

*Pacific  
Journal of  
Mathematics*

**IMMERSIONS UP TO JOINT-BORDISM**

GUI SONG LI

## IMMERSIONS UP TO JOINT-BORDISM

GUI-SONG LI

**A necessary and sufficient condition for a map to be joint-bordant to an immersion is given in terms of Stiefel-Whitney numbers.**

**1. Introduction.** This note is devoted to a study of immersions of manifolds into manifolds up to joint-bordism. We will work throughout in the category of smooth manifolds and smooth maps.

A map of dimension  $(n, k)$  is a map of a closed  $n$ -manifold into a closed  $(n+k)$ -manifold. Two maps  $f_0 : M_0 \rightarrow N_0$  and  $f_1 : M_1 \rightarrow N_1$  of dimension  $(n, k)$  are said to be *joint-bordant* if there is a map  $F : V \rightarrow W$  extending  $f_0 \cup f_1$  where  $V$  and  $W$  are compact manifolds with  $\partial V = M_0 \cup M_1$  and  $\partial W = N_0 \cup N_1$ . Joint-bordism classes of maps of dimension  $(n, k)$  form an abelian group under the disjoint union which we denote by  $M(n, k)$ . It is well known that Stiefel-Whitney numbers form a complete system of invariants for the joint-bordism theory [9]. So one may hope to characterize maps joint-bordant to immersions or embeddings in terms of these numbers whenever  $k > 0$ . For the case of embeddings this has already been settled by Brown [3]; his proof is based on a construction suggested by Stong. In this note, using the *model construction* of Koschorke [6], we shall give such a criterion for maps joint-bordant to immersions in the “metastable” range  $n \leq 2k$ .

Our method of proof can also be applied to study immersions up to various oriented joint-bordism relations. These are naturally defined for the following restricted classes of maps (see Stong [9]):

- $C_1$ : maps with oriented source manifolds;
- $C_2$ : maps with oriented target manifolds;
- $C_3$ : maps with oriented stable normal bundles;
- $C_4$ : maps with oriented source and target manifolds.

In fact, let  $M_r(n, k)$  be the resulting oriented joint-bordism group of maps of dimension  $(n, k)$  belonging to  $C_r$  and let  $\rho_r : M_r(n, k) \rightarrow M(n, k)$  be the natural forgetful homomorphism. Then  $x \in M_r(n, k)$  contains an immersion if  $\rho_r(x)$  does provided  $n \leq 2k$ .

We now summarize our main results by the following

**MAIN THEOREM.** *Let  $n \leq 2k$ . Then under all five orientedness assumptions above a map  $f : M \rightarrow N$  of dimension  $(n, k)$  is joint-bordant to an immersion if and only if*

$$w_m(f)f^*w_{\mu_1}(N)w_{\mu_2}(M) \cap [M] = 0$$

for all  $m > k$  and all partitions  $\mu_1, \mu_2$  with  $|\mu_1| + |\mu_2| = n - m$ . Here  $w(f) = f^*w(N)\bar{w}(M)$  is the total Stiefel-Whitney class of the stable normal bundle of  $f$  and  $[M]$  is the  $\mathbf{Z}_2$ -fundamental class of  $M$ .

**REMARK.** Stong [9] showed that the unoriented joint-bordism class of  $f$  is completely determined by Stiefel-Whitney numbers of the form

$$w_{\mu_1}(N)f_1w_{\mu_2}(M) \cdots f_lw_{\mu_l}(M) \cap [N]$$

which, in the case of  $l > 1$ , are equal to

$$(*) \quad f^*w_{\mu_1}(N)w_{\mu_2}(M)f^*f_1w_{\mu_3}(M) \cdots f^*f_lw_{\mu_l}(M) \cap [M].$$

Here  $\mu_i$  are partitions with  $\sum |\mu_i| = n - (l-2)k$  and  $f_l : H^*(M, \mathbf{Z}_2) \rightarrow H^{*+k}(N, \mathbf{Z}_2)$  is the *Umkehr* homomorphism defined by taking a cohomology class  $x$  into the Poincaré dual of  $f_*(x \cap [M])$ . In the case of  $n \leq 2k$  the  $f^*f_lw_{\mu_i}(M)$  factors above disappear if  $|\mu_1| + |\mu_2| > k$ . So our main theorem is equivalent to saying that: “if  $n \leq 2k$  then under all five orientedness assumptions above  $f$  is joint-bordant to an immersion if and only if all Stiefel-Whitney numbers of the form (\*) involving  $w_m(f)$  where  $m > k$  are zero.” I do not know whether this statement remains true if  $n > 2k$ .

**REMARK.** It was conjectured by Olk [8] that in a certain “meta-stable” range (probably  $n \leq 2k - 1$ ) a closed  $n$ -manifold  $M$  can be immersed into  $\mathbb{R}^{n+k}$  up to bordism, or equivalently, a map  $f : M \rightarrow \mathbb{S}^{n+k}$  is bordant in the sense of Atiyah [1] to an immersion if and only if

$$\bar{w}_m(M)w_\mu(M) \cap [M] = 0$$

for all  $m > k$  and all partitions  $\mu$  of  $n - m$ . By our results, these numbers are zero if and only if  $f$  is joint-bordant in  $M_2(n, k)$  (even in  $M_4(n, k)$  if  $M$  is oriented) to an immersion provided  $n \leq 2k$ . Olk [8] showed that the above statement is always true if  $k \geq n - 7$ . So in these cases “joint-bordant to an immersion” and “bordant to an immersion” are the same for a map into a sphere.

I would like to thank the Alexander von Humboldt-Stiftung for financial support and Ulrich Koschorke for his hospitality. I also want to thank M.A. Aguilar for drawing my attention to the work of R.L.W. Brown.

**2. The unoriented case.** We start by describing the behavior of a generic map around its singularity subset. Recall that a map is *generic* if its self-intersections are transversal and its jet sections are transversal to the Boardman manifolds [2]. Given a generic map  $f : M \rightarrow N$  of dimension  $(n, k)$ , we denote by  $\Sigma_f$  the set of singular points of  $f$  and by  $\bar{\Sigma}_f = f(\Sigma_f)$  its image. Hereafter we always assume  $k > 0$ . In the case of  $n \leq 2k + 2$ ,  $\Sigma_f$  is a closed submanifold of dimension  $n - k - 1$  containing only points over which  $df$  is of rank  $n - 1$ ,  $f$  is an embedding restricted to  $\Sigma_f$  and the bordism class of  $\Sigma_f$  depends only on the joint-bordism class of  $f$ .

Let us denote by  $\text{Ker}$  and  $\text{Coker}$  the 1- and  $(k + 1)$ -dimensional vector bundles over  $\Sigma_f$  and  $\bar{\Sigma}_f$  which are determined by kernels and cokernels of  $df|_{\Sigma_f}$  respectively. By Koschorke [6] the normal bundle  $\nu$  of  $\Sigma_f$  in  $M$  is canonically isomorphic to  $\text{Ker} \otimes \text{Coker}$ . Here and henceforth we shall always omit the pull-backs of vector bundles by  $f|_{\Sigma_f}$ . Now let  $\bar{\nu}$  be the normal bundle of  $\bar{\Sigma}_f$  in  $N$  and let  $\bar{f} : E\nu \rightarrow E\bar{\nu}$  be the map between total spaces of normal bundles determined by the  $k$ -morphism  $\nu \rightarrow \bar{\nu}$  induced by  $df$ . Then by applying the model construction given in Section 1 of Koschorke [6] to the bundle homomorphism  $df|_{\Sigma_f}$  of constant rank  $n - 1$ , we obtain a nondegenerate  $(n - 1)$ -morphism  $T(E\nu) \rightarrow T(E\bar{\nu})$  covering  $\bar{f}$  with  $\Sigma_f$  its singularity subset, and hence a generic map

$$\alpha_f : E\nu \rightarrow E\bar{\nu}$$

with  $\Sigma_f$  its singularity subset by Feit [5]. Moreover, suitably identifying  $E\nu$  and  $E\bar{\nu}$  with certain tubular neighborhoods of  $\Sigma_f$  in  $M$

and of  $\overline{\Sigma}_f$  in  $N$  respectively, we can identify  $\alpha_f$  with  $f$  restricted to a small tubular neighborhood of  $\Sigma_f$  in  $M$  (see [6]).

Now let  $x \in M(n, k)$  be represented by a generic map  $f : M \rightarrow N$  and let  $n \leq 2k + 2$ . We denote by

$$\beta_f : \Sigma_f \rightarrow BO_{k+1} \times \mathbb{P}^\infty$$

the map defined by the classifying maps of Coker and Ker. The bordism obstruction  $\Phi(x)$  is defined to be the unoriented bordism class of  $\beta_f$ . It is readily seen that

$$\Phi : M(n, k) \rightarrow \mathfrak{R}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty)$$

is a well defined homomorphism.

**THEOREM .** *If  $n \leq 2k$  then  $x \in M(n, k)$  contains an immersion if and only if  $\Phi(x) = 0$ .*

*Proof.* If  $x$  contains an immersion then clearly  $\Phi(x) = 0$ . To prove the contrary let  $f : M \rightarrow N$  be a generic map of dimension  $(n, k)$  so that  $\beta_f$  is null-bordant. We have to show that  $f$  is joint-bordant to an immersion.

In the dimensional range  $n \leq 2k + 1$  there is a  $k$ -dimensional vector bundle  $\nu_0$  over  $\Sigma_f$  so that  $\nu \cong \nu_0 \oplus \text{Ker}$ . In this case it is easy to see that  $\overline{\nu} \cong \nu_0 \oplus \text{Coker}$  and that the  $k$ -morphism  $\nu \rightarrow \overline{\nu}$  induced by  $df$  then corresponds to the  $k$ -morphism

$$\nu_0 \oplus \text{Ker} \rightarrow \nu_0 \oplus \text{Coker}$$

defined by  $a \oplus b \rightarrow a$ . Let us denote by  $\lambda$  and  $\gamma$  the canonical vector bundles over  $\mathbb{P}^\infty$  and  $BO_{k+1}$  respectively, and by  $G : \Sigma \rightarrow BO_{k+1} \times \mathbb{P}^\infty$  a null-bordism of  $\beta_f$ . For dimensional reasons  $\nu_0$ , which is stably isomorphic to  $G^*\lambda \otimes G^*\gamma - G^*\lambda$  restricted over  $\Sigma_f$ , can be extended to a  $k$ -dimensional vector bundle  $\eta_0$  over  $\Sigma$  so that  $G^*\lambda \otimes G^*\gamma \cong \eta_0 \oplus G^*\lambda$ . Now let  $\eta = G^*\lambda \otimes G^*\gamma$  and let  $\overline{\eta} = \eta_0 \oplus G^*\gamma$ . Then  $\eta$  and  $\overline{\eta}$  restrict to  $\nu$  and  $\overline{\nu}$  over  $\Sigma_f$  respectively, and the  $k$ -morphism

$$\eta \cong \eta_0 \oplus G^*\lambda \rightarrow \eta_0 \oplus G^*\gamma \cong \overline{\eta}$$

defined by  $a \oplus b \rightarrow a$  restricts to the  $k$ -morphism  $\nu \rightarrow \overline{\nu}$  induced by  $df$ . Applying the model construction of Koschorke to the obvious

bundle homomorphism  $\eta \oplus T\Sigma_f \rightarrow \bar{\eta} \oplus T\Sigma_f$  of constant rank  $n - 1$ , we obtain a generic map

$$F : E\eta \rightarrow E\bar{\eta}$$

with  $\Sigma_f$  its singularity subset, which, after identifying  $\Sigma_f \subset E\bar{\eta}$  with  $\bar{\Sigma}_f$  via  $f$ , is readily seen to be an extension of the map  $\alpha_f$ .

Now let  $M_1$  be defined by glueing  $M - \text{int}T$  and a sphere bundle  $S\eta$  of  $\eta$  along the boundaries via a diffeomorphism between  $\partial T$  and  $S\nu$  where  $T$  is a small tubular neighborhood of  $\Sigma_f$  in  $M$ , and let  $N_1$  be defined similarly. Then  $M_1$  and  $N_1$  are closed smooth manifolds after straightening possible angles. Moreover, if the above constructions are suitably done, then  $f$  and  $F$  can be fitted to yield an immersion  $f_1 : M_1 \rightarrow N_1$  which is readily seen to be joint-bordant to  $f$ . This completes the proof of the theorem.  $\square$

We now calculate the Stiefel-Whitney numbers of the map  $\beta_f : \Sigma_f \rightarrow BO_{k+1} \times \mathbb{P}^\infty$  associated to a given generic map  $f : M \rightarrow N$  of dimension  $(n, k)$ . By definition the Stiefel-Whitney numbers of  $\beta_f$  take the form

$$w_{\mu_1}(\text{Ker})w_{\mu_2}(\text{Coker})w_{\mu_3}(\Sigma_f) \cap [\Sigma_f]$$

where  $\mu_i$  are partitions with  $\sum |\mu_i| = n - k - 1$ . We have the following simple relations among total Stiefel-Whitney classes:

$$w(\text{Ker})w(\text{Coker}) = w(f)|_{\Sigma_f}$$

$$w(\text{Ker} \otimes \text{Coker})w(\Sigma_f) = w(M)|_{\Sigma_f}.$$

It follows that every Stiefel-Whitney number of  $\beta_f$  is the sum of numbers of the form

$$w_1(\text{Ker})^m(f^*w_{\mu'_1}(N)w_{\mu'_2}(M))|_{\Sigma_f} \cap [\Sigma_f]$$

which by 9.11 of Koschorke [6] are equal to

$$w_{m+k+1}(f)f^*w_{\mu'_1}(N)w_{\mu'_2}(M) \cap [M].$$

This proves our main theorem for the unoriented case.

**3. Oriented cases.** We now extend the argument above to study various oriented cases. For our purpose we shall need the notion of *oriented bordism group with coefficients*. Let  $\phi$  be a vector bundle over  $X$ , we denote by  $\overline{\Omega}_n(X, \phi)$  the bordism group of triples  $(M, f, \text{or})$  where  $M$  is a closed  $n$ -manifold,  $f : M \rightarrow X$  is a map and  $\text{or}$  is an isomorphism between orientation line bundles of  $TM$  and  $f^*\phi$ . This kind of bordism groups was first studied by Atiyah [1] and then by Koschorke [6]. Our notational convention follows from that of Koschorke. It should be noticed that  $\overline{\Omega}_*(X, \phi) \cong \Omega_*(X)$  if  $\phi$  is orientable and that

$$\overline{\Omega}_*(X \times \mathbb{P}^\infty, \lambda + \phi) \cong \mathfrak{R}_*(X).$$

The bordism group  $\overline{\Omega}_*(X, \phi)$  has several properties analogous to the usual oriented bordism group, including a generalization of Rochlin's theorem given in [4].

Similar to the unoriented case, if  $n \leq 2k + 2$  then for each  $1 \leq r \leq 4$  we can define a homomorphism  $\phi_r : M_r(n, k) \rightarrow B_r$  where

$$\begin{aligned} B_1 &= \overline{\Omega}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty, \lambda \otimes \gamma_{k+1}), \\ B_2 &= \overline{\Omega}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty, \lambda \otimes \gamma_{k+1} + \gamma_{k+1} - \lambda), \\ B_3 &= \mathfrak{R}_{n-k-1}(BSO_l \times \mathbb{P}^\infty), \quad l \gg 0, \\ B_4 &= \overline{\Omega}_{n-k-1}(BSO_l \times \mathbb{P}^\infty, \lambda \otimes (\tilde{\gamma}_l - \mathbb{R}^{l-k-1})), \quad l \gg 0, \end{aligned}$$

$\gamma_{k+1}$  and  $\tilde{\gamma}_l$  are canonical vector bundles over  $BO_{k+1}$  and  $BSO_l$  respectively. To see this notice that if  $f$  is a generic map belonging to the class  $C_r$  then the map  $\beta_f$  defined in Section 1 also represents a bordism class of  $B_r$ . For example, if  $r = 3$  then in the definition of  $\beta_f$  the bundle  $\text{Coker} - \text{Ker}$  can be canonically oriented. Let  $\tau : BO_{k+1} \times \mathbb{P}^\infty \rightarrow BO_l \times \mathbb{P}^\infty$  be defined by the classifying map of  $\gamma_{k+1} - \lambda$  and  $\lambda$ . Then  $\tau \circ \beta_f$  can be lifted to a map into  $BSO_l \times \mathbb{P}^\infty$  which represents a bordism class of  $B_3$ .

Now, a similar argument as in the unoriented case leads to the following

**THEOREM .** *If  $n \leq 2k$  then for each  $1 \leq r \leq 4$  a joint-bordism class  $x \in M_r(n, k)$  contains an immersion if and only if  $\Phi_r(x) = 0$ .*

**REMARK.** In [7] we defined a normal bordism obstruction for a map to be bordant in the sense of Atiyah to an immersion as

well as its various oriented versions. Since the invariants  $\Phi_r$  here are images under certain forgetful homomorphisms and projections of our invariants in [7], the images of  $\Phi_r$  contain only 2-primary torsion elements. Later we will see that all these elements are in fact of order 2.

We now proceed to prove our main theorem for the remaining four oriented cases. In what follows, when  $l$  is large enough,  $B(S)O_l$  and  $\gamma_l$  will be simply denoted by  $B(S)O$  and  $\gamma$  respectively.

If  $r = 1$  and  $k$  is even, or if  $r = 2$  and  $k$  is odd, or if  $r = 3$ , or if  $r = 4$  and  $k$  is even, then  $B_r$  is canonically isomorphic to  $\mathfrak{R}_{n-k-1}(B(S)O)$  or to  $\mathfrak{R}_{n-k-1}(BSO \times \mathbb{P}^\infty)$ . In these cases, our result follows by calculating characteristic numbers as in the unoriented case.

If  $r = 2$  and  $k$  is even, or if  $r = 4$  and  $k$  is odd, then  $B_r$  is canonically isomorphic to  $\Omega_{n-k-1}(B(S)O \times \mathbb{P}^\infty)$  which contains only  $\mathbb{Z}$  and  $\mathbb{Z}_2$  factors. It follows that  $\Phi_r(x)$  is an element of order 2 and hence is determined by its Stiefel-Whitney numbers (see [4]). In these cases, our result can again be proved by calculating characteristic numbers.

It remains to consider the case when  $r = 1$  and  $k$  is odd. Since  $B_1 \cong \overline{\Omega}_{n-k-1}(BO \times \mathbb{P}^\infty, \gamma)$  if  $k$  is odd and since  $\Phi_1(x)$  is a 2-primary torsion by our last remark, it suffices to show that the natural forgetful homomorphism

$$\rho : \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) \rightarrow \mathfrak{R}_*(BO \times \mathbb{P}^\infty)$$

is injective restricted to the 2-primary torsion subgroup of  $\overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma)$ . For this purpose we fit  $\rho$  into the following commutative diagram

$$\begin{array}{ccccc} \Omega_*(BSO \times \mathbb{P}^\infty) & \xrightarrow{\pi_*} & \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) & \longrightarrow & \Omega_{*-1}(BO \times \mathbb{P}^\infty) \\ & \searrow \bar{e} & \downarrow e & & \downarrow i_* \\ & & \mathfrak{R}_*(BO \times \mathbb{P}^\infty) & \longrightarrow & \Omega_{*-1}(BO \times \mathbb{P}^\infty \times \mathbb{P}^\infty) \end{array}$$

where  $\bar{e}$  is the natural forgetful homomorphism,  $i_*$  is the injection induced by the natural inclusion and  $\pi_*$  is induced by the projection. The top line is the exact Gysin sequence corresponding to the orientation line bundle of  $\gamma$  and the lower line is defined similarly

by identifying  $\mathfrak{R}_*(BO \times \mathbb{P}^\infty)$  with  $\overline{\Omega}_*(BO \times \mathbb{P}^\infty \times \mathbb{P}^\infty, \lambda + \gamma)$  (see 9.21(ii) of [6]).

Now let  $x \in \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma)$  be a 2-primary torsion so that  $\rho(x) = 0$ . To prove  $x = 0$  we first conclude from the exactness of the top line that there exists  $y \in \Omega_*(BSO \times \mathbb{P}^\infty)$  with  $\pi_*(y) = x$ . Since  $\bar{\rho}$  is injective restricted to the 2-primary torsion part of  $\Omega_*(BSO \times \mathbb{P}^\infty)$ , it suffices to show that  $y$  is a 2-primary torsion. For this, let us consider the homomorphism

$$d : \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) \longrightarrow \Omega_*(BSO \times \mathbb{P}^\infty)$$

defined by taking double covers. By definition  $d \circ \pi_* = id + T$  where  $T$  is the involution on  $\Omega_*(BSO \times \mathbb{P}^\infty)$  induced by  $t$  and  $t$  is the involution on  $BSO$  defined by interchanging the two sheets. Since the involution on  $H_*(BSO, \mathbb{Z})$  induced by  $t$  is the identity, so is the involution  $T$ . It follows that  $2y = d \circ \pi_*(y) = d(x)$ , and hence  $y$  itself, is a 2-primary torsion as desired.

This completes the proof of our main theorem for the oriented cases.

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Received April 2, 1992, revised July 9, 1993 and accepted for publication August 19, 1993.

INSTITUTE OF SYSTEMS SCIENCE  
ACADEMIA SINICA  
BEIJING 100080, CHINA





# PACIFIC JOURNAL OF MATHEMATICS

Volume 166 No. 2 December 1994

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0030-8730(1994)166:2;1-F