega_A \omega_B \omega_C$  the covariant derivative of  $h$  by

$$(\nabla h) \quad \sum_C h_{ABC} \omega_C = dh_{AB} + \sum_C h_{CB} \omega_{CA} + \sum_C h_{AC} \omega_{CB}.$$

Then, by virtue of (h),  $(\nabla h)$  can be interpreted as

$$(1.1) \quad \sum_C h_{ijC} \omega_C = \delta_{ij} d\lambda,$$

$$(1.2) \quad \sum_C h_{\alpha\beta C} \omega_C = \delta_{\alpha\beta} d\mu,$$

$$(1.3) \quad \sum_C h_{abC} \omega_C = \delta_{ab} d\sigma,$$

$$(2.1) \quad \sum_C h_{i\alpha C} \omega_C = (\lambda - \mu) \omega_{i\alpha},$$

$$(2.2) \quad \sum_C h_{inC} \omega_C = (\lambda - \sigma) \omega_{in},$$

$$(2.3) \quad \sum_C h_{\alpha\alpha C} \omega_C = (\mu - \sigma) \omega_{\alpha\alpha}.$$

Recall that  $h_{ABC}$  is symmetric in all the indices since the ambient space  $S^{n+1}$  is of constant curvature and (cf. [4])

$$(S) \quad \sum_{A,B,C} h_{ABC}^2 = S(S-n) + H^2 - Hf$$

where  $f = \sum_{A,B,C} h_{AB} h_{BC} h_{CA}$ .

Note that  $S = n(n-1) + H^2 - R$  (cf. [4]) is constant and all the principal curvatures  $\lambda$ ,  $\mu$  and  $\sigma$  are smooth functions on  $M^n$ .

By differentiating both (\*) and  $f = p\lambda^3 + q\mu^3 + r\sigma^3$ , we have

$$\begin{aligned} pd\lambda + qd\mu + rd\sigma &= 0, \\ p\lambda d\lambda + q\mu d\mu + r\sigma d\sigma &= 0, \\ p\lambda^2 d\lambda + q\mu^2 d\mu + r\sigma^2 d\sigma &= \frac{1}{3} df. \end{aligned}$$

It follows that

$$(\#) \quad \frac{pd\lambda}{\sigma - \mu} = \frac{qd\mu}{\lambda - \sigma} = \frac{rd\sigma}{\mu - \lambda} = \frac{df}{3D},$$

where  $D = (\sigma - \mu)(\sigma - \lambda)(\mu - \lambda)$ .

In the case when all the principal curvatures are non-simple, from the Miyaoka theorem [7], we immediately assert that  $M^n$  is isoparametric.

And the case when  $p = q = r = 1$  was already verified by the author in [4]. It therefore suffices to show that all the principal curvatures are simple if so is one of them, say,  $r = 1$ .

To this aim, we need the following:

**KEY LEMMA.** *With the same notations as above. If  $r = 1$  and  $pq \geq 2$ , then  $h_{i\alpha n} = 0, \forall i, \alpha$ .*

The proof of this lemma itself will be given in §2. We will finish the current section by showing how to achieve our aim from the Key Lemma.

Consider a point  $x_0 \in M^n$  where  $df = 0$ , from (#) we have

$$d\lambda = d\mu = d\sigma = 0,$$

i.e.  $h_{ijA} = h_{\alpha\beta A} = h_{abA} = 0, \quad \forall i, j, \alpha, \beta, a, b, A.$

Now suppose otherwise that  $r = 1$  and  $pq \geq 2$ .

From the Key Lemma, the left-hand side of (S) would vanish at  $x_0$  and then

$$S(S - n) + H^2 - Hf(x_0) = 0.$$

When  $H \neq 0$ , since  $df = 0$  at both maximum and minimum points of  $f$ , it would follow that  $f = \frac{1}{H}(S(S - n) + H^2)$  identically. From (#), this in turn would yield that  $\lambda, \mu$  and  $\sigma$  were constant and then  $M^n$  be isoparametric, contradicting the classification by Cartan.

When  $H = 0$ , it would follow that  $S(S - n) = 0$  and then  $M^n$  be either an equator or a product of spheres, due to Chern-do Carmo-Kobayashi and Lawson [5, 6], contradicting the assumption that  $g = 3$ .

**2. Proof of the Key Lemma.** At each point  $x \in M^n$ , denote by  $Y$  the  $p \times q$  matrix  $(h_{i\alpha n}) \in M_{p \times q}$ . We are supposed to show that  $Y = 0$  everywhere if  $r = 1$  and  $pq \geq 2$ .

We will employ the following [8]:

**THEOREM [Otsuki, 1970].** *Let  $M^n$  be a hypersurface immersed in an  $(n + 1)$ -dimensional Riemannian manifold of constant curvature such that the multiplicities of principal curvatures are all constant.*

*Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. Moreover, if the*

*multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

Now, without loss of generality, assume that  $q \geq 2$ .

Applying the Otsuki theorem to  $\mu$  and noting that  $d\lambda = \frac{\mu-\sigma}{\sigma-\lambda} d\mu$  and  $d\sigma = \frac{\mu-\lambda}{\lambda-\sigma} d\mu$  from (#), we have

$$\lambda_\alpha = \mu_\alpha = \sigma_\alpha = 0, \quad \forall \alpha.$$

*Case 1.  $p = 1$ .*

Rewrite (1.1)  $\rightarrow$  (2.3) as

$$(I.1) \quad h_{11C} = \lambda_C, \quad h_{\alpha\alpha C} = \mu_C, \quad h_{nnC} = \sigma_C, \quad \forall \alpha, C,$$

$$(I.2) \quad h_{\alpha\beta C} = 0, \quad \forall \alpha \neq \beta,$$

$$(II.1) \quad \omega_{1\alpha} = \frac{1}{\lambda - \mu} (\mu_1 \omega_\alpha + h_{1\alpha n} \omega_n),$$

$$(II.2) \quad \omega_{1n} = \frac{1}{\lambda - \sigma} \left( \lambda_n \omega_1 + \sum_{\beta} h_{1\beta n} \omega_\beta + \sigma_1 \omega_n \right),$$

$$(II.3) \quad \omega_{\alpha n} = \frac{1}{\mu - \sigma} (h_{1\alpha n} \omega_1 + \mu_n \omega_\alpha).$$

Recall that the curvature tensor of  $M^n$  is given by  $R_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + h_{AC}h_{BD} - h_{AD}h_{BC}$ .

Differentiating (II.3) and applying equations (II.1)–(II.3) and the structure equations of  $M^n$  to the resulting equation, we compute

$$\begin{aligned} \text{LHS} &= d\omega_{\alpha n} = \omega_{\alpha 1} \wedge \omega_{1n} + \sum_{\beta} \omega_{\alpha\beta} \wedge \omega_{\beta n} - (1 + \mu\sigma)\omega_\alpha \wedge \omega_n \\ &= -\frac{1}{\lambda - \mu} (\mu_1 \omega_\alpha + h_{1\alpha n} \omega_n) \\ &\quad \wedge \frac{1}{\lambda - \sigma} \left( \lambda_n \omega_1 + \sum_{\beta} h_{1\beta n} \omega_\beta + \sigma_1 \omega_n \right) \\ &\quad + \sum_{\beta} \omega_{\alpha\beta} \wedge \frac{1}{\mu - \sigma} (h_{1\beta n} \omega_1 + \mu_n \omega_\beta) - (1 + \mu\sigma)\omega_\alpha \wedge \omega_n, \end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \left( d \frac{1}{\mu - \sigma} \right) \wedge (h_{1\alpha n} \omega_1 + \mu_n \omega_\alpha) \\
&\quad + \frac{1}{\mu - \sigma} (d h_{1\alpha n} \wedge \omega_1 + h_{1\alpha n} d \omega_1 + d \mu_n \wedge \omega_\alpha + \mu_n d \omega_\alpha) \\
&= \left( d \frac{1}{\mu - \sigma} \right) \wedge (h_{1\alpha n} \omega_1 + \mu_n \omega_\alpha) \\
&\quad + \frac{1}{\mu - \sigma} \left[ d h_{1\alpha n} \wedge \omega_1 + h_{1\alpha n} \left( \sum_{\beta} \omega_{1\beta} \wedge \omega_\beta + \omega_{1n} \wedge \omega_n \right) \right. \\
&\quad\quad\quad + d \mu_n \wedge \omega_\alpha \\
&\quad\quad\quad \left. + \mu_n \left( \omega_{\alpha 1} \wedge \omega_1 + \sum_{\beta} \omega_{\alpha\beta} \wedge \omega_\beta + \omega_{\alpha n} \wedge \omega_n \right) \right].
\end{aligned}$$

Picking up only those terms of the type of  $\omega_\beta \wedge \omega_n$ , we get

$$\begin{aligned}
\text{LHS} &= - \frac{1}{(\lambda - \mu)(\lambda - \sigma)} \left( \mu_1 \omega_\alpha \wedge \sigma_1 \omega_n + h_{1\alpha n} \omega_n \wedge \sum_{\beta} h_{1\beta n} \omega_\beta \right) \\
&\quad - (1 + \mu\sigma) \omega_\alpha \wedge \omega_n, \\
\text{RHS} &= - \frac{\mu_n - \sigma_n}{(\mu - \sigma)^2} \omega_n \wedge \mu_n \omega_\alpha \\
&\quad + \frac{1}{\mu - \sigma} \left[ h_{1\alpha n} \left( \sum_{\beta} \frac{h_{1\beta n}}{\lambda - \mu} \omega_n \wedge \omega_\beta + \sum_{\beta} \frac{h_{1\beta n}}{\lambda - \sigma} \omega_\beta \wedge \omega_n \right) \right. \\
&\quad\quad\quad \left. + \left( \mu_{nn} \omega_n \wedge \omega_\alpha + \frac{\mu_n}{\mu - \sigma} \omega_\alpha \wedge \omega_n \right) \right].
\end{aligned}$$

Compare the coefficients of  $\omega_\beta \wedge \omega_n$  and note that  $-\frac{1}{(\mu - \sigma)(\lambda - \mu)} + \frac{1}{(\mu - \sigma)(\lambda - \sigma)} = -\frac{1}{(\lambda - \mu)(\lambda - \sigma)}$ , we find  $\forall \alpha, \beta$ ,

$$\begin{aligned}
& - \frac{\mu_1 \sigma_1}{(\lambda - \mu)(\lambda - \sigma)} \delta_{\alpha\beta} + \frac{h_{1\alpha n} h_{1\beta n}}{(\lambda - \mu)(\lambda - \sigma)} - (1 + \mu\sigma) \delta_{\alpha\beta} \\
&= \frac{\mu_n (\mu_n - \sigma_n)}{(\mu - \sigma)^2} \delta_{\alpha\beta} - \frac{h_{1\alpha n} h_{1\beta n}}{(\lambda - \mu)(\lambda - \sigma)} - \frac{\mu_{nn}}{\mu - \sigma} \delta_{\alpha\beta} + \frac{\mu_n}{(\mu - \sigma)^2} \delta_{\alpha\beta}.
\end{aligned}$$

Hence

$$2h_{1\alpha n} h_{1\beta n} = z \delta_{\alpha\beta}$$

where  $z$  is a smooth function on  $M^n$  defined as

$$z = (1 + \mu\sigma)(\lambda - \mu)(\lambda - \sigma) + \mu_1\sigma_1 + \frac{\mu_n(\mu_n - \sigma_n)}{(\mu - \sigma)^2}(\lambda - \mu)(\lambda - \sigma) \\ - \frac{\mu_{nn}}{\mu - \sigma}(\lambda - \mu)(\lambda - \sigma) + \frac{\mu_n}{(\mu - \sigma)^2}(\lambda - \mu)(\lambda - \sigma).$$

Let  $Y^t$  denote the transpose of  $Y$ . In the form of matrix, the above equation reads as

$$Y^t Y = \frac{1}{2} z I_q.$$

Since  $Y \in M_{1 \times q}$  with  $q \geq 2$ , it follows that  $Y = 0$  everywhere as desired.

*Case 2.*  $p \geq 2$ .

Arguing as before, we further have

$$\lambda_i = \mu_i = \sigma_i = 0, \quad \forall i = 1, \dots, p.$$

And equations (2.1)–(2.3) now read as

$$(II.1)' \quad \omega_{i\alpha} = \frac{1}{\lambda - \mu} h_{i\alpha n} \omega_n,$$

$$(II.2)' \quad \omega_{in} = \frac{1}{\lambda - \sigma} \left( \lambda_n \omega_i + \sum_{\beta} h_{i\beta n} \omega_{\beta} \right),$$

$$(II.3)' \quad \omega_{\alpha n} = \frac{1}{\mu - \sigma} \left( \sum_j h_{j\alpha n} \omega_j + \mu_n \omega_{\alpha} \right).$$

Similarly, by differentiating (II.1)' we have

$$\text{LHS} = \sum_j \omega_{ij} \wedge \omega_{j\alpha} + \sum_{\beta} \omega_{i\beta} \wedge \omega_{\beta\alpha} + \omega_{in} \wedge \omega_{n\alpha} - (1 + \lambda\mu) \omega_i \wedge \omega_{\alpha} \\ \sim \frac{1}{\lambda - \sigma} \left( \lambda_n \omega_i + \sum_{\beta} h_{i\beta n} \omega_{\beta} \right) \wedge \left( -\frac{1}{\mu - \sigma} \right) \left( \mu_n \omega_{\alpha} + \sum_j h_{j\alpha n} \omega_j \right) \\ - (1 + \lambda\mu) \omega_i \wedge \omega_{\alpha},$$

$$\begin{aligned}
\text{RHS} &= \left( d \frac{1}{\lambda - \mu} \right) \wedge h_{i\alpha n} \omega_n \\
&+ \frac{1}{\lambda - \mu} \left[ dh_{i\alpha n} \wedge \omega_n + h_{i\alpha n} \left( \sum_j \omega_{nj} \wedge \omega_j + \sum_\beta \omega_{n\beta} \wedge \omega_\beta \right) \right] \\
&\sim \frac{h_{i\alpha n}}{\lambda - \mu} \left( -\frac{1}{\lambda - \sigma} \sum_{j, \beta} h_{j\beta n} \omega_\beta \wedge \omega_j - \frac{1}{\mu - \sigma} \sum_{j, \beta} h_{j\beta n} \omega_j \wedge \omega_\beta \right) \\
&= \sum_{j, \beta} \frac{h_{i\alpha n} h_{j\beta n}}{\lambda - \mu} \left( \frac{1}{\lambda - \sigma} - \frac{1}{\mu - \sigma} \right) \omega_j \wedge \omega_\beta \\
&= - \sum_{j, \beta} \frac{h_{i\alpha n} h_{j\beta n}}{(\lambda - \sigma)(\mu - \sigma)} \omega_j \wedge \omega_\beta,
\end{aligned}$$

where for any two given 2-forms  $\psi$  and  $\psi'$ , by  $\psi \sim \psi'$  we mean  $\psi \equiv \psi' \pmod{\omega_n}$ , i.e.,  $\psi - \psi' = \omega \wedge \omega_n$  for some 1-form  $\omega$ .

Now, by picking up those terms of the type of  $\omega_j \wedge \omega_\beta$  we have

$$\begin{aligned}
&- \frac{1}{(\lambda - \sigma)(\mu - \sigma)} \left( \lambda_n \omega_i \wedge \mu_n \omega_\alpha + \sum_\beta h_{i\beta n} \wedge \sum_j h_{j\alpha n} \omega_j \right) \\
&- (1 + \lambda\mu) \omega_i \wedge \omega_\alpha \\
&= - \sum_{j, \beta} \frac{h_{i\alpha n} h_{j\beta n}}{(\lambda - \sigma)(\mu - \sigma)} \omega_j \wedge \omega_\beta.
\end{aligned}$$

Then,

$$2h_{i\alpha n} h_{j\beta n} = \bar{z} \delta_{ij} \delta_{\alpha\beta}, \quad \forall i, j, \alpha, \beta,$$

where  $\bar{z} = \lambda_n \mu_n + (1 + \lambda\mu)(\lambda - \sigma)(\mu - \sigma)$ .

In particular,

$$h_{i\alpha n} h_{i\beta n} = \frac{1}{2} \bar{z} \delta_{\alpha\beta}, \quad \forall i, \alpha, \beta.$$

Again, since  $q \geq 2$  we have  $h_{i\alpha n} = 0$ ,  $\forall i, \alpha$ , i.e.  $Y = 0$  everywhere.

This establishes the Key Lemma and thus completes the proof of the Main Theorem.

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christ@math.ucla.edu

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clemens@math.utah.edu

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University of California, San Diego  
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tenright@ucsd.edu

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spk@gauss.stanford.edu

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