ON SYSTEMS OF GENERATORS OF ARITHMETIC SUBGROUPS OF HIGHER RANK GROUPS

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We show that any two maximal opposing unipotent subgroups of
an irreducible lattice in a higher rank Lie Group, generate a lattice
in the Lie Group. The method of proof is to use certain techniques
of the solution of the congruence subgroup problem of arithmetic
lattices in higher rank groups.

We freely use the notation and results of [3] without giving ex-

cplicit references therein.

Let \( G \) be a simply connected absolutely almost simple linear al-
gerbroic group defined and isotropic over a global field \( K \). Let \( U^+ \)
be the unipotent radical (which is defined over \( K \)) of a minimal
parabolic \( K \)-subgroup \( P^+ \) of \( G \). Let \( U^- \) be the unipotent radical of
another minimal parabolic \( K \)-subgroup \( P^- \) of \( G \) which is opposed to
\( P^+ \) in the sense that \( U^+ \cap U^- = \{1\} \). Let \( S \) be a finite set of places
of \( K \) including all the archimedian ones, if any. We call the ring
\( \mathcal{O}_S = \{ x \in K; |x|_v \leq 1 \text{ for all places } v \text{ of } K, \text{ not in } S \} \)
the ring of \( S \)-integers in \( K \). Choose a faithful representation \( G \hookrightarrow GL_N \)
defined over \( K \) and define \( G(\mathcal{O}_S) = \{ g \in G; g_{ij} \in \mathcal{O}_S, 1 \leq i,j \leq N \} \). The subgroups in \( G \) which are of finite index in \( G(\mathcal{O}_S) \) are
called \( S \)-arithmetic groups. Define the \( S \)-rank of \( G \) to be the sum
\( \sum_{v \in S} K_v - \text{rank}(G) \). Given a non-zero ideal \( \mathfrak{a} \) of \( \mathcal{O}_S \) and an algebraic
\( \mathfrak{a} \)-subgroup \( H \) of \( G \) define \( H(\mathfrak{a}) = \{ h \in H; h_{ij} \equiv \delta_{ij}(\text{mod } \mathfrak{a}) \} \), where
\( \delta_{ij} = 0 \) if \( i \neq j \) and \( \delta_{ij} = 1 \) if \( i = j \). Let \( A(S) \) denote the ring
of \( S \)-adèles of \( K \).
Theorem. With the notation as above, let \( E(a) \) denote the group generated by \( U^+(a) \) and \( U^-(a) \). Then \( E(a) \) is an \( S \)-arithmetic subgroup of \( G(a) \) provided \( S - \text{rank}(G) \geq 2 \) and \( K - \text{rank}(G) = 1 \), \( \text{Char}(K) \neq 2 \).

Remark. The theorem holds also when \( K - \text{rank}(G) \geq 2 \) and is proved for \( G \) a classical group of \( K - \text{rank}(G) \geq 2 \) in [17], \( G \) is a Chevalley group of \( K - \text{rank}(G) \geq 2 \) [15] and for \( G \) an arbitrary group of \( K - \text{rank}(G) \geq 2 \) [11].

The theorem is proved for \( G = SL_2 \) in [18] and Vaserstein has informed us that he has a proof (unpublished) of the theorem when \( G = SU(2,1) \).

We now give an outline of the proof. The proposition of Section 1 says: a subgroup \( F(a) \) which is closely related to \( E(a) \) (and normalises \( E(a) \)) has the property that given \( g \in G(K) \) there exists a nonzero ideal \( \mathfrak{a} \) of \( A \) such that \( gF(b)g^{-1} \subset F(a) \). This is used to show that there is a completion \( \widetilde{G} \) of \( G(K) \) with respect to which the subgroups \( G(a) \) have open closures in \( \widetilde{G} \). We then show that there is a continuous surjection \( \pi \) from \( \widetilde{G} \) onto \( G(A(S)) \) where \( (A(S)) \) is the ring of \( S \)-adèles of \( K \). The main point is then to show that the kernel \( C \) of \( \pi \) is central in \( \widetilde{G} \). Then by appealing to [13], we are done.

In Section 2, we show that \( C \) is central when the semi-simple part of the Levi component of the minimal parabolic subgroup \( P^+ \) of \( G \) is isotropic over \( K_v \) for some \( v \in S \). In Section 3, we prove the same when \( G = SU(2,1) \) and in Section 4, by looking at suitable embeddings of \( G = SU(2,1) \) and \( SL_2 \) in \( G \), we prove that \( C \) is central even in the case of \( G \) for which the semisimple part mentioned above is anisotropic over \( K_v \) for all \( v \in S \).

1. Construction of a completion \( \widetilde{G} \) of \( G(K) \).

Notation 1.1. Let \( G, E(a) \) be as in the introduction. Assume that \( S - \text{rank}(G) \geq 2 \) and that \( K - \text{rank}(G) = 1 \). Let \( F(a) \) denote the group generated by \( U^+(a), U^-(a), \) and \( M(a) \) where \( M + P^+ \cap P^- \).
Lemma 1.2.

(i) The group $F(\alpha)$ is Zariski dense in $G$.

(ii) More generally, if $\rho : G(K) \to GL_n(C)$ or if $\rho : G(K)^+ \to GL_n(C)$ is a homomorphism of abstract groups ($C$ is algebraically closed), then the Zariski closure of $\rho(F(\alpha) \cap G(K)^+)$ is equal to the Zariski closure of $\rho(G(K)^+)$.

(iii) The Zariski closure of $\rho(G(K)^+)$ is connected.

Proof. The proof of (i) is easy: the Zariski closure of $F(\alpha)$ in $G$ contains $U^+(\alpha)$ and $U^-(\alpha)$, therefore contains $U^+$ and $U^-$ and therefore equals $G$.

Given a non-zero ideal $I$ of $A$, let $U^+_I$ denote the Zariski closure of $\rho(U^+(I))$. Clearly, if $I \subset C$ and $c$ is a non-zero ideal of $A$ then $U^+_c \subset U^+_b$. Since $U^+_A$ is Noetherian, there exists a non-zero ideal $I$ of $A$ such that $U^+_b$ is minimal. Given any non-zero ideal $I$ of $A$, we have $c \cap b \subset c$ and by minimality of $U^+_b$, we have $U^+_b = U^+_I \cap c \subset U^+_c$.

If $a \in P^+(K)$ is given, then there exists a non-zero ideal $I$ of $A$ such that $c \subset b$ and $aU^+(I)a^{-1} \subset U^+(b)$. Taking Zariski closures in $GL_n(C)$, we obtain: $\rho(a)U^+_c \rho(a)^+ \subset U^+_b \subset U^+_c = U^+_b$ which means that $U^+_b$ is normalised by $\rho(P^+(K))$. It is easy to show that

$$\bigcup_{a \in P^+(K)} aU^+(b)a^{-1} = U^+(K),$$

using the facts that for any non-zero integer $m$,

$$\bigcup_{\lambda \in K^*} \lambda^m b = K$$

and that $P^+$ contains a $K$-split torus. Therefore $U^+_b$ contains $\rho(U^+(K))$. We thus get: the Zariski closure of $\rho(F(I) \cap G(K)^+)$ contains $U^+ \supset U^+_b = U^+_I$ which contains $\rho(U^+(K))$ and by symmetry the Zariski closure of $\rho(F(I) \cap G(K)^+)$ contains $\rho(U^-(K))$.

This proves part (ii). Now (iii) follows from the fact that $(G(K)^+/\text{centre})$ is an abstract simple group [16].

Definition 1.3. Let $L/K$ be an algebraic extension and $k \subset L$ a subfield. Suppose $f : G(K)^+ \to k$ is a function whose $G(K)^+$-translates on the left (or right) span a finite dimensional vector space (over $k$). We call $f$ a $G(K)^+$-finite function.
COROLLARY 1.4. Let $L,k$ be as above, $f : G(K)^+ \to k$ a $G(K)^+$-finite function. Suppose $f$ vanishes on $F(a) \cap G(K)^+$. (A) Then $f$ vanishes on $G(K)^+$. (B) Moreover, $G(K)^+$-finite functions with values in $k$ form an integral domain.

Proof. (A) Immediate from (II) of Lemma 1.2. (B) Follows from (iii) of Lemma 1.2. \hfill \Box

Lemma 1.5. The group $M(A)$ is infinite.

Proof. Suppose $\text{Card}(S) \geq 2$. The group $M$ contains a $K$-split torus $G_m$ since $1 = K - \text{rank}(M)$, and $G_m(A)$ contains, by the Dirichlet unit theorem, a free abelian group of rank $= (\text{Card}(S) - 1) \geq 1$.

Suppose $\text{Card}(S) = 1$, $S = \{v\}$. We have a nontrivial $K$-homomorphism $M \to G_m$, with kernel $M_0$. Now, $S - \text{rank}(M_0) = S - \text{rank}(M) - S - \text{rank}(G_m) \geq 2 - 1 = 1$. Therefore $M_0(K_v)$ is not compact; but, (by [2] and [4]), $M_0(A)$ is a cocompact lattice in $M_0(K_v)$ and so $M_0(A)$ is infinite. In particular $M(A)$ is infinite. \hfill \Box

We now state the main result of this section. We will prove it later in the section after proving some preliminary results.

Proposition 1.6. Given a nonzero ideal $a$ of $A$ and an element of $g \in G(K)$, there exists a nonzero ideal $b$ of $A$ such that

$$gF(b)g^{-1} \subseteq F(a).$$

Notation 1.7. The map $U^- \times M \times U^+ \to G$ given by $(u^-, m, u^+) \mapsto u^-mu^+$ is a $K$-isomorphism onto an open subset $\Omega$ of $G$. Given $x \in \Omega$, we may write $x = u^-xu_x$ with $u_x^- \subseteq U^-$, $m_x \in M$, $u_x \in U^+$ and $x \mapsto u_x$ is a $K$-rational function from $\Omega$ into $U^+$.

Given a nonzero ideal $a$ of $A$ and $g \in G(K)$, consider the conjugate $g^{-1}F(a)g = g^{-1}h^{-1}F(a)hg$ for all $h \in F(a)$. By Lemma 1.2, there exists an $h \in F(a)$ such that $hg \in \Omega$ and so, by replacing $g$ by $hg$ if necessary we assume, as we may, that $g \in \Omega$,
while looking at $g^{-1}F(a)g$. Let $b_1$ be a nonzero ideal of $A$ such that $g^{-1}G(a)g \supset G(b_1)$. Then for any $h \in F(a) \cap g^{-1}$ we have

$$g^{-1}h^{-1}F(a)h \supset g^{-1}h^{-1}P^{-}(a)h \cap P^+ \supset (g^{-1}h^{-1}P^{-}h \cap P^+)(b_1)$$

$$= (u_{h_1}^{-1}Pu_{h_1} \cap P^+)(b_1) = (u_{h_1}^{-1}Pu_{h_1} \cap P^+)(b_1) = (u_{h_1}^{-1}Mu_{h_1})(b_1).$$

For $x \in \Omega$ denote by $M_x$ the group $x^{-1}P-x \cap P^+ = u_x^{-1}MU_x$. Then $g^{-1}f(a)g$ contains $\{M_{h_1}(b_1); h \in F(a) \cap \Omega g^{-1}\}$. Denote by $\Delta_g$ the subgroup of $P^{-}(b)$ generated by $\{M_{h_1}(b_1); h \in F(a) \cap \Omega g^{-1}\}$. We aim to show that $\Delta_g$ contains $P^+(b)$ for some nonzero ideal $b_g$ of $A$, with $b_g \subset b$. Let $h \in \{F(a) \cap \Omega g^{-1}\}$, with $M_{h_1}(b_1) = (u_{h_1}^{-1}Mu_{h_1})(b_1), v_h = u_{h_1}^{-1}u_g$ whence $M_{h_1}(b_1) = (v_hM_gV_h^{-1})(b_1)$. There exists a nonzero ideal $b_h$ of $A$ such that $b_h \subset b_1$ and such that $\Delta_g \supset (v_hM_gv_h^{-1})(b_1) \supset v_hM_g(b_h)v_h^{-1}$. We also have $M_g(b_h) \subset M_g(b_1) \subset \Delta_g$. Denote by $[M_g(b_h), v_h]$ the subgroup of $\Delta_g$ generated by $\{mv_hm^{-1}v_h^{-1}; m \in M_g(b_h)\}$. Then $\Delta_g$ contains $[M_g(b_H), v_h]$. Observe that $M_g(b_h)$ is normalised by $M_g(b_1) \subset \Delta_g$. Denote by $M_g(b_1)(v_h)$ the set $\{mv_hm^{-1}; m \in M_g(b_1)\}$. Then we get: $\Delta_g \supset [M_g(b_h), M_g(b_1)(v_h)],$ and therefore $\Delta_g \supset [M_g(b_h), [M_g(b_1), v_h]].$ Let $H_g$ be the subgroup generated by $\{[M_g(b_1), v_h]; h \in F(a) \cap \Omega g^{-1}\}$. Define $V^+ = [U^+, U^+], W^+ = U^+/V^+$ and $pr : U^+ \to W^+$ the quotient map. Then $V^+$ and $W^+$ are finite dimensional $K$-vector spaces, on which $M(K)$ acts by $K$-linear transformations. We have the unique quotient $\pi : M_g \to G_m$ defined over $K$. Let

$$M^0_g = \left\{ x = (x_v)_{v \in S} \in \prod_{v \in S} M(K_v); \prod_{v \in S} |\pi(x_v)| = 1 \right\}.$$

Then (i) by [2] and [4], we have: $M_g(b_1)$ is a cocompact lattice in $M^0_g$; (ii) $M^0_g$ is compactly generated [1]; (iii) $M_g(b_1)$ is a finitely generated group. (This follows from (i), (ii) and [5].) Moreover every element of $M_g(b_1)$ is semisimple.

We assume, as we may, that $b_1$ is an ideal so deep that no nontrivial element of $M_g(b_1) - \{1\}$ has a nontrivial root of unity as an eigenvalue in its action on $W^+$. Let $\{\gamma; \gamma \in F\}$ be a finite nontrivial set of generators of $M_g(b_1)$. For $\Theta \in M_g(b_1)$ let $\theta_*$ denote the linear transformation induced by $\theta$ on $W^+$. Let
If $\gamma \in F$, write $W_{\gamma} = (\gamma_\ast - 1)W^+$. Then $w \mapsto (\gamma_\ast - 1)w$ is a $K$-linear map of $W^+$ onto $W_{\gamma}$. Since $\gamma$ is semisimple; (i) $w \mapsto (\gamma_\ast - 1)w$ is an isomorphism of $W_{\gamma}$ onto itself (ii); $W_{\gamma}$ is a direct sum of irreducible $K[\gamma]$-modules: $W_{\gamma} = \oplus W_j$; (iii) if $F_p[\gamma_\ast, \gamma^{-1}_\ast]$ denotes the subring generated by $\gamma_\ast$ and $\gamma^{-1}_\ast$ in $End_K(W_{\gamma})$, then $F_p[\gamma_\ast, \gamma^{-1}_\ast]$ is a ring without nilpotent elements. By Schur's lemma, the commutant of the image of $F_p[\gamma_\ast, \gamma^{-1}_\ast]$ in $End_K(W_j)$ is a division algebra over $K$ and therefore $\exists m\{F_p[\gamma_\ast, \gamma^{-1}_\ast] \cong End(W_j)\}$ is an integral domain and thus defines a prime ideal $p_j$ of $F_p[\gamma_\ast, \gamma^{-1}_\ast]$. 

**Remark 1.8.** Let $H \subset U^+(K)$ be a subgroup normalised by $M_g(b)$. Then $pr_p(H) \subset W_p$ is an $(F_p[\gamma_\ast, \gamma^{-1}_\ast]/p)$-module. Moreover, $pr_p(H)$ contains $pr_p(H)$. 

**Proof.** Clearly $pr(H)$ is a subgroup of $W^+$, is therefore $F_p$-stable and hence $pr(H)$ is an $F_p[\gamma_\ast, \gamma^{-1}_\ast]$-module. Since $pr_p : W_\gamma \to W_p$ and $W^+ \to W_\gamma$ are given by multiplication by elements of $F_p[\gamma_\ast, \gamma^{-1}_\ast]$, we have: $pr_p(H) \subset F_p[\gamma_\ast, \gamma^{-1}_\ast](pr_p(H)) \subset F_p[\gamma_\ast, \gamma^{-1}_\ast](pr(H)) \subset pr(H)$. 

**Notation 1.9.** Let $k_p$ denote the quotient field of the domain $A_p = F_p[\gamma_\ast, \gamma^{-1}_\ast]/p$. Now, $W_p$ is an $(F_p[\gamma_\ast, \gamma^{-1}_\ast]/p)$-module and hence is a $k_p$-vector space as well as a $K$-vector space. [We use the fact that $W_p$ splits as a direct sum of irreducible $K[A_p]$ modules $W_i$ and $A_p$ acts faithfully on each $W_i$ ($\gamma$ does not have roots of unity as eigenvalue) and by Schur's lemma, the commutant of $A_p$ is a division ring $D$ whence $k_p \subset D$ and acts on $W_i$. Thus $k_p$ acts on $W_p$ too.] Let $R_p$ be the subring of $End_K(W_p)$ generated by $k_p$. 
and $K$. Then $R_p$ is a finite dimensional $K$-vector space and since $\gamma$ acts semisimply on $W_p$, $R_p$ is a product of finite field extensions of $K$. Now $K$ is a global field and $k_p$ is an infinite field (again, $k_p$ is infinite because $\gamma$ has no root of unity as an eigenvalue) and so $k_p$ is also a global field, whence $R_p$ is finite dimensional over $k_p$. Now $W_p = \bigoplus W_i \cong \bigoplus \text{Im}(R_p)$ and so $W_p$ is finite dimensional over $R_p$ and therefore $W_p$ is a finite dimensional $k_p$-vector space.

**Lemma 1.10.** Given $\gamma \in F$ and $p \in X(\gamma)$, there exists a finite set $\{v_h\}$ of elements of $U^+(K)$, for $h \in F(\alpha) \cap \Omega g^{-1} \cap G(K)^+$, such that the $k_p$-span of $\{pr_p[\gamma, v_h]\}$ is all of $W_p$.

**Proof.** Let $\lambda : W_p \to k_p$ be a linear form over $k_p$ which vanishes on all $\{pr_p[\gamma, v_h] ; h \in F(\alpha) \cap \Omega g^{-1}\}$. Then $\varphi(h) = \lambda \circ pr_p[\gamma, v_h]$ (for $h \in \Omega g^{-1}$) has the property: $\varphi(h) = 0$ for $h \in F(\alpha) \cap \Omega g^{-1} \cap G(K)^+$. Now $pr_p[\gamma, v_h] = pr_p[\gamma*, pr(v_h)]$ and $pr(v_h) = A(h) B(h)$ with $A(h) \in W^+$ and $B(h) \in K$; both $A(h)$ and $B(h)$ are polynomial functions on $G(K)$ with $B(h) \neq 0$ for all $h \in \Omega g^{-1}$. We think of $K$ as embedded in $R_p$. Let $\{\xi\}$ be a $k_p$-basis of $R_p$ and write $B(h) = \sum \xi B(\xi) \in \xi$. The function $B(h)$ is polynomial on $G(K)^+$ and hence the $k_p$-valued function $B_\xi(h)$ is a $G(K)^+$-finite function. Now $B(h)^{-1} \in K$ hence $B(h)^{-1} = \sum X_\xi(h) \in X_\xi(h) \in k_p$. Thus $\{X_\xi(h)\}$ are solutions of linear equations whose coefficients are $k_p$-valued $G(K)^+$-finite functions on $G(K)^+$, and by Corollary 1.4, such functions form a domain. We may thus assume that $X_\xi(h)$ belong to the quotient field of $R_p$, and write $x_\xi(h) = Y_\xi(h) Z(h)$, where $Y_\xi, Z : G(K)^+ \to k_p$ are $G(K)^+$-finite functions. We finally get

$$\varphi(h) = \lambda \circ pr_p[\gamma* - 1][pr(v_h)] = \lambda \circ pr_p[\gamma* - 1]. \begin{bmatrix} A(h) \\ B(h) \end{bmatrix} = \lambda \circ pr_p(\gamma* - 1) \begin{bmatrix} C(h) \\ Z(h) \end{bmatrix} = \lambda \circ pr_p((\gamma* - 1)C(h)) \frac{Z(h)}{Z(h)}$$

(since $\lambda$ is $k_p$-linear) and $\varphi(h)$ vanishes on $F(\alpha) \cap \Omega g^{-1} \cap G(K)^+$. Therefore $\lambda \circ pr_p(\gamma* - 1)C(h) = \psi(h)$ is a $G(K)^+$-finite function which vanishes on $F(\alpha) \cap \Omega g^{-1} \cap G(K)^+$, and $G(K) - \Omega g^{-1}$ is the
set of zeros of a polynomial \( \eta(h) = \sum \eta_e(h) \). Thus, for all \( \psi(h) \eta_e(h) \equiv 0 \) on \( G(K)^+ \cap F(a) \) and by Corollary 1.4, part (ii), \( \psi(h) \equiv 0 \) on \( G(K)^+ \). This means that \( \varphi(h) = \lambda \circ (pr_p(\gamma_* - 1)pr(v_h)) = 0 \) for all \( h \in G(K)^+ \cap \Omega g^{-1}, v_h = u_h^{-1}u_g \). Taking \( h = Zg^{-1}, Z \in U^+(K) \), we get: \( V_h = Z^{-1}u_g \) represents an arbitrary element of \( U(K)^+ \), and

\[
0 = \lambda \circ (pr_p(\gamma_* - 1)pr(U^+(K))) = \lambda \circ (pr_p(\gamma_* - 1)W^+) = \lambda(W_p).
\]

Thus \( \lambda = 0 \) on \( W_p \) whenever \( 0 = \lambda \circ (pr_p[\gamma, v_h], (h \in F(a) \cap \Omega g^{-1} \cap G(K)^+)) \). Hence \( \{pr_p[\gamma, V_h]; h \in F(a) \cap \Omega g^{-1}\} \) contains a \( k_p \)-basis of \( W_p \), but \( W_p \) is finite dimensional, whence the lemma follows. \( \square \)

**Lemma 1.11.** There exists a finite set \( \{v_h\} = X \) of elements of \( U^+(K) \) with \( h \in F(a) \cap G(K)^+ \cap \Omega g^{-1} \) such that (i) for every \( \gamma \in F \) and \( p \in X(\gamma) \), we have, the \( k_p \)-span of \( pr_p[\gamma, v_h] \) is all of \( W_p \) (ii) \( [\gamma, v_h] \in H_g \) for all \( \gamma \in F, v_h \in X \). We denote by \( H_X \) the group generated by \( \{[\theta, v_h]: v_h \in X \theta, \in M_b(b_1)\} \).

**Proof.** We get a finite set \( X_{\gamma,p} = \{v_h\} \) satisfying the conditions of Lemma 1.10. Take \( X = U X_{\gamma,p} \). \( \square \)

**Lemma 1.12.** Let \( c \) be a nonzero ideal of \( A \), contained in \( b \). Then there exist a nonzero ideal \( c_p \) of \( (F_p[\gamma_*, \gamma_*^{-1}]/p) \) and a subgroup \( H_c \subset U^+(c) \cap H_X \) such that

\[
pr_p(H_c) = \sum_{v_h \in X} c_p(pr_p[\gamma, v_h]).
\]

**Proof.** We have \( pr_p[\gamma, v_h] = (\gamma_* - 1)pr_p(v_h) \). Now, for any integer \( N, pr_p[\gamma^N, v_h] = (1 + \gamma_* + \cdots + \gamma_*^{N-1})pr_p[\gamma, v_h] \in k_ppr_p[\gamma, v_h] \) because \( \gamma_* \) has no torsion eigenvalues. Therefore the \( k_p \)-span of \( \{pr_p[\gamma^N, v_h]; v_h \in X\} \) is \( k_p \)-span of \( \{pr_p[r; v_h]; v_h \in X\} \) which by Lemma 1.11 is all of \( W_p \). Choose now an integer \( N \) such that \( [\gamma^N, v_h] \in U^+(c) \) for all \( v_h \in X \). Let \( c_p \subset F_p[\gamma_*, \gamma_*^{-1}]/p \) be the ideal generated by \( (1 + \gamma_* + \cdots + \gamma_*^{N-1}) \). Let \( H_c \) be the smallest subgroup of \( U^+(c) \) containing \( \{[\gamma^N, v_h]; v_h \in X\} \) and normalised by
Then $H_c$ is clearly contained in $U^+(c) \cap \Delta_g(\gamma^{-1}v_h\gamma v_h^{-1} \in \Delta_g$ for $\gamma \in M_g(b_1))$. Moreover, by Remark 1.8,

$$
pr_p(H_c) \supset \sum_{v_h \in X} (F_p[\gamma*, \gamma_*^{-1}]/p) \cdot (1 + \gamma* + \cdots + \gamma_*^{N-1})pr_p[\gamma, v_h] \supset \sum_{v_h \in X} c_p pr_p[\gamma, v_h].
$$

\[ \Box \]

**Lemma 1.13.** Given a nonzero ideal $C$ of $A$ contained in $b$, there is a subgroup $H_c \subset U^+(c) \cap H_X$ such that $pr(H_c)$ contains $W^+(b_3)$ where $b_3$ is a nonzero ideal of $A$ contained in $c$.

**Proof.** We have: $R_p \bigotimes_K \left( \prod_{v \in S} K_v \right)$ is a product of local fields $L_w$.

Let $S_1 = \{w; |\gamma|_{L_w} < 1\}, S_2 = \{w; |\gamma|_{L_w} > 1\},$ and $S_3 = \{w; |\gamma|_{L_w} = 1\}.$

Let $k_1$ (resp. $k_2$) be the closure of $k_p$ in $\prod_{w \in S_1} L_w$ (resp. in $\prod_{w \in S_2} L_w$). Now $F_p[\gamma*, \gamma_*^{-1}]/p$ is a lattice in $k_1 \times k_2$. Since $p$ is a (faithful) $R_p$-module, the module $W_p \bigotimes_K \prod_{v \in S} K_v$ is a direct sum of $\{L_w\}$, with multiplicity $m_w$. Let $U_3$ be a compact open subgroup of $\bigoplus_{w \in S_3} m_w L_w$. Then $\sum_{v_h \in X} c_p pr_p[\gamma, v_h]$ contains a lattice in $U_p = \sum k_1 pr_p[\gamma, v_h] + \sum k_2 pr_p[\gamma, v_h] + U_3$.

Now $\{pr_p([\gamma, v_h]): v_h \in X\}$ contains a $k_p$-basis of $w_p$ and since $k_i$ is a local field ($i = 1, 2$), each $\{L_w: w \in S_i\}$ is a finite dimensional vector space over $k_i$ and $\sum k_i pr_p[\gamma, v_h] = \sum w \in S_i m_w L_w$. Thus, $U_p$ contains the nonzero $\prod_{v \in S} K_v$-submodule $\sum_{w \in S_1 U S_2} m_w L_w = E_p$. Write $E_g = \sum_{p \in X(\gamma) x \in M_g(b)} x_*(E_p)$. This is also a $\left( \prod_{v \in S} K_v \right)$-submodule of
$W \otimes \prod_{K_v} K_v$, which is stable under $M_g^0$ (since it is stable under the Zariski closure of $M_g(b)$: by finite dimensionality, the sum over $x \in M_g(b)$ is really a finite sum). We look at the action of $\gamma$ on $(W^+ \otimes \prod K_v)/E_\gamma$. By the definition of $S_1$ and $S_2$, $\gamma$ has only bounded eigenvalues in its action on $W^+/E_\gamma$. Now the space $\sum_{\gamma \in F} E_\gamma$ is also $M_g^0$ stable and on $\left(\prod_{v \in S} U^+(K_v)\right) \rtimes M_g^0$ is compactly generated, to conclude $\left(\prod_{v \in S} K_v \otimes W^+\right)/\sum_{\gamma \in F} E_\gamma \rtimes M_g^0$ is compactly generated. On the other hand, the image of $M_g^0$ in $\text{Aut} \left(\prod_{v \in S} K_v \otimes W^+ / \sum_{\gamma \in F} E_\gamma\right)$ is bounded, as we have just observed. This implies that

$$W^+ \otimes \prod_{v \in S} K_v = \sum_{\gamma \in F} E_\gamma = \sum_{\gamma \in F} \sum_{p \in X(r)} \sum_{x \in M_g(b)} x_*(E_p) = \sum_{\gamma, p, x} x_*(U_p)$$

\left( E_p \subset U_p \subset W \otimes \prod_{K_v} K_v \right).$$

We have: $pr_p(H_c) \cap U_p(\subset pr(H_c) \cap U_p)$ contains a lattice in $U_p$ whence $pr(H_c)$ contains a lattice $L_c$ in $W^+ \otimes \prod_{v \in S} K_v$, such that (i) $L_c \subset \sum_{\gamma, p} pr_p(H_c) \cap U_p \subset W^+(c\lambda)$ by Lemma 1.12. (Here $\lambda \in K^*$ is such that for all ideals $a$ of $A$, $pr(U^+(a)) \subset W^+(a, \lambda)$), (ii) $L_c$ is $M_g(b)$-stable. It can then be shown easily (see [13], Section (2.10)) that $\overline{L_c} \supset W^+(b_2)$, whenever $L_c \subset W^+ \otimes \prod_{v \in S} K_v$ is a lattice satisfying (i) and (ii), where $b_2 \subset A$ is a nonzero ideal. Take $b_3 = b_2 \cap C$. The proof of the lemma is over.

**Lemma 1.14.** There exists a non-zero ideal $b_4$ of $A$ such that $\Delta_g$ contains $U^+(b_4)$.

**Proof.** From Lemma 1.13, we have any nonzero ideal $C$ of $A$, an $H_c$ such that $H \supset H_c, H_c \subset U^+(c)$ and $pr(H_c) \supset W^+(b_3)$, with
b_3 \subset C \subset b. Now, \([W^+(a), W^+(a)]\) contains \(V^+(a^2 \mu)\) for a fixed \(\mu \in K^*\) and varying nonzero ideals \(a\) of \(A\). In particular, \([H_c, H_c]\) contains \([W^+(b_3), W^+(b_3)] \supset V^+(b_3^2 \mu)\), whence \(H_X \supset V^+(b_3^2 \mu)\). Let \(b_4 \subset b_3^2 \mu \cap b_3 \lambda^{-1} \cap A\). Then for \(x \in U^+(b_4)\) we have \(pr(x) \in W^+(b_3)\) (by definition of \(\lambda\)) \(\subset pr(H_c)\) and so (14) there exists an \(h \in H_c\) such that \(xh^{-1} \in V^+(c)\). Thus \(U^+(b_4) \subset H_c V^+(c) \subset H_X V^+(c)\). Now we replace \(c\) by \(c \cap b_3 \mu = c'\). Then for the corresponding ideals \(b'_3, b'_3\) we have: \(U^+(b'_4) \subset H_X V^+(c') \subset H_X V^+(b_3 \mu) \subset H_X\). We recall that \(A = [M^p(b), [M^p, \nu]_j\) and that for \(v_h \in X\), we have \([M^p(b_1), v_h] \in H_X\). Therefore if \(b_5 = \cup v_h \in x b_h\), then \(\Delta_g \supset [M^p(b_5), H_X]\) and we have just shown that \(H_X \supset U^+(b_4)\). Thus \(\Delta_g \supset [M^p(b_5), U^+(b_4)]\). Again, by arguments similar to (2.10) of [13], it is easy to show that \([M^p(b_5), U^+(b_4)] \supset U^+(b_5)\), for a nonzero ideal \(b_5\) of \(A\). This proves the lemma.

We now complete the proof of Proposition 1.6: we have seen that \(g^{-1}F(a)g \supset \Delta_g \supset U^+(b_5)\) and \(\Delta_g \supset M^p(b_1)\). The group generated by \(U^+(b_6)\) and \(M^p(b_1)\) contains \(P^+(b_+)^{\pm}\) for some nonzero ideal \(b_+\) of \(A\), hence \(g^{-1}F(a)g \supset P^+(b_+)\). By symmetry \(g^{-1}F(a)g \supset P^-(b_+)\). It is now routine to check that equivalence classes of Cauchy sequences in \(G(K)^+\) form a topological group \( \mathcal{G}\) with \(G(K)^+\) being a dense subgroup. Let \(C\) be the kernel of the map \( \mathcal{G} \to G(A(S))\). By [12], Lemma

**NOTATION AND DEFINITIONS 1.15.** We construct \( \mathcal{G}\). Consider the group \(G(K)^+\) (instead of \(G(K)\)). By [7], \(G(K)/G(K)^+\) is finite and therefore we may (and we do) replace \(G(A), G(a)\) and \(F(a)\) by their intersections with \(G(K)^+\) without affecting questions of \(S\)-arithmeticity. We denote the intersections by \(G(A), G(a)\) and \(F(a)\). Define a topology on \(G(K)^+\) by taking the sets \(\{gF(a); g \in G(K)^+, a \neq 0\}\) to be open. We then get a left uniform structure and a right uniform structure on \(G(K)^+\). We now call a sequence \(\{x_n\}\) in \(G(K)^+\) to be Cauchy if and only if \(\{x_n\}\) is Cauchy with respect to both the uniform structures on \(G(K)^+\) i.e. if and only if for every non-zero ideal \(a\) of \(A\), there exists an integer \(l = l(a) \geq 0\) such that \(x_{n+m} \in F(a), x_{n}x_{n}^{-1} \in F(a)\) for all \(m, n \geq l\). Define two Cauchy sequences \(\{x_n\}, \{y_n\}\) to be equivalent if and only if for every nonzero ideal \(a\) of \(A\), there exists an integer \(l = l(a) \geq 0\) such that \(x_n y_n \in F(a), x_{n} y_{n}^{-1} \in F(a)\) for all \(n \geq l\). It is now routine to check that equivalence classes of Cauchy sequences in \(G(K)^+\) form a topological group \( \mathcal{G}\) with \(G(K)^+\) being a dense subgroup. Let \(C\) be the kernel of the map \( \mathcal{G} \to G(A(S))\). By [12], Lemma
(2.10), the map is surjective. Clearly $C$ is a closed normal subgroup of $G$. We also observe that $U^+(A(S))$ and $U^-(A(S))$ are embedded in $G$ (as closures of $U^+(K)$ and $U^-(K)$ respectively).

**Lemma 1.16.** Suppose $C$ is centralised by $G(K)^+$. Then $C$ is centralised by $G(K)^{+_{\text{a}}}$ and $C$ is finite and $F(a)$ is an $S$-arithmetic subgroup of $G(K)^{+_{\text{a}}}$.

*Proof.* Let $\overline{F(a)}$ be the closure of $F(a)$ in $G$. Then $\overline{F(a)}$ is open in $G$ and therefore $C \cap \overline{F(a)}$ is open in $C$. By assumption, and density of $G(K)^+$ in $G$, we see that $C$ is central in $G$ and so we get a central extension

$$1 \to C/C \cap \overline{F(a)} \to G/C \cap \overline{F(a)} \to G(A(S)) \to 1,$$

where $C/C \cap \overline{F(a)}$ is a discrete group. Thus $G/C \cap \overline{F(a)}$ is a locally compact central extension of $G(A(S))$, split over $G(K)^+$ and by [10], $C/C \cap \overline{F(a)}$ is a quotient of $\mu(K)$ the group of $n^{th}$-roots of unity in $K$ for all $n$. This shows that $C$ itself is finite, and so, $F(a)$ is $S$-arithmetic (see proof of (1.10) in [11]). $\square$

**Notation 1.17.** Let $G_a$ denote the closure of $F(a)$ in $G(K)^+$ in the $S$-congruence topology on $G(K)^+$. Then, by [12], $G_a$ is a congruence subgroup. Since $F(a)$ is stable under conjugation by $M(A)$, we see that $M(A)$ acts by conjugation on the double coset $F(a) \backslash G(a)/F(a)$. Let $H$ be a $K$-isotropic $K$-simple algebraic $K$-subgroup of $G$, let $\mathcal{H}$ be the closure of $H \cap G(K)^+$ in $G$, $H_0$ the closure of $H \cap G(K)^+$ in $G(a(S))$, $H_0$ is the closure of $H \cap F(a)$ in the $S$-congruence topology on $G(K)^+$. We get an extension

$$1 \to C \cap \mathcal{H} \to \mathcal{H} \to H_0 \to 1.$$
1, i.e. $h_m \in H$ if $m \geq l(a)$, for any fixed nonzero ideal $a \subset A$ which is sufficiently deep. Therefore $bh_m b^{-1} = \xi_m h_m \eta_m$ (by assumption) where $\xi_m, \eta_m \in F(a)$. This shows that $\xi_m \to 1$ and $\eta_m \to 1$ in $G$, which implies that $bcb^{-1} = c$. 

2. Centrality of $C$ when $M_0$ is not abelian.

**Notation 2.1.** We denote by $M_0$, the connected component of identity of the Zariski closure $M_1$ of $M(A)$. We assume in this section that $[M_0, M_0]$ is not trivial. Therefore $[M_0, M_0]$ is a semisimple $K$-group, let $\widetilde{M}_1$ denote the simply connected cover of $[M_0, M_0]$. Now, $\widetilde{M}_1(A)$ is Zariski dense in $\widetilde{M}_1$ since $M_1(A) \cap M_0$ is Zariski dense in $M_0$. Since $\widetilde{M}_1$ is simply connected, by [8] and [9], $\widetilde{M}_1(A)$ has strong approximation.

**Lemma 2.2.** There exists a congruence subgroup $B$ of $\widetilde{M}_1(A)$ such that for any two nonzero ideals $a$ and $b$ of $A$ with $a + b = A$, the group generated by $\widetilde{M}_1(a)$ and $\widetilde{M}_1(b)$ contains $B$.

**Proof.** This is an easy consequence of strong approximation. For details see [12], Section (4.12). 

2.3. Proof of Centrality. We borrow the notation of (4.8) of [12]. Let $f(g)$ be the function defined there. Write, as in (1.5), $g = u_g^m m_g u_g$ for $g \in G_a$. This can be done if $a$ is a sufficiently deep ideal. With respect to the representation $W$ in (4.8) of [12], $f(g)$ is defined, and $u_g^−, m_g, u_g$ have the properties: $f(g) \equiv 1(mod a)$,

$$f(g)^N((u_g^−)_{ij} - \delta_{ij}), f(g)^N[(m_g)_{ij} - \delta_{ij}], f(g)^N[(u_g)_{ij} - \delta_{ij}] \in$$

where $N$ is a large integer depending only on $(G, W)$ and if $T \in End(W), T_{ij}$ denotes its $(ij)^{th}$ entry of viewed as a matrix, with respect to the basis defined in (4.8), [12]. Let $B$ be as in (2.2). Take $\theta \in \widetilde{M}_1(f(g)^{2N})$. Then $\theta g \theta^{-1} = (\theta u_g^− \theta^{-1})(\theta m_g \theta^{-1})(\theta u_g \theta^{-1}) = [\theta, u_g^−] u_g^m m_g u_g[(m_g u_g)^{-1}, \theta]$ and $[\theta, u_g^−]$ and $[(m_g u_g)^{-1}, \theta]$ lie in $F(a)$. This shows that in $F(a) \setminus G(a)/F(a)$, $\widetilde{M}_1(f(g)^{2N})$ acts trivially on $g$. Since $F(a)$ is dense in $G_a$ in the $S$-congruence topology on $G(K)^+$,
one can find $h \in F(a)$ such that $gh \equiv 1 \mod(f(g)^{2N})$, and therefore $f(g)^{2N}$ and $f(gh)^{2N}$ are coprime. We have proved that $\overline{M}_1(f(gh)^{2N})$ fixes $gh \equiv g$ in the double coset. Apply Lemma 2.2 to conclude that $B$ fixes every $g$ in the double coset $F(a) \backslash G(a)/F(a)$. Now, by Lemma 1.18, $C$ is centralised by $B \subset G(K)^+$, and since $G(K)^+$ is simple modulo its centre, $G(K)^+$ centralises $C$.

3. Centrality of $C$ when $G = SU(2,1)$. We first prove a lemma which is very similar to Lemma (2.1) of [14].

**Lemma 3.1.** Suppose $a$ and $b$ are two elements of $A$ such that $aA + bB = A$. Let $L/K$ be a finite separable extension. Consider the $K$-group $T = R_{L/K}(G_m)$ where $R_{L/K}$ is the Weil restriction of scalars, let $N$ be a positive integer and consider the group $T_{a,b,N}$ in $T(A)$ generated by $\{T((a + bx)^N); x \in A\}$. Then the index $f_{a,b,N}$ of $T_{a,b,N}$ in $T(A)$ is bounded by a constant independent of $a, b$.

**Proof.** It is easy to reduce to the case when $L/K$ is a Galois extension. Let $d = \text{degree of } (L/K)$. We will show that $f_{a,b,N} \leq G(d, N, K)$ where $G$ is a function of $d, N$ and $K$. Let $\mathcal{S}$ be the places of $L$ lying above the places of $S$. Then $T(A) = T(O_{\mathcal{S}})$ is commensurable with $O_{\mathcal{S}}^* = G_m(O_{\mathcal{S}})$. Moreover if $\mathfrak{a} \subset A$ is a nonzero ideal and $\mathfrak{a} = \mathfrak{a} \otimes_{O_S} O_{\mathcal{S}}$ denotes the ideal in $O_{\mathcal{S}}$ generated by $\mathfrak{a}$, then $T(\mathfrak{a}) = G_m(\mathfrak{a})$. Now $T(A)/T_{a,b,N}$ is a quotient of $T(A)/T[(a + bx)^N] = G_m(O_{\mathcal{S}})/G_m[(a + bx)^N]$ and the latter is a subgroup of $(O_{\mathcal{S}}/(a + bx)^N)^*$ the group of units in $(O_{\mathcal{S}}/(a + bx)^N)$. This shows that if $\varphi[(a + bx)^N]$ denotes the cardinality of $(O_{\mathcal{S}}/(a + bx)^N)^*$, then $f_{a,b,N}$ divides $\varphi[(a + bx)^N]$ for all $x \in A$. We have, therefore:

$$f_{a,b,N} \leq gcd\{\varphi(a + bx)^N : x \in A\}.$$ 

Write $a + bx = p_1 \cdots p_k$, a product of primes of $A$. Each $p_i$ decomposes as a product of primes $\mathcal{B}$ of $O_{\mathcal{S}}$. Hence

$$\left(\frac{a + bx}{\beta}\right)^N = \prod_{i=1}^{k} \left(\prod_{\beta | p_i} \beta\right)^N.$$
Therefore
\[ \varphi[(a + bx)^N] = \prod_{i=1}^{k} \prod_{\beta | p_i} (\text{Norm } \beta)^{N-1}(\text{Norm }\beta - 1). \]

Since \( L/K \) is Galois, for a given \( p_i \), we have \( \text{Norm } \beta = (\text{Norm } p_i)^{f_i} \) for some integer \( f_i \) dividing \( d \), for each \( \beta | p_i \). Moreover the number \( f_i \) of \( \beta \) lying above \( p_i \) also divides \( d \) (in fact \( f_i r_i \) divides \( d \)). We thus get:

(1) \[ \varphi[(a + bx)^M] = \prod_{i=1}^{k} (\text{Norm } p_i)^{(N-1)f_i r_i}([\text{Norm } p_i]^{f_i} - 1)^{r_i}. \]

Let \( l > 1 \) be a prime and suppose \( l^e | \text{gcd}\{\varphi[(a + bx)^N]; x \in O_S\} \). Then by (1) we have:

(2) \[ l^e | \prod_{i} (Np_i)^{(N-1)f_i r_i}([\text{Norm } p_i]^{f_i} - 1)^{r_i}. \]

Case 1: \( \text{Char}(K) = 0 \) or \( N = 1 \):

Let \( e' \) be the smallest integer such that \( 4de' \geq e \). Then \( e' = [\frac{e}{4d}] + 1 \), where \( x \mapsto [x] \) is the "integral part" function. Let \( q > 1 \) be a prime, suppose \( q^h \) divides the degree \( d(e') \) of \( K(\sqrt{1}/K) \). We may write

(3) \[ (a, K(\sqrt{1}/K)) = \sigma^m \]

where \( (a, K(\sqrt{1}/K)) \) is the Artin symbol and \( \sigma \in Gal(K(\sqrt{1}/K)) \) is a generator. Let \( K_b/K \) be the classifield corresponding to (b) in A. Let \( E \) be the compositum of \( K_b \) and \( K(\sqrt{i}/K) \) is \( \sigma \).

(A) If \( q \) is odd or if \( q = 2 \) and \( m \) even, then \( x(m - x) \not\equiv 0 \) (mod \( q \)) has solutions. We write

(4) \[ (a, E/K) = (\bar{\sigma}^x)(\bar{\sigma}^{(m-x)})(\bar{\sigma}^{-1}\xi)(\bar{\sigma}) \]

where \( \xi \) restricted to \( K(\sqrt{1}/K) \) is trivial. This can be done by (3). We may represent each of the bracketed terms in (4) by (an infinite family of) prime ideals \( p_1, p_2, p_3, p_4 \) by the Cebotarev density theorem. Then from (4) we get:

(5) \[ (a, K_b/K) = (p_1 p_2 p_3 p_4, K_b/K) \]
and so there exists \( \lambda \in K^* \) such that \( \lambda \equiv 1 \pmod{b} \) and \( a' = a\lambda = p_1p_2p_3p_4 \) (by Artin Reciprocity) i.e. \( a' \in A \) and \( a' \equiv a \pmod{b} \), therefore \( a' = a\lambda = p_1p_2p_3p_4 \). This shows from (2), that

\[
\varphi[(a + bx)^{N}] = \prod_{i=1}^{4}(\text{Norm } p_i)^{(N-1)i}((\text{Norm } wp_i)^i - 1)^r.
\]

We have \( l^e | \varphi(a + bx)^N \). We are in the case \( 0 = \text{Char}(K) \) or \( N = 1 \).

If \( \text{Char}(K) = p > 0 \) then \( N = 1 \), whence \( l^e \prod_{i=1}^{4}((N p_i)^{i} - 1)^r \). If \( \text{Char}(K) = 0 \), we use the infinitude of the solutions \( \{p_i\} \) to (4), to pick \( p_i \) such that \( \text{Norm}(p_i) \) is a power of a prime \( p_i > l \). Then again \( l^e \prod_{i=1}^{4}((N p_i)^{i} - 1)^r \). Let \( e_2 \) be the largest power of \( l \) dividing \( (N p_i)^{i} - 1 \). We have \( e \leq r_2e_1 + r_1e_2 + r_3e_3 + r_4e_4 \). Let \( e_M \) be the maximum of \( e_1, e_2, e_3, e_4 \). Then \( e \leq 4e_Md \) (since \( r_i \leq d \)) which shows that \( e_M \geq e' \). Therefore, for some \( i, l^e \) divides \( (N p_i)^{i} - 1 \).

But \( (p_i, K(\sqrt[1]{i})/K) \) sends \( i^r \sqrt[1]{i} \) into \( (i^r \sqrt[1]{i})^{\text{Norm}(p_i)} \), therefore the order of \( (p_i, K(\sqrt[1]{i})/K) \) is equal to 1. From (4) and the fact that \( \sigma | K(\sqrt[1]{i}) = \sigma \), we know that

\[
(p_i, K(\sqrt[1]{i})/K) = \sigma^x \text{ or } \sigma^{m-x} \text{ or } \sigma^{-1} \text{ or } \sigma.
\]

We therefore get: one of the numbers \( xf_1, (m - x)f_2, -f_3 \text{ or } f_4 \) is divisible by the degree \( d(e) \) of \( K(\sqrt[1]{i})/K \) and hence by \( q^h \). By the choice of \( x \), \( x(m - x) \neq 0 \), this means that \( d(e) \) divides \( f_i \) for some \( i \) and \( f_i | d \) for each \( i \). Therefore \( q^h \) divides \( d \) if \( q \) is an odd prime or else if \( (a, K(\sqrt[1]{i})/K) = \sigma^m \) where \( m \) is even.

(B) If \( q = 2 \) but \( m \) is odd, and

\[
(a, K(\sqrt[1]{i})/K) = \sigma^m,
\]

we write

\[
(a, E/K) = (\sigma^m)(\sigma^{-1})(\sigma)
\]

and represent \( \sigma^m, \sigma^{-1} \) and \( \sigma \) by primes \( p_1, p_2, p_3 \). Then \( (p_1, K(\sqrt[1]{i})/K) = \sigma^m \), \( (p_2, K(\sqrt[1]{i})/K) = \sigma^{-1} \), \( (p_3, K(\sqrt[1]{i})/K) = \sigma \).
By (2), we have \( l^e \big| \prod_{i=1}^{3}((Np_i)^{f_i} - 1)^{r_i} \) (we use an argument similar to the one in (A) to choose \( p_i \) such that \( l|(Norm p_i) \)). If \( e_i \) is the largest power of \( l \) dividing \( (Np_i)^{f_i} - 1 \) and \( e_M = \max\{e_1, e_2, e_3\} \), then \( e \leq \sum r_i e_i \leq 3de_M \leq 4de_M \) whence \( e' \leq e_M \), i.e. \( l^{e'} \) divides \( (Np_i)^{f_i} - 1 \) for some \( i \). Now (6) shows that one of the numbers \( f_1m, -f_2, f_3 \) is divisible by \( q^h \), and since \( m \) is odd and \( q = 2 \), this means \( q^h \mid f_1 \) or \( f_2 \) or \( f_3 \) and each \( f_i \) divides \( d \). We have thus proved in all cases that if \( q^h \) is a prime power dividing the degree \( d(e') \), then \( q^h \) divides by a constant depending only on \( (K, d) \) provided \( O = \text{Char}(K) \) or \( N = 1 \). Now \( l^{e'} = l^{[e/4d-1]+1} \) is bounded which means that \( l^{e'} \) is bounded by \( G(K, d, 1) \) whenever \( l^{e'} \mid \gcd\{\varphi(a + bx)^N; x \in A\} \) and we are in case 1. Thus in Case 1, \( f_{a,b,1} \) is bounded by \( G(K, d, 1) \) and if \( \text{Char}(K) = 0 \) then \( f_{a,b,N} \) is bounded by \( G(K, d, 1) \).

Case 2: \( \text{Char}(K) = p > 0 \), and \( N > 1 \). Let \( p^M \geq N \)

\[
\text{(choose } M = 1 + \left\lceil \frac{\log N}{\log p} \right\rceil \text{). Then } T_{a,b,1}^{p^M} \subset T_{a,b,N}, \text{ because } (T(a + bx))^{p^M} \subset T((a + bx)^N) \text{ which shows that } \text{Card } (T(A)/T_{a,b,N}) \leq p^{M(\text{Card}(S)-1)} \text{ Card } (T(A)/T_{a,b,N}) \text{ i.e. } f_{a,b,N} \leq p^{M(\text{Card}(S)-1)} f_{a,b,1} \text{ and we have shown in Case 1 that } f_{a,b,1} \leq G(d, K, 1). \text{ Therefore } f_{a,b,N} \leq G(K, d, N) \text{ in all cases.} \quad \square \]

**NOTATION 3.2.** We observe that for \( G = SU(2,1) \), for any \( K \)-algebra \( A \), we have

\[
G(A) = \left\{ g \in SL_3(L \otimes A); \quad \sigma^t(g) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},
\]

where \( L/K \) is a Galois extension of degree 2 whose Galois group is generated by \( \sigma, \sigma \) acts on \( L \otimes A \) by its action on \( L \) and on the group \( SL_3(L \otimes A) \) by acting entrywise, \( g^t \) is the transpose of the matrix \( g \). Then

\[
U^+(K) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}; \quad N(x) = tr_{L/K}(y), \quad x, y \in L \right\},
\]

\[
U^-(K) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -\bar{x} & 1 \end{pmatrix}; \quad N(x) = tr_{L/K}(y), \quad x, y \in L \right\},
\]

where \( \text{Card}(S) \) is a constant depending only on \( (K, d) \).
Thus, $M = R_{L/K}(G_m)$. Take $W$ to be the standard representation of $G$ on $L^3$ and $f(g) = a_{11}(g)$ i.e. the $(1,1)$-th entry of $g$. Look at the action of $M(OS)$ on $F(a) \backslash G(a)/F(a)$, one can write

\[
(*) \quad g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_{21}/a_{11} & 1 & 0 \\ a_{31}/a_{11} & -(a_{21}/a_{11}) & 1 \end{pmatrix} 
\times \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \bar{a}_{11}/a_{11} & 0 \\ 0 & 0 & \bar{a}_{31}^{-1} \end{pmatrix} \begin{pmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ 0 & 1 & -a_{12}/a_{11} \\ 0 & 0 & 1 \end{pmatrix}.
\]

Suppose $\rho : SL_2 \to SU(2,1)$ is a $K$-representation (nontrivial) such that

\[
\rho \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in G \right\} \subset U^+, \quad \rho \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}; x \in G \right\} \subset U^+ 
\]

then $\rho \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}; t \in G \right\} \subset M$. Moreover, $f \left[ \rho \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \right] = a$ or $a^2$ as can be easily seen. If $g \in [\rho(SL_2)]$, then (*) shows that $M(a^2)$ acts trivially on $g = \rho \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$. Therefore $M((a + bx)^2)$ acts trivially on $\bar{g}$ in $F(a) \backslash F(a)\rho(SL_2)F(a)/F(a)$. Now Lemma 2.2 shows that there is a fixed subgroup $T_0$ of $M(OS)$ such that $T_0$ acts trivially on $F(a) \backslash F(a)[\rho(SL_2)]aF(a)/F(a)$. Hence, if $H_\rho = \rho(SL_2)$, then by Lemma 1.18, $T_0$ acts trivially on $C \cap H_\rho$.

Then $T_0$ acts trivially on $H_\rho \cap C$, and $H_\rho(K)$ acts on $C \cap H_\rho$. But, if $2 \neq \text{Char } (K)$ then $T_0$ and $H_\rho(K)$ generate $G(K)^+$, which shows that $G(K)^+$ acts, and acts trivially on $C \cap H_\rho$. Thus $H_\rho(K) \cap U^+(K_w)$ and $H_\rho(K) \cap U(K_{\bar{w}})$ commute if $w, w' \notin S$, $\ w \neq w'$. By [12], this implies $U^+(K_w)$ and $U(K_w)$ commute, whence $C$ is centralised by $G(K)^+$.

4. The case when $M_0$ is abelian. In this case we have embeddings of $H = R_{L/K}SU(2,1)$ or of $H = SL_2$ in $G$ where $R_{L/K}(SU(2,1))$
of $SL_2$ has $S$-rank at least 2 ([12]). Therefore $(U^+ \cap H)(K_v)$ and $(U^- \cap H)(K_w)$ commute if $v \neq w, v, w \notin S$. By Lemma (2.1) of [12], this means that $[U^+(K_v), U^-(K_w)] = 1$ in $G$ for all $v, w \in S, \ v \neq w$, i.e. $C$ is central.

**Conclusion.** We have shown that

$$1 \to C \to G \to G(A(S)) \to 1$$

is a central extension in all cases, whence $C$ is finite by [11], i.e. $F(a)$ is a subgroup of finite index in $G(a)$. Now $F(a)$ normalises $E(a)$. Hence, again by [12], $E(a)$ has finite index in $G(a)$. This completes the proof of the theorem of the introduction.

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**References**


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