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It is proved that if Dehn surgery on a strongly invertible knot, which is not a satellite knot yields a manifold containing an incompressible torus, then the slope of the surgery consists of a certain number of meridians and at most two longitudes. Furthermore, if the slope has two longitudes, then there is an incompressible torus which meets the surgered solid torus twice.

Introduction. Let k be a knot in S^3, and consider the following construction: Take a solid torus neighborhood $\eta(k)$ of k, remove it, and glue it back differently. Let $M_k = S^3 - int\eta(k)$. The different regluings are parameterized by the isotopy class r, slope, of the simple closed curve on the torus ∂M_k that bounds a meridional disk in the reglued solid torus. Denote the resulting closed 3-manifold by $M_k(r)$, we say that it is obtained by r-Dehn surgery on k. Slopes on ∂M_k are parameterized by $Q \bigcup \{1/0\}$, using a meridian-longitude basis $\{\mu, \lambda\}$ for $H_1(\partial M_k)$. Then r corresponds to p/q if and only if $[r] = p\mu + q\lambda$ in $H_1(\partial M_k)$. $\Delta(r, s)$ denotes the minimal geometric intersection number of two slopes r, s on ∂M_k. If r, s correspond to p/q and a/b respectively, then it can be shown that $\Delta(r, s) = |pb - qa|$. For an excellent exposition of the main problems on Dehn surgery on knots see the survey paper of C. McA. Gordon [**7**].

We consider the following problem: Suppose k is not a satellite knot, i.e. M_k does not contain any incompressible, non-boundary parallel torus. When is possible that $M_k(r)$ does contain an incompressible torus? i.e. when an essential torus can be created after surgery?

If k is not a satellite and not a torus knot, then by results of W. Thurston [**20**], k is hyperbolic and $M_k(r)$ is hyperbolic for all but finitely many r. Also, results of S.A. Bleiler and C.G. Hodgson [**1**] show that if k is hyperbolic then $M_k(r)$ has a Riemannian metric of negative curvature for all but at most 24 values of r. These results imply that if k is not a satellite knot then $M_k(r)$ may contain incompressible tori at most for 24 values of r. A result of C.McA. Gordon [**8**] says that if both $M_k(r)$ and $M_k(s)$ contain an incompressible torus then $\Delta(r, s) \leq 8$; this also implies that for all but finitely many r, $M_k(r)$ does not contain incompressible tori. But these results do not give information about which values of the slope r are possible, in case $M_k(r)$ does contain an incompressible torus. In [**7**], Gordon conjectured that if $M_k(r)$ contains an incompressible torus then $\Delta(r, \mu) \leq 2$, in other words, r is homologous to several meridians and at most two longitudes.

In this paper we prove Gordon's conjecture for the case when k is a strongly invertible knot (Theorem 6.3); we show also that if $M_k(r)$ contains incompressible tori and $\Delta(r, \mu) = 2$, then there is an incompressible torus T in $M_k(r)$ which intersects the reglued solid torus in two meridional disks, or in other words, there is a properly embedded, incompressible, punctured torus in M_k whose boundary consist of two curves on ∂M_k of slope r. This last statement can be seen as a kind of generalization of the cabling conjecture, which says that only certain surgery on cable knots yields reducible manifolds, or more explicitly, if $M_k(r)$ is reducible then there is a properly embedded, essential annulus on M_k whose boundary has slope r on ∂M_k. There are examples of knots k, where k is not satellite, $M_k(r)$ contains an incompressible torus for some r so that $\Delta(r, \mu) = 2$ and there is an incompressible torus hitting the surgered solid torus twice. The simplest example I know is 37/2-surgery on the (-2,3,7) pretzel knot (see [**10**]). We have an infinite family of such examples; in those knots there is a non-integral surgery producing a manifold containing an incompressible torus, which hits the surgered solid torus twice, and which divides the manifold into two Seifert fiber spaces with base a disk and 2 exceptional fibers. Those knots also admit two integral surgeries which yield Seifert fiber spaces with base a sphere and at most 3 exceptional fibers. Then for those knots there are 3 different surgeries producing non-hyperbolic manifolds. It is also satisfied that any two of those exceptional surgeries are at distance 1. Those examples will be explained in a forthcoming paper.

C. McA. Gordon and J. Luecke have announced a proof of Gordon's conjecture for all knots. It is still unknown in the general case, if when $\Delta(r, \mu) = 2$ there is an incompressible torus hitting the surgered solid torus twice. In the case when $\Delta(r, \mu) = 1$ not too much is known; it is not known for example if there is an upper bound for the number of times an incompressible torus may hit the surgered solid torus.

The proof of Theorem 6.3 is somehow inspired by the solution of the cabling conjecture for strongly invertible knots [**6**]. First, if for a strongly invertible knot k, $M_k(r)$ contains an incompressible torus, then by the equivariant torus Theorem of W.H. Holzmann [**11**], there is an incompressible torus equivariant under the involution of $M_k(r)$. By taking quotients, the surgery problem is tranlated into a problem of sums of tangles. In §1 we state the required results about sums of tangles, which are Theorems 1.3, 1.4 and Corollaries 1.5 and 1.6. In §2, 3 and 4 we prove Theorems 1.3 and 1.4 by using sutured manifold theory and a combinatorial argument. Corollary 1.5 is proved in §5, and Corollary 1.6 in §1, for it follows easily from 1.4. Through the paper we assume familiarity with [**16**] and [**6**]. In §6 we apply the Theorems on tangles to get a proof of Theorem 6.3.

1. Theorems from tangle theory.

1.1. A tangle (B, t) is a pair that consists of a 3-ball B and a pair of disjoint arcs and simple closed curves t properly embedded in B. Let B_1 be the unit ball in R^3, and let a, b, c, d, be four points in ∂B_1 lying in the lines $Y = Z$, $X = 0$ and $Y = -Z$, $X = 0$.

A tangle (B_1, t) is rational if:

(a) It is a trivial tangle, i.e. there is an homeomorphism of pairs from (B_1, t) to the tangle $(D^2 \times I, \{x, y\} \times I)$ where D^2 is the unit ball in R^2 and x, y are distinct points in the interior of D^2.

(b) $t \bigcap \partial B_1 = \{a,\ b,\ c,\ d\}$.

Two rational tangles (B_1, t), (B_1, t') are equivalent if there is an homeomorphism of pairs $h : (B_1, t) \to (B_1, t')$ such that $h|_{\partial B_1} = id$.

There is a "natural" one to one correspondence between rational tangles and $Q \bigcup \{1/0\}$ (see [**2**], [**4**], [**14**]). Denote by $(B_1, p/q)$ the rational tangle determined by $p/q \in Q \bigcup \{1/0\}$. As $(B_1, p/q)$ is a trivial tangle, there is a disk $D_{p/q}$ properly embedded in B_1 which separates the strings of $(B_1, p/q)$. Let $J_{p/q} = \partial D_{p/q}$, it is a simple closed curve in $\partial B_1 - \{a, b, c, d\}$. Define the distance between two rational tangles $(B_1, p/q)$ and $(B_1, r/s)$, denoted by $\Delta(p/q, r/s)$, as half of the minimal number of intersection between the curves $J_{p/q}$ and $J_{r/s}$. It can be shown that $\Delta(p/q, r/s) = |ps - qr|$.

A tangle (B, t) is *prime* if has the following properties:

(a) It has no local knots, i.e. any S^2 in B which meets t transversally in two points, bounds in B a ball meeting t in an unknotted spanning arc;

(b) There is no disk properly embedded in B which separates the strings of (B, t);

(c) $B - t$ is irreducible, i.e. any sphere in B disjoint from t bounds a 3-ball disjoint from t.

A knot or link k is *doubly composite* if it can be expressed as the sum of two prime tangles, i.e., there is a sphere S meeting k transversally in four points, such that each of the balls bounded by S determines, with its intersection with k, a prime tangle. Such a sphere is called a *tangle-decomposing* sphere, or simply a *decomposing sphere.* A knot or link k is *doubly prime* if it is prime and is not doubly composite. A knot or link k is a *satellite knot* or *link* if there is an incompressible torus in $S^3 - k$ which is not parallel to a component of $\partial\eta(k)$. Such a torus is called a *satellite torus.*

A *Seifert surface* for a link k is a compact, orientable surface none of whose components is closed and whose boundary is the link. Define $\chi(k)$ to be the maximal Euler characteristic of all Seifert surfaces for k.

1.2. Let k be a link in S^3. Let B be a 3-ball in S^3 which intersects k in two arcs, and such that $(B, B \cap k)$ is a trivial tangle. Suppose also that $(B', B' \cap k)$, where $B' = cl(S^3 - B)$, is a prime tangle. Fix an homeomorphism of pairs $h : (B_1, 1/0) \to (B, B \cap k)$. Define a new link $k(B, p/q)$ by changing $(B, B \cap k)$ by $h\big((B_1, p/q)\big)$. $k(B, 1/0)$ is just k. For simplicity denote $h((B_1, p/q))$ by $(B, p/q)$, and $h(J_{p/q})$ by $J_{p/q}$.

Let S be a Seifert surface for k with $\chi(S) = \chi(k)$; it is incompressible. S can be isotoped so that it intersects the 3-ball B in a collection of disks; two of them have as boundary one arc of $B \cap k$ plus one arc in ∂B; the other disks have as boundary a curve parallel to $J_{1/0}$, so $S \cap \partial B$ consist of two arcs and a collection of simple closed curves. If S is given an orientation, this induces an orientation on each simple closed curve of $S \bigcap \partial B$. We say that B intersects S always in the same direction if all the curves $S \bigcap \partial B$, with the induced orientation, are homologous in $\partial B - h(\{a, b, c, d\})$.

If $k(B, p/q)$ is doubly composite or satellite and P is a decomposing sphere or satellite torus, then P can be isotoped so that it intersects B in a collection of disks, whose boundaries are parallel to the curve $J_{p/q}$ on ∂B.

THEOREM 1.3. *Let k be a link and B a 3-ball as before. Suppose that $k(B, p/q)$ is a doubly composite link. Let P be a tangle-decomposing sphere for $k(B, p/q)$, isotoped to intersect $(B, p/q)$ in a minimal number of disks. If $\Delta(p/q, 1/0) > 3$ then one of the following holds*

(a) P is disjoint from B.

(b) $S^3 - k$ is irreducible and there is a Seifert surface S for k with $\chi(S) = \chi(k)$, and such that S intersects ∂B only in two arcs, which join the points $h(\{a, b, c, d\})$.

If $\Delta(p/q, 1/0) = 3$, then either (a), (b), or

(c) $S^3 - k$ is irreducible and there is a Seifert surface S for k with $\chi(S) = \chi(k)$, and such that B intersects S always in the same direction. Furthermore, P meets ∂B exactly in one curve parallel to $J_{p/q}$.

If $\Delta(p/q, 1/0) = 2$, then either (a), (b), or

(d) $S^3 - k$ is irreducible and there is a Seifert surface S for k with $\chi(S) = \chi(k)$, and such that B intersects S always in the same direction. Furthermore, P meets ∂B exactly in two curves parallel to $J_{p/q}$; or

(e) P meets ∂B exactly in one curve parallel to $J_{p/q}$.

THEOREM 1.4. *Let k be a link and B a 3-ball as before. Suppose that $k(B, p/q)$ is a satellite link, and T is a satellite torus, T not a swallow-follow torus. Suppose T has been isotoped to intersect $(B, p/q)$ in a minimal number of disks. If $\Delta(p/q, 1/0) > 1$ then one of the following holds*

(a) T is disjoint from B.

(b) $S^3 - k$ is irreducible and there is a Seifert surface S for k with $\chi(S) = \chi(k)$, and such that S intersects ∂B only in two arcs.

REMARK. Theorem 1.4 is a generalization of Theorem 3.1 in [**18**].

In our terminology they consider only the case $\Delta = 2$.

COROLLARY 1.5. *Let k, B, $k(B,p/q)$, P as in Theorem 1.3. Suppose that k is the trivial knot or a split link. If $\Delta(p/q,1/0) \geq 2$ then one of the following holds*

(*a*) *P is disjoint from B; or*

(*b*) *$\Delta(p/q,1/0) = 2$ and P crosses ∂B in one curve parallel to $J_{p/q}$.*

COROLLARY 1.6. *Let k, B, $k(B,p/q)$, T as in Theorem 1.4. Suppose that k is the trivial knot or a split link. If $\Delta(p/q,1/0) \geq 2$ then T is disjoint from B.*

Note that when k is a split link, cases (b), (c), (d) of Theorem 1.3, and case (b) of Theorem 1.4 cannot happen, i.e. Corollaries 1.5 and 1.6 are obvious when k is a split link. The proof of 1.5 when k is a trivial knot is given in §5.

Note that Corollaries 1.5 and 1.6 are a kind of generalization of Theorems 2 and 4 in [**5**], and Theorems 2 and 3 in [**6**], by replacing the fact of being a composite link by the fact of being a doubly composite or satellite link. Corollaries 1.5 and 1.6 are potentially useful in determining if a given link is doubly prime or non-satellite. We have examples of case (b) of 1.5, which produce via double branched covers, examples for Theorem 6.3 (b). In the case $\Delta = 1$ not too much is known; it is not known for example if there is an upper bound for the number of disks of intersection between a tangle decomposing sphere or satellite torus and the ball B.

1.7. *Proof of Corollary 1.6.* Suppose T is a satellite torus in $k(B,p/q)$ isotoped to intersect $(B,p/q)$ a minimum number of times. If (a) of 1.4 happens we are done. If (b) happens then there is a disk D with $\partial D = k$, such that D intersects ∂B only in two arcs. Let α be an arc in D joining two points lying in the distinct arcs of intersection between D and ∂B. The two strings of the tangle $(B', B' \cap k)$ are parallel to α, so because $(B', B' \cap k)$ is not a trivial tangle, α is a knotted arc in B'. Let $T_1 = \partial\eta(B \cup D)$. T_1 is an incompressible torus in $S^3 - k(B,p/q)$, disjoint from B. Looking at the curves of intersection between T and T_1 it is not difficult to conclude that T is disjoint from B, except possibly when $\Delta = 1$, and T_1 is the boundary of a neighborhood of a cable knot. □

2. Preliminary arguments from sutured manifolds.

2.1. Let β be the planar *eyeglass* 1-complex consisting of two circles β_0 and β_1 and an arc β_α joining them. Regard β as a complex in $R^2 \subset R^3$, and let U and W be regular neighborhoods of β in R^2 and R^3 respectively, so that U is a properly embedded planar surface in the genus two handlebody W. $\partial U \subset \partial W$ has three components; two of them, denoted by c_0 and c_1, are parallel in U to β_0 and β_1 respectively. Denote the third by c_α. Denote the cocore of β_α by $c_\beta \subset \partial W$. Let λ_0, λ_1 be two circles in ∂W, parallel to c_β, which bound properly embedded disks in W which separate it in three parts, say a neighborhood of each of β_0, β_1 and β_α. See Figure 2 in [**6**].

2.2. Let k, B and (B,t), where $t = B \bigcap k$, be as in 1.2. Let $W' = B - \operatorname{int} \eta(t)$. Clearly W' is homeomorphic to a regular neighborhood of β; i.e., there is an homeomorphism $f : W \to W'$. Assume that $f(c_0)$ is the cocore of one of the arcs of t, and $f(c_1)$ is the cocore of the other arc. Also assume that $f(c_\alpha) = J_0$ and $f(c_\beta) = J_{1/0}$. For the sake of simplicity we will write c_i instead of $f(c_i)$, $i = 0,\ 1,\ \alpha,\ \beta$ and λ_i instead of $f(\lambda_i)$, $i = 0,\ 1$; that is, consider β as embedded in B, and W as a neighborhood of β in S^3. The curve $J_{p/q}$ on ∂W intersects the regular neighborhood of β_0 (β_1) in a collection of $\Delta = \Delta(p/q, 1/0)$ essential arcs, disjoint from c_0 (c_1), and intersects a neighborhood of β_α in 2Δ arcs, each joining λ_0 and λ_1.

2.3. Let $\eta(k)$ be a neighborhood of k disjoint from W. Let $M = S^3 - int\eta(k)$. Note that β_0 and β_1 are parallel to meridians of k. Note that $\beta \bigcap \partial M = \emptyset$. Consider M as a sutured manifold, all of whose boundary is in R_+ or R_-; denote it by (M, γ, β) (cf. [**16**]). By hypothesis $S^3 - (k \bigcup B)$ is irreducible, for $(B', B' \bigcap k)$ is a prime tangle; this implies that (M, γ, β) is β-irreducible, so (M, γ, β) is β-taut. (M, γ) may not be $\emptyset$-taut, i.e. it may be reducible or ∂M may be compressible. $\emptyset$ denotes the empty set, to be $\emptyset$-taut means to be taut in the Thurston norm.

The proof of Theorems 1.3 and 1.4 will be as follows: First we take a β-taut Seifert surface S for k, and a decomposing sphere or satellite torus Q for $k(B, p/q)$, which will be considered as a parameterizing surface; then construct a sutured manifold hierarchy, starting with S and respecting Q. The goal is to prove that the final step in the hierarchy, i.e. (M_n, γ_n) is $\emptyset$-taut, for in this case [**17**, 2.7] and [**16**, 3.3] imply that (M, γ) is $\emptyset$-taut unless k is the trivial knot, and S is $\emptyset$-taut, so $\chi(S) = \chi(k)$, $S^3 - k$ is irreducible and β crosses S always in the same direction. In this section we show that (M_n, γ_n) is $\emptyset$-taut if Q is a torus, and if Q is a decomposing sphere we show the same, except in three cases, one of them being (e) of 1.3. This implies 1.4 and 1.3 (b), (e); the remaining two cases are treated in §3 and §4.

2.4. Suppose $k(B, p/q)$ is a doubly composite or a satellite link, and let P be a decomposing sphere or satellite torus, not a swallow-follow torus, for $k(B, p/q)$. Consider P as a properly embedded surface in $S^3 - \operatorname{int} \eta(k(B, p/q))$; note that P is incompressible and ∂-incompressible. P can be isotoped to intersect $(B, p/q)$ in a collection of disks D_i properly embedded in $(B, p/q)$, and such that ∂D_i is a curve parallel to $J_{p/q}$ in ∂B. Assume this number of disks is minimal among all the surfaces isotopic to P, and that the intersection is not empty, for otherwise we are done.

Let $Q_\alpha = P - intB$. Let $(Q, \partial Q) \subset (M - intW, \partial M \cup \partial W)$ be a surface consisting of three components, Q_0, Q_1 and Q_α with the following properties:

(a) Q_0 (Q_1) is an annulus for which one boundary component is $c_0 \subset \partial W$ $(c_1 \subset \partial W)$ and the other is a meridian of $\partial\eta(k)$; and either

(b) Q_α is a connected planar surface, four of its boundary components are meridians in ∂M, and the others are parallel to $J_{p/q}$ in ∂W; or

(c) Q_α is a connected genus one surface, and all of its boundary components are parallel to $J_{p/q}$ in ∂W.

According to [**16**, 7.1], Q is a parameterizing surface for (M, γ, β). Note that Q is incompressible and, because P is not a swallow-follow torus, Q is also ∂-incompressible.

2.5. Let S be a β-taut Seifert surface for k; put S in normal position with respect to Q ([**16**, 7.2]). (See [**16**, 7.7] for the definition of a sutured manifold decomposition respecting a parameterizing surface). It is not difficult to see and is implicit in [**16**] that the sutured manifold decomposition $(M, \gamma) \xrightarrow{S} (M_1, \gamma_1)$ is β-taut and respects Q. Here we use the notion of sutured manifold hierarchy as presented in [**17**, 2.1]. Construct a β-taut sutured manifold hierarchy

$$(M,\gamma,\beta) \xrightarrow{S=S_1} (M_1,\gamma_1,\beta_1) \xrightarrow{S_2} \ldots \xrightarrow{S_n} (M_n,\gamma_n,\beta_n)$$

respecting Q. (See [**17**, 2.5], [**16**, 4.19; 7.8]). ∂M_n is a collection of spheres. The surface S meets β_0 (β_1) in one point and Q_0 (Q_1) in one arc. Following [**16**, 2.4(c)], [**17**, 2.1], the edges β_0, β_1 and β_α can be oriented so that at any point of intersection with an S_i (hence with $R(\gamma_i) = R_+(\gamma_i) \bigcup R_-(\gamma_i)$) the orientation points in the direction of the normal vector to S_i. We can suppose β has one of the orientations showed in figure 3 in [**6**] (the choice of one of them depends on the orientation of S_1). Consider $R_+(\gamma_i)$ ($R_-(\gamma_i)$) as the part of ∂M_i in which the orientation points out of (into) M_i. Denote by Q_i and β_i the remnants of Q and β in M_i. Sometimes for simplicity β_i, $R_+(\gamma_i)$, $R_-(\gamma_i)$, $R(\gamma_i)$ will be denoted by β, R_+, R_-, $R(\gamma)$ respectively.

2.6. Recall from [**16**, 7.4] what the index of a parameterizing surface is, $I(Q) = \nu + \mu + K - 2\chi(Q)$, where ν is the number of sutures and μ the number of edges that ∂Q crosses. For each arc δ of $\partial Q \bigcap \eta(v)$, where v is a vertex of β, define $\kappa(\delta)$ to be -1 if δ passes between an edge of β pointing into the vertex and one pointing out, and define $\kappa(\delta) = -2$ if δ passes between edges of β both pointing into the vertex or both pointing out (see figure 4 in [**6**]). Let $K = \sum \kappa$. The curve $J_{p/q} \subset W$ has $\nu = 0$, $\mu = 4\Delta$, $\kappa = -6\Delta$, and the curves c_0, c_1 each have $\nu = 0$, $\mu = 1$, $\kappa = -1$. Hence if Q_α has p boundary components in ∂W, $\chi(Q_\alpha) = \chi(P) - p$, and so $I(Q_\alpha) = 2p(1-\Delta) - 2\chi(P)$. Also $I(Q_0) = I(Q_1) = 0$.

2.7. Suppose that for an arc λ of β_n, which contains no vertices, there is a disk $D \subset Q_n$, such that $\partial D = \delta_1 \bigcup \delta_2$, where $\delta_1 \subset \eta(\lambda)$, $\delta_2 \subset \partial M_n$, and δ_2 crosses a suture; λ is called a cancellable arc, and D is called a cancelling disk. Assume w.l.o.g. that the components of M_n which contain a vertex of β do not contain cancellable arcs (see [**6**, 1.6]).

2.8. Denote by $Q_{i,n}$, $i = 0,1$, the parameterizing surface obtained from Q_i at the end of the hierarchy. Similarly denote by $\beta_{i,n}$, $i =$

$0, 1$, the 1-complex obtained from β_i at the end of the hierarchy. $I(Q_{i,n}) \leq 0$, by [**16**, 7.5; 7.6].

CLAIM . *Each component q of $Q_{i,n}$ for which $\partial Q_{i,n} \bigcap \eta(\beta_{i,n}) \neq \emptyset$, is a disk for which $I(q) = 0$. Indeed $\partial q \bigcap \beta_{i,n}$ is a single arc and $\partial q \bigcap \partial M_n$ is a single arc crossing a single suture.*

The proof is similar to [**18**, claim 2]. Note that the surface S_1 meets β_0 and β_1.

This claim implies that there is a component q of $Q_{i,n}$, q a disk, such that ∂q runs through one of the vertices; all the other components of $Q_{i,n}$ are cancelling disks for an arc of $\beta_{i,n}$.

2.9. A component of M_n which does not meet β is a 3-ball with a single suture on its boundary. We will disregard these trivial components and suppose with no loss of generality that all the components of M_n meet β. As a consequence of that we conclude that every disk component of $R_\pm$ intersects β. Note also that any component of ∂M_n has sutures (see [**6**, 1.8] for details).

∂M_n is a collection of spheres, so (M_n, γ_n) is $\emptyset$-taut if and only if each sphere has only one suture and is the boundary of a 3-ball. Note that if a component of M_n has connected boundary, then this sphere bounds a 3-ball, for M_n is contained in S^3.

LEMMA 2.10. *If none of the surfaces S_i intersects β_α then (M_n, γ_n) is $\emptyset$-taut.*

For the proof see [**6**, 1.13].

2.11. If (M_n, γ_n) is $\emptyset$-taut then by [**17**, 2.7] (M_i, γ_i) is $\emptyset$-taut for all $i \geq 1$, and either (M, γ) is $\emptyset$-taut and so is irreducible and ∂-incompressible, or M is a solid torus (i.e. k is the trivial knot). Now in any case, by [**16**, 3.3] the surface S is $\emptyset$-taut. It is not difficult to see that this implies that $\chi(S) = \chi(k)$ (for a proof see [**19**, 1.2]).

2.12. Suppose that some surface S_i intersects β_α. Then there are two components of β_n which have a vertex. Denote the component which has two ends in R_+ (R_-) and one in R_- (R_+) by A_+ (A_-). Denote the ends of A_+ by a_+, a_-, b, and the ends of A_- by d_+, d_-, c, where b, c are the ends which are part of β_α. a_+, a_- (d_+, d_-) are part of β_0 (β_1). a_+, d_+, b lie on R_+, and a_-, d_-, c lie on R_-. 2.8

shows that a_+ and a_- (d_+ and d_-) lie in adjacent components of $R_\pm$ and there is a disk q component of $Q_{0,n}$ ($Q_{1,n}$) such that $\partial q \cap \partial M_n$ is an arc joining a_+ and a_- (d_+ and d_-) and which crosses a suture. See figure 5 in [**6**].

There is a collection of arcs on A_+ going from a_+ to b; each such arc λ is contained in the boundary of a component of Q_n, and for this arc $\kappa(\lambda) = -2$. Call the part of A_+ which contains these arcs the negative side of A_+. Analogously, there is a collection of arcs on A_+ going from a_- to b. For each such arc λ, $\kappa(\lambda) = -1$. Call the part of A_+ which contains these arcs the positive side of A_+. In a similar way we define the negative and positive side of A_-

LEMMA 2.13. *Suppose that some surface S_i intersects β_α. Then no component of Q_n has negative index.*

The proof is as in [**6**, 1.15].

2.14. If P is a torus then $I(Q) = 2p(1 - \Delta) < 0$ whenever $\Delta > 1$. This would contradict lemma 2.13 if some S_i meets β_α. So in this case none of the S_i intersects β_α, and then by 2.10 and 2.11 the surface S is $\emptyset$-taut, $S^3 - k$ is irreducible and $\chi(S) = \chi(k)$, so we have case (b) of Theorem 1.4. This completes the proof of Theorem 1.4.

If P is a decomposing sphere then $I(Q) = 2p(1 - \Delta) + 4$. So $I(Q) < 0$ whenever $\Delta > 3$. If $\Delta = 3$ then $I(Q) \leq 0$, with equality occurring only if $p = 1$. If $\Delta = 2$ then $I(Q) \leq 2$, and note that $I(Q) = 2$ only if $p = 1$, $I(Q) = 0$ only if $p = 2$, and $I(Q) < 0$ if $p > 2$. If $I(Q) < 0$, then 2.13 implies that none of the S_i intersects β_α, and by 2.10 and 2.11 the surface S is $\emptyset$-taut, $\chi(S) = \chi(k)$ and $S^3 - k$ is irreducible. This implies case (b) of Theorem 1.3. If $\Delta = 2$, $I(Q) = 2$ and $p = 1$ we get case (e) of 1.3.

From now on we will assume that P is a decomposing sphere, and that $I(Q) = 0$, i.e. $\Delta = 3$ and $p = 1$, or $\Delta = 2$ and $p = 2$.

REMARK. The above proof for the case $I(Q) < 0$ proves essentially the same thing that is proved in [**18**] and [**12**]. Instead of using an eyeglass complex they use a complex consisting of the wedge of two circles; it seems that with that complex it is not possible to do the case when $I(Q) = 0$. The proof here for the case $I(Q) < 0$ is like the one in [**16**, §8].

3. The combinatorial setting. Our goal in this and next section is to prove that (M_n, γ_n) is $\emptyset$-taut. We first introduce a combinatorial structure on ∂M_n, and then prove Lemmas 3.4 and 3.5. These Lemmas are used to prove Lemmas 3.6, 3.7, 3.13, 3.14 and 3.15. Lemmas 3.6 and 3.7 are used in §4 to show that if (M_n, γ_n) is not $\emptyset$-taut then there is a finite number of possible configurations of the special vertices (Lemma 4.1). Then it is showed that all these configurations are in contradiction with 3.13-3.15 (Proposition 4.2). This will imply 1.3 (c) and (d).

3.1. Let T be one of the sphere components of ∂M_n. The points $\beta \cap T$ and the arcs $\partial Q_n \cap T$ can be regarded as a graph Γ in T. A vertex of Γ is a point of $\beta \cap T$ and an edge is an arc component of $\partial Q_n \cap T$, each one of its ends is at a vertex. Denote the components of $\partial Q_\alpha - \partial M$ by a_1, a_2 if $\Delta = 2$, and by a_1 if $\Delta = 3$. Denote by $\epsilon_a = \partial Q_{0,n} \cap T$ ($\epsilon_d = \partial Q_{1,n} \cap T$) the edge joining a_+ and a_- (d_+ and d_-), this edge exists for 2.8; it crosses a suture.

λ_j intersects a_i in 2Δ points, $j = 0, 1$; label them i, i^*, i, i^*,... alternately around λ_j, and so that a point i (i^*) in λ_0 is connected, via a subarc of a_i contained in $\partial\eta(\beta_\alpha)$, to a point labeled i (i^*) in λ_1. Let v be a vertex in Γ, then v in W is a circle; label the end of an edge incident to v, other than ϵ_a and ϵ_d, with i (i^*) if this point is connected to a point in λ_0 labeled with i (i^*), via a subarc of a_i whose interior misses λ_0, v, $a_\pm$ and $d_\pm$. See Figure 1 for the case $\Delta = 2$, and Figure 2 for the case $\Delta = 3$.

Let $\Delta = 2$. The curve a_1 separates ∂B in two parts; one of them

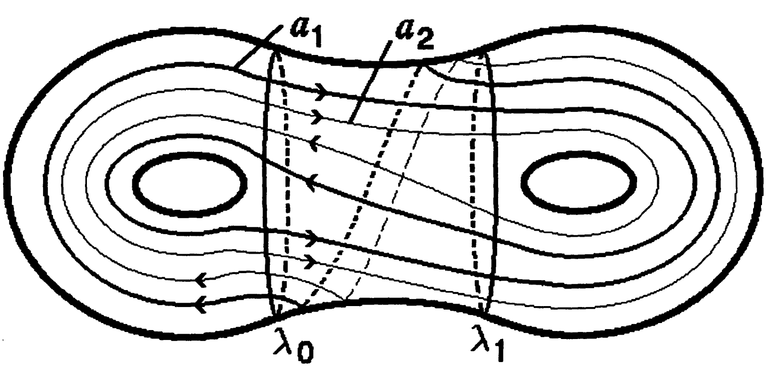

FIGURE 1.

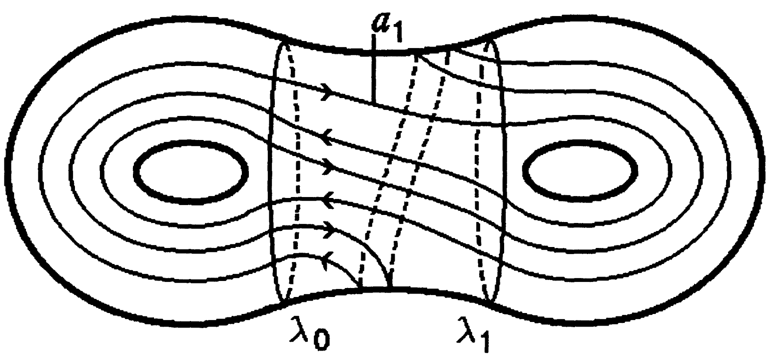

FIGURE 2.

does not contain a_2, call it V_1. V_1 intersects k twice. Choose a rectangle $R_1 \subset V_1$, such that ∂R_1 consists of four arcs e_1, e_2, e_3, e_4, where e_1, e_3 are contained in $\partial V_1 = a_1$, and e_2, e_4 are arcs properly embedded in V_1. Suppose that $R_1 \bigcap \eta(k) = \emptyset$, then $R_1 \subset \partial W$; note that $V_1 - R_1$ has two components, each one of them intersecting $\eta(k)$ once, and assume that for any vertex v in T, $v \neq a_\pm, d_\pm$, we have that $v \bigcap (V_1 - R_1) = \emptyset$. It can be assumed that if $v = a_\pm, d_\pm$, then $v \bigcap \partial V_1 \subset e_1 \cup e_3 \subset \partial R_1$. Analogously, there is a disk V_2 and rectangle R_2 for a_2, with $\partial R_2 = f_1, f_2, f_3, f_4$. Label the end of an edge ϵ in Γ, labeled previously 1 or 1^*, by 1_1 or 1_1^* (1_3 or 1_3^*) if $\epsilon \bigcap a_1 \subset e_1$ ($\epsilon \bigcap a_1 \subset e_3$). Analogously change $2, 2^*$ to $2_1, 2_1^*$ or $2_3, 2_3^*$. It can be assumed w.l.o.g. that the collection of labels around a vertex in Γ, other than $a_\pm$, $d_\pm$, looks like $1_1, 2_1, 2_3^*, 1_1^*, 1_3, 2_3, 2_1^*, 1_3^*$. We may assume that a_+ is labeled by $1_1, 2_1, 2_3^*, 1_1^*$, and a_- is labeled by $1_3, 2_3, 2_1^*, 1_3^*$. See Figure 3.

Let $\Delta = 3$. The curve a_1 separates ∂B in two parts, V_1, and V_2, and there are rectangles $R_1 \subset V_1$, and $R_2 \subset V_2$, as in the previous case. Label the end of an edge ϵ in Γ, labeled previously 1, by 1_{13} if $\epsilon \bigcap a_1 \subset e_1$ and $\epsilon \bigcap a_1 \subset f_3$. Analogously change the labels in all the other cases. It can be assumed w.l.o.g. that the collection of labels around a vertex in Γ, other than $a_\pm$, $d_\pm$, looks like $1_{13}, 1_{11}^*, 1_{33}, 1_{31}^*, 1_{11}, 1_{33}^*$. To avoid cumbersome notation change these labels for the labels $1, 2, 3, 4, 5, 6$. We may assume that a_+ is labeled by $1, 2, 3$, and a_- is labeled by $4, 5, 6$. See Figure 4.

The vertices $a_\pm$, $d_\pm$, b, c are named special vertices; any other is called a simple vertex. Call the part of a special vertex which is contained in the negative (positive) side of A_+ or A_- the negative

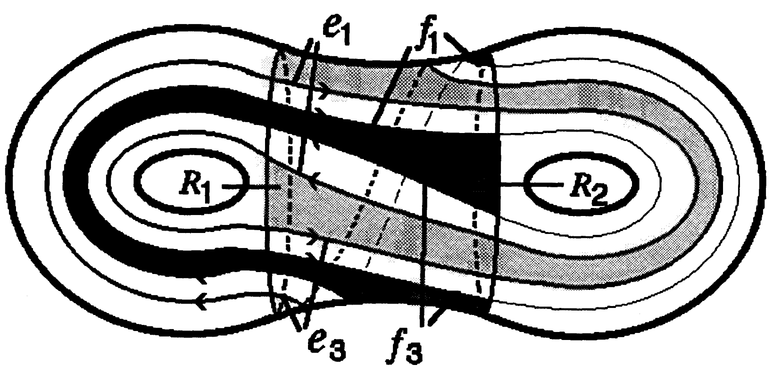

FIGURE 3.

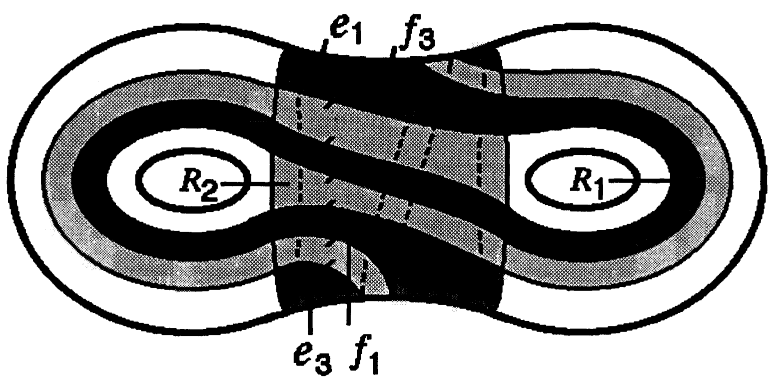

FIGURE 4.

(positive) side of such vertex (see 2.12). b and c have both negative and positive sides, a_+ and d_- have only a negative side, a_- and d_+ have only a positive side.

3.2. Give an orientation to Q_α; this orientation induces an orientation on the boundary components of Q_α. Two components of $\partial Q_\alpha - \partial M$ are parallel if with the induced orientation they are homologous in ∂W; otherwise they are antiparallel. Analogously we define paralellism of vertices in T (see [**6**, 2.1] for the definition). All the vertices of T contained in R_+ (R_-) with the exception of a_+ (a_-) are parallel. A vertex in R_+ is antiparallel to a vertex in R_- (other than a_+, d_-). The vertex a_+, (d_-) is parallel to vertices in R_- (R_+), other than d_- (d_+).

We have the following parity rule [**6**, 2.1]: Let v_1, v_2 be vertices of Γ, and let $i, j \in \{1, 2\}$.

(1) If an edge joins parallel vertices v_1 and v_2 with labels i and j (i and j^*, or i^* and j^*) respectively, then a_i and a_j are antiparallel (resp. parallel, resp.antiparallel).
(2) If an edge joins antiparallel vertices v_1 and v_2 with labels i and j (i and j^*, or i^* and j^*) respectively, then a_i and a_j are parallel (resp. antiparallel, resp. parallel).

In the present case if $\Delta = 2$ then $\partial Q_\alpha - \partial M$ has two components, a_1 and a_2, which are antiparallel. If $\Delta = 3$, then $\partial Q_\alpha - \partial M$ has only one component, i.e. a_1; in this case the parity rule traduces to: an edge joins parallel vertices v_1 and v_2 in Γ with labels $i,\ j \in \{1,2,3,4,5,6\}$, if and only if i and j have different parity.

We make the convention that in all our Figures the vertices parallel to b have labels ordered $1_1, 2_1, 2_3^*, 1_1^*, 1_3, 2_3, 2_1^*, 1_3^*$ or $1,2,3,4,5,6$ in an anticlockwise direction. The vertices antiparallel to b have labels ordered in the opposite direction.

3.3. An edge is level if its ends are equally labeled (ignoring the asterisk and the 1,3 subindex). Define a path and cycle in Γ in the usual way.

For $x \in \{1_1, 2_1, 2_3^*, 1_1^*, 1_3, 2_3, 2_1^*, 1_3^*\}$ or $x \in \{1,2,3,4,5,6\}$, an x-path is a path in Λ so that the beginning point of each edge is labeled with x, and all its vertices are parallel. An x-cycle is an x-path which is a cycle. An innermost cycle is a cycle which is the boundary of a disk without edges or vertices in its interior. A Scharlemann cycle is an x-cycle which is an innermost cycle. Let γ be a cycle with edges $\epsilon_1, \ldots, \epsilon_n$, and vertices $v_1, \ldots, v_n$, and so that ϵ_i is incident to v_i and v_{i+1} (mod n), with labels x_i and y_i respectively; the label sequence of γ is the sequence $y_n, x_1, y_1, x_2, \ldots, y_{n-1}, x_n$.

Let Λ be a subgraph of Γ, and let $x \in \{1_1, 2_1, 2_3^*, 1_1^*, 1_3, 2_3, 2_1^*, 1_3^*\}$ or $x \in \{1,2,3,4,5,6\}$. We say that Λ satisfies $P(x)$ if (see [**3**, 2.6]): For each vertex v in Λ there exist an edge of Λ incident to v with label x and connecting v to a parallel vertex.

If Λ satisfies $P(x)$ for some x, then it is possible to construct an x-path beginning at any vertex, and this path will close, forming an x-cycle in Λ.

Lemma 3.4. *Let Λ be a subgraph of Γ which consists of the intersection of Γ with a disk, such that all its vertices are parallel,* *and that satisfies $P(x)$ for some x. Then Γ contains a Scharlemann cycle.*

Proof. The proof is a straightforward modification of [**3**, 2.6.1, 2.6.2]. □

LEMMA 3.5. *There is no innermost cycle in Γ whose label sequence is one of the following*
(a) $1, 2, 1, 2, \ldots, 1, 2$, *in the case $\Delta = 2$, where the asterisk and the subindex have been ignored.*
(b) $I_1, I_3, I_1, I_3, \ldots, I_1, I_3$, *in the case $\Delta = 2$, where $I = 1$ or 2 but the same on each label, and the asterisk has been ignored.*
(c) $1_{1-}, 1_{3-}, 1_{1-}, 1_{3-}, \ldots, 1_{1-}, 1_{3-}$ *or* $1_{-1}, 1_{-3}, 1_{-1}, 1_{-3}, \ldots, 1_{-1}, 1_{-3}$, *in the case $\Delta = 3$, where $-$ stands for 1 or 3, but not necessarily the same on each label, and the asterisk has been ignored.*
In particular there is no Scharlemann cycle in Γ.

Proof. Suppose σ is an innermost cycle, say σ consists of vertices $v_1, v_2, \ldots, v_n$, and edges $\epsilon_1, \ldots, \epsilon_n$, where ϵ_i joins v_i and v_{i+1} (mod n). There is an arc ρ_i in v_i, joining the labels incident to ϵ_{i-1} and ϵ_i, and which does not contain any label in its interior. Let D be the disk in T bounded by σ, it follows that $\partial D = (\cup\epsilon_i)\bigcup(\cup\rho_i)$. If σ is a loop then Q_α is ∂-compressible, which is a contradiction, so suppose σ has at least two edges.

If the edges ϵ_i have endpoints labeled 1-2, then all the arcs ρ_i lie over the annulus E cobounded by a_1 and a_2, and when traveling along σ in a given direction all the ρ_i run from a_1 to a_2 (or vice versa). The annulus E and the two disks in the interior of B bounded by a_1 and a_2 cobound a 3-ball $C \subset B$ whose interior is disjoint from Q_α. Note that $\partial D \subset Q_\alpha \bigcup E$, and then a regular neighborhood of $Q_\alpha \bigcup C \bigcup D$ is a punctured lens space, which is impossible.

Suppose the edges ϵ_i are level. First note that the arcs ρ_i lie all over R_1 or all over R_2. To see that suppose the opposite; then there is an arc ρ_i which lies over R_1 and an arc ρ_j which lies over R_2, this is possible only if $\Delta = 3$. Take two points, one in the interior of each of ρ_i and ρ_j; there is an arc lying over ∂W which joins them and which intersects transversally Q_α exactly once, and there is also an arc lying over D joining these points and which is disjoint from Q_α. So there is a simple closed curve meeting transversally the sphere

P in one point, which is impossible. Therefore the arcs ρ_i lie all over R_1 or R_2, say R_1. ρ_i has its endpoints in e_1 and e_3 (or f_1 and f_3) and when traveling along σ in a given direction all the ρ_i run from e_1 to e_3 (or vice versa). Then $\partial D \subset Q_\alpha \bigcup R_1$, and a regular neighborhood of $Q_\alpha \bigcup R_1 \bigcup D$ is a punctured lens space, which is not possible.

Finally, note that the label sequence of a Scharlemann cycle is like one of the above sequences. □

LEMMA 3.6. *Suppose that D is a disk component of $R(\gamma)$ which does not contain $a_\pm$ nor $d_\pm$. Let $\Delta = 2$ ($\Delta = 3$). Then for each $x \in \{1_1, 2_1, 2_3^*, 1_1^*, 1_3, 2_3, 2_1^*, 1_3^*\}$ ($x \in \{1,2,3,4,5,6\}$) there is at least one edge with label x at a vertex in D which crosses the suture ∂D. So there are at least* 8 (6) *edges crossing ∂D.*

Proof. If this does not happen for some x then $\Lambda = D \bigcap \Gamma$ has $P(x)$, contradicting 3.4 and 3.5. □

LEMMA 3.7. *Suppose that D is a disk component of $R(\gamma)$ which contains only one of $a_\pm$, $d_\pm$. Let $\Delta = 2$ ($\Delta = 3$), then there are at least* 4 (3) *edges which cross ∂D, other than ϵ_a, ϵ_d.*

The proof is as in [**6**, 2.6].

LEMMA 3.8. *Let $\Delta = 2$. $d_\pm$ is labeled in one of the following ordered ways:*

$$2_3^*, 1_1^*, 1_3, 2_3; \quad or \quad 2_1^*, 1_3^*, 1_1, 2_1.$$

The positive and negative sides of c are labeled in the same way as d_+ and d_- respectively.

Proof. The sequence of labels in d_+ has to be a subsequence of $1, 2, 2^*, 1^*, 1, 2, 2^*, 1^*$. Note that, in the given order of the labels, the first label in d_+ is connected to the last label in d_-, via a subarc of a_1 or a_2 contained in $\partial\eta(\beta_1)$. So one of these labels is 1 and the other 1^* (or 2^* and 2), see Figure 1. This implies the sequence of labels in d_+ is like $1, 2, 2^*, 1^*$ or $2^*, 1^*, 1, 2$. The label 1_1 at a_+ is joined to the label 1_3^* at a_-, via a subarc of a_1 contained in $\partial\eta(\beta_0)$; also, these two labels are joined, via subarcs of a_1 contained in $\partial\eta(\beta_\alpha)$ to labels 1_1 and 1_3^* at $d_\pm$. If d_+ has labels $1, 2, 2^*, 1^*$, then there is a label 1_1 at d_+ and a label 1_3^* at d_- (or vice versa), so there is a subarc of a_1 contained in $\partial\eta(\beta_1)$ which joins these two points. This implies that a_1 meets $\lambda_0 \bigcup \lambda_1$ in four points, a contradiction. □

LEMMA 3.9. *Let $\Delta = 3$. a_1 meets the six points of intersection between a_1 and λ_0 in one of the following ways, cyclically,* $1, 2, 5, 4, 3, 6$ *or* $1, 4, 3, 2, 5, 6$. *(The former is shown in Figure 4.)*

Proof. Travel around a_1 starting at the point labeled 1 in $a_1 \bigcap \lambda_0$, and in the direction of the orientation of β. Note that the odd and even numbers in $a_1 \bigcap \lambda_0$ have opposite orientation. It follows from Figure 2 that traveling in that direction, 6 is followed by 1, 2 by 5, and 4 by 3. So the sequence of labels in a_1 has to be one of those written above. □

LEMMA 3.10. *If a_1 meets its points of intersection with λ_0 in the cyclic order* 1, 2, 5, 4, 3, 6 *then d_+ (d_-) is labeled* 5, 6, 1 (2, 3, 4) *or* 2, 3, 4 (5, 6, 1). *If a_1 meets its points of intersection with λ_0 in the cyclic order* 1, 4, 3, 2, 5, 6, *then d_+ (d_-) is labeled* 3, 4, 5 (6, 1, 2) *or* 6, 1, 2 (3, 4, 5).

Proof. Note that if a pair of labels, say 1,2, are consecutive in λ_0 (and then in λ_1) and also in a_1, then one of these labels lies in d_+, and the other in d_-. This implies that the labels of $d_\pm$ are as desired. See Figure 2. □

3.11. Let D be a disk contained in T so that ∂D cuts transversally two special vertices. Suppose all the vertices in D are parallel, except possibly one of the special vertices meeting ∂D. Suppose that $a_\pm, d_\pm$ are not in the interior of D, and that no edge crosses ∂D. Let $\Lambda = \Gamma \bigcap D$. Call the labels of the two vertices which meet ∂D, the labels of ∂D. Suppose the labels of ∂D in the case $\Delta = 2$ are $x_1, x_2, x_3, x_4, k_4, k_3, k_2, k_1$, in this cyclic order, as shown in Figure 5, where the labels x_i correspond to one of the vertices which meet ∂D, and the labels k_i to the other one. In the case $\Delta = 3$ consider six

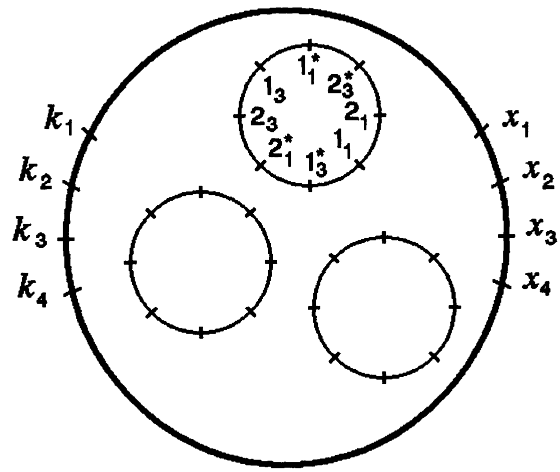

FIGURE 5.

labels in ∂D, three for each vertex meeting ∂D. Suppose the vertex containing the labels x_i's is parallel to all vertices in the interior of D.

LEMMA 3.12. *Let D be as in* 3.11. *Then for each x_i, $i = 1, 2, 3, 4$, there is a x_i-path starting at ∂D, and finishing at the other side of ∂D with label k_i.*

Proof. For the label x_1 take a x_1-path beginning at ∂D, this path has to finish at the other side of ∂D, otherwise there is a x_1-cycle, and then by 3.4 a Scharlemann cycle, which contradicts 3.5, so the path finishes at a label k_{i_1} in ∂D, $i_1 \geq 1$. The complementary part of γ_1 which contains the others labels x_i's has $P(x_2)$. Now take a x_2-path γ_2 beginning at ∂D, it finishes at ∂D with label k_{i_2}, $i_2 > i_1$. Repeating the argument for x_3 and x_4, we conclude that each x_i-path has to finish in ∂D at the label k_i. □

LEMMA 3.13. *Let $\Delta = 2$. The following configuration of vertices is not possible: Suppose we have a situation as in* 3.11, *where D cuts one of b or $a_\pm$, and one of c or $d_\pm$, and the labels of ∂D are in one side the positive or negative side of b or $a_\pm$, and in the other side the positive or negative side of c or $d_\pm$.*

Proof. Suppose first that D cuts b and c, and assume w.l.o.g. that all the vertices lying in the interior of D are parallel to b.

Therefore x_1, x_2, x_3, x_4 are $1_3, 2_3, 2_1^*, 1_3^*$ or $1_1, 2_1, 2_3^*, 1_1^*$ respectively, and k_1, k_2, k_3, k_4 are $2_3^*, 1_1^*, 1_3, 2_3$ or $2_1^*, 1_3^*, 1_1, 2_1$ respectively, for b and c are antiparallel. See Figure 6. Then by 3.12 for each i, $1 \leq i \leq 4$, there is a x_i-path γ_i starting at ∂D with label x_i and finishing at ∂D with label k_i. Incident to the label k_2 ($= 1_1^*$ or 1_3^*) is an edge ϵ whose other end is at a vertex v with label 2_3 (or 2_1). Construct a 1_3-path (or a 1_1-path) γ starting at v; this path will close forming a cycle or will reach a vertex incident to the path γ_1. So the path γ_1-ϵ-γ will contain a cycle, and by an argument as in 3.4 there will an innermost cycle σ (σ can be chosen to be a Scharlemann cyle if it does not contain c, and if it does contain c, then can be chosen to be a x-cycle, where x is one of the labels $1, 1^*, 2, 2^*$, and we have ignored the subindices 1, 3). If the vertex c is part of the cycle σ, then all the edges of σ have ends labeled 1-2 (i.e. σ does not contain level edges, for no edge incident to c is level); if c is not in the cycle σ, then σ is a Scharlemann cycle. Both cases contradict 3.5.

Now suppose D cuts b and $d_\pm$. Then the labels x_1, x_2, x_3, x_4 correspond to the labels $1_3, 2_3, 2_1^*, 1_3^*$ or $1_1, 2_1, 2_3^*, 1_1^*$, and k_1, k_2, k_3, k_4 to the labels $2_3, 1_3, 1_1^*, 2_3^*$ or $2_1, 1_1, 1_3^*, 2_1^*$, for b and $d_\pm$ are parallel. Note that in any of the four possible choices of labels there is a label x in common in b and in $d_\pm$, which implies that $\Gamma \bigcup D$ has $P(x)$, which contradicts 3.4 and 3.5.

If D cuts $a_\pm$ and one of c or $d_\pm$, an argument as in the previous cases yields a contradiction. Note that $a_\pm$ is parallel to c and

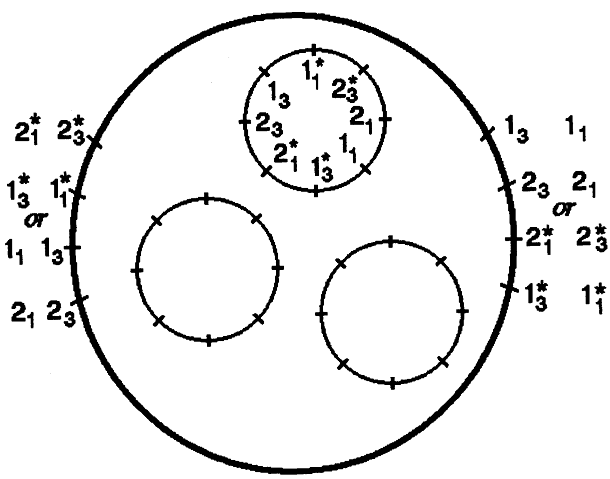

FIGURE 6.

antiparallel to $d_\pm$. □

LEMMA 3.14. *Let* $\Delta = 3$. *The following configuration of vertices is not possible: Suppose we have a situation as in* 3.11, *where* D *cuts one of* b *or* $a_\pm$ *and one of* c *or* $d_\pm$, *and the labels of* ∂D *are in one side the positive or negative side of* b *or* $a_\pm$, *and in the other side the positive or negative side of* c *or* $d_\pm$.

Proof. Suppose first that D cuts b and c, and assume w.l.o.g. all the vertices lying in D are parallel to b.

Case 1. The labels x_i's on ∂D correspond to the negative side of b and the k_i's to the positive or negative side of c.

Then x_1, x_2, x_3 are $1, 2, 3$ respectively and k_1, k_2, k_3 are $5, 6, 1$, or $2, 3, 4$, or $3, 4, 5$, or $6, 1, 2$. See Figure 7. Then by 3.12, for each $i, i = 1, 2, 3$, there is a x_i-path γ_i starting in ∂D at label x_i and finishing in ∂D at label k_i. If k_1, k_2, k_3 are $2, 3, 4$ or $6, 1, 2$, this will contradict the parity rule, for b and any vertex in the interior of D are antiparallel to c. So suppose first k_1, k_2, k_3 are $5, 6, 1$. Let ϵ be the edge incident to c at label k_1; the other end of ϵ is at a vertex v at a label 1. Start a 2-path γ at v. This path will close forming a cycle, or will reach a vertex incident to the path γ_2. So the path γ_2-ϵ-γ will contain a cycle, and by an argument as in Lemma 3.4 there will be an innermost cycle σ. If c is not in σ, then it will be a Scharlemann cycle, which contradicts Lemma 3.5. If c is part of σ, so is the arc in c joining the labels k_1 and k_2. Remember that $1, 2, 3$ represent in a short notation the labels $1_{13}, 1^*_{11}, 1_{33}$, and $4, 5, 6$ represent the labels $1^*_{31}, 1_{11}, 1^*_{33}$. The label sequence of σ is like $1, 2, 1, 2, ..., 1, 2, 6, 5$, i.e. is like $1_{13}, 1^*_{11}, 1_{13}, 1^*_{11}, \dots, 1^*_{33}, 1_{11}$. This contradicts 3.5.

Suppose now k_1, k_2, k_3 are $3, 4, 5$ respectively. Let ϵ be the edge incident to c at label k_3; its other end has label 3 at a vertex v. Take a 2-path γ starting at v. Then the path γ_2-ϵ-γ will form a cycle, and as before there is an innermost cycle σ. If c is not in σ, then it is a Scharlemann cycle. If c is in σ, then σ has a label sequence $3, 2, 3, 2, ..., 3, 2, 4, 5$, note that this sequence contradicts 3.5.

Case 2. The labels x_i's on ∂D correspond to the positive side of b.

Then x_1, x_2, x_3 are $4, 5, 6$ respectively, and as in Case 1, there are four possibilities for the labels k_1, k_2, k_3. As in Case 1, the

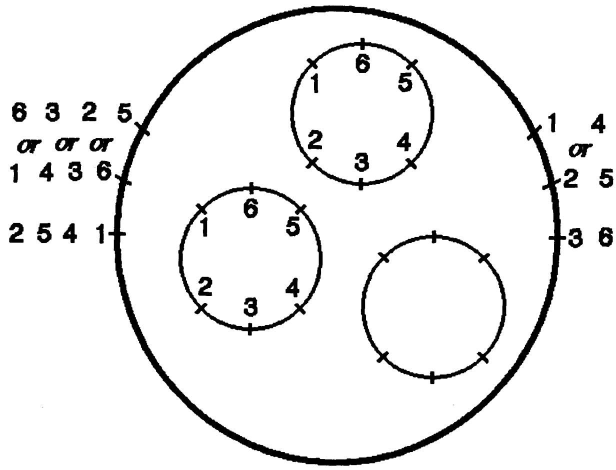

FIGURE 7.

sequences $5, 6, 1$ and $3, 4, 5$ are not possible since they contradict the parity rule. The sequences $2, 3, 4$ and $6, 1, 2$ are not possible for they would imply the existence of a lens space summand in S^3. In the case $2, 3, 4$ there is an innermost cycle with label sequence $4, 5, 4, 5, ..., 3, 2$, and in the case $6, 1, 2$ there is an innermost cycle with label sequence $6, 5, 6, 5, ..., 1, 2$; both contradict 3.5.

Suppose now D cuts b and $d_\pm$.

The labels x_i's on ∂D correspond to the negative or positive side of b and the labels k_i's to d_+ or d_-. There are eight possible choices of labels. Note that in any choice of labels there is a label x_i equal to some k_j, and because b and $d_\pm$ are parallel this implies that $\Gamma \bigcap D$ has $P(x)$ for some label x, which contradicts 3.4 and 3.5.

If D cuts $a_\pm$ and one of c or $d_\pm$, an argument as in the previous cases yields a contradiction. Note that $a_\pm$ is parallel to c and antiparallel to $d_\pm$. □

LEMMA 3.15. *The following configuration of vertices is not possible: Suppose we have a situation as in* 3.11, *where D cuts b and $a_\pm$, and the labels of ∂D are in one side the positive* (*negative*) *side of b, and in the other side a_+* (a_-).

Proof. Let $\Delta = 2$. Suppose first that the labels of ∂D correspond to the positive side of b and to a_+, and assume w.l.o.g. that all the vertices lying in the interior of D are parallel to b. Therefore x_1, x_2, x_3, x_4 are $1_3, 2_3, 2_1^*, 1_3^*$ respectively, and k_1, k_2, k_3, k_4 are $1_1, 2_1, 2_3^*, 1_1^*$ respectively, for b and a_+ are antiparallel. Here we do an argument as in 3.13. By 3.12 for each i, $1 \leq i \leq 4$, there is a x_i-path γ_i starting at ∂D with label x_i and finishing at ∂D with label k_i. Incident to the label $k_2 = 2_1$ is an edge ϵ whose other end is at a vertex v with label 2_3. Construct a 2_1^*-path γ starting at v; this path will close forming a cycle or will reach a vertex incident to the path γ_3. So the path γ_3-ϵ-γ will contain a cycle, and by an argument as in 3.4 there will an innermost cycle σ. If the vertex a_+ is part of the cycle σ, then all the edges of σ have ends labeled $2_3 - 2_1$ (ignoring the asterisk), if a_+ is not in the cycle σ, then σ is a Scharlemann cycle; both cases contradict 3.5. If the labels of ∂D correspond to the negative side of b and a_-, a similar argument yields a contradiction.

Let $\Delta = 3$. Suppose that the labels of ∂D correspond to the positive side of b and to a_+, and assume w.l.o.g. that all the vertices lying in the interior of D are parallel to b. Therefore x_1, x_2, x_3 are $4, 5, 6$ respectively, and k_1, k_2, k_3, are $1, 2, 3$ respectively, for b and a_- are antiparallel. Then as in the previous case there is an edge with ends labeled 4 and 1, which lie on antiparallel vertices, which contradict the parity rule. If the labels of ∂D correspond to the negative side of b and a_-, a similar argument yields a contradiction. □

4. (M_n, γ_n) is $\emptyset$-taut.

LEMMA 4.1. *Suppose that (M_n, γ_n) is not $\emptyset$-taut. Then ∂M_n has at most two boundary components. Suppose first that ∂M_n has only one boundary component. This boundary component has two innermost sutures, which determine two disks: one is in R_+ and one in R_-, denoted by D_+ and D_- respectively. Denote by E_-, E_+ the annulus incident to D_+ and D_- respectively. Note that there may be more annuli between E_+ and E_-. In this case the following are all the possible configurations of the special vertices (see Figure 9 in* [6]*):*

Case 1. *a_+ is in D_+, c is in D_-, a_- and d_- are in E_-, b and d_+ are in E_+.*

Case 2. *a_+ is in D_+, d_- is in D_-, a_- and c are in E_-, b and d_+ are in E_+.*

Case 3. *b is in D_+, d_- is in D_-, a_- and c are in E_-, a_+ and d_+ are in E_+.*

Case 4. *b is in D_+, a_- and d_- are in D_-, c is in E_-, a_+ and d_+ are in E_+.*

Case 5. *b is in D_+, a_- and c are in D_-, d_- is in E_-, a_+ and d_+ are in E_+.*

Case 6. *b is in D_+, c is in D_-, a_- and d_- are in E_-, a_+ and d_+ are in E_+.*

Case 7. *a_+ and d_+ are in D_+, c is in D_-, a_- and d_- are in E_-, b is in E_+.*

Case 8. *b and d_+ are in D_+, c is in D_-, a_- and d_- are in E_-, a_+ is in E_+.*

Case 9. *b and d_+ are in D_+, a_- and c are in D_-, d_- is in E_-, a_+ is in E_+.*

Case 10. *a_+ and b are in D_+, c and d_- are in D_-, a_- is in E_-, d_+ is in E_+.*

Case 11. *d_+ is in D_+, a_- is in D_-, c and d_- are in E_-, b and a_+ are in E_+.*

Suppose ∂M_n has two boundary components, say T_1, T_2; each has only one suture, dividing T_1 in $D_{1,+}$ and $D_{1,-}$, and T_2 in $D_{2,+}$ and $D_{2,-}$. The following are all the possible configurations of the special vertices (see Figure 10 *in* [6]*):*

Case 12. *b and d_+ are in $D_{1,+}$, d_- is in $D_{1,-}$, a_+ is in $D_{2,+}$, a_- and c are in $D_{2,-}$.*

Case 13. *b is in $D_{1,+}$, c is in $D_{1,-}$, a_+ and d_+ are in $D_{2,+}$, a_- and d_- are in $D_{2,-}$.*

Case 14. *a_+ and b are in $D_{1,+}$, a_- is in $D_{1,-}$, d_+ is in $D_{2,+}$, c and d_- are in $D_{2,-}$.*

Proof. Lemmas 3.6, 3.7 and the fact that the negative side of a vertex has 4 or 3 labels imply the following facts, whose proofs are analogous to the corresponding in [**6**]:

(a) A component N of ∂M_n which does not contain a vertex of β is a 3-ball with one suture on its boundary, so is $\emptyset$-taut [**6**, 3.1].

(b) A disk D in $R_\pm$ contains special vertices [**6**, 3.2].

(c) A component of $R_\pm$ adjacent to a disk also contains a special vertex [**6**, 3.3].

(d) A disk D in $R_{\pm}$ contains at most two special vertices [**6**, 3.4].

(e) A component T of ∂M_n has at most three disks components of $R(\gamma)$ [**6**, 3.5].

(f) Let D be a disk in $R_{\pm}$. Suppose that the area adjacent to D is an annulus E. Suppose a_+ (d_-) is the only special vertex in D. Then D contains simple vertices and E contains either c or d_- (b or a_+) [**6**, 3.6].

(g) A component T of ∂M_n has at most two innermost sutures [**6**, 3.8].

(h) Let D be a disk in $R_{\pm}$. Suppose that the area adjacent to D is an annulus E. Suppose a_- (d_+) is in E, and b (c) is the only special vertex in D. Then either c or d_- (b or a_+) are in E [**6**, 3.9].

Note however that the proofs of [**6**, 3.7; 3.11] do not follow from 3.6, 3.7. Suppose ∂M_n has only one component, then claims (a)-(h) almost imply which are all the configurations of the special vertices. Doing an argument similar to [**6**, 3.12-3.15] we get that Cases 1-11 are all the possible configurations of the special vertices. Cases 10 and 11 do not appear in [**6**, 3.16], but appear here for we are not using [**6**, 3.7; 3.11].

Suppose T is a component of ∂M_n. Then doing an argument as in [**6**, 3.17], T cannot meet only a_+ and a_-, or only d_+ and d_-. This implies that ∂M_n has at most two components. If ∂M_n has two components, say T_1 and T_2, is not difficult to see that Cases 12, 13, 14 are all the possible configurations of the special vertices. Case 14 appears here and not in [**6**, 3.17] because [**6**, 3.7; 3.11] are not being used. □

PROPOSITION 4.2. *(M_n, γ_n) is $\emptyset$-taut.*

Proof. We will show that any of the Cases of 4.1 yields a contradiction.

Case 1. In this Case a_+ is the only special vertex in D_+. Doing an argument as in [**6**, 3.6], we see that there are simple vertices in D_+, and no edge incident to a_+ crosses ∂D_+. Then there are exactly 4 edges (or 3, depending if $\Delta = 2$ or $\Delta = 3$) crossing ∂D_+ which come from simple vertices. These edges have to meet d_-. It is not difficult to see that there is a disk D which cuts $a_\pm$ and $d_\pm$, as in 3.11, where the labels of d_- and a_+ correspond to the labels of ∂D.

Note also that no edge crosses ∂D. An application of 3.13 and 3.14 yields a contradiction.

Case 2. As in Case 1, a_+ is the only special vertex in D_+, there are simple vertices in D_+, and no edge incident to a_+ crosses ∂D_+. Then there are exactly 4 edges (or 3, depending if $\Delta = 2$ or $\Delta = 3$) crossing ∂D_+ which come from simple vertices. These edges have to meet the negative side of c. Let k_1, k_2, k_3, k_4 be the labels corresponding to the negative side of c, ordered according to their cyclical occurrence in c. Let γ_1 (γ_4) = (edge incident at c at label k_1 $(k_4))\bigcap E_-$, and let α be the arc in c which goes through the negative side of c, and which joins the labels k_1 and k_4. There is an arc $\beta \subset \partial D_+$ joining the endpoints of γ_1 and γ_4, and so that there is a disk $F \subset E_-$, whose boundary is $\partial F = \gamma_1 \cup \gamma_4 \cup \alpha \cup \beta$. There are two possibilities:

(1) a_- lies in F. In this Case we have a situation as in 3.11, with a_- and the positive side of c as the vertices on the boundary of a disk D which contains F. An application of 3.13 and 3.14 yields a contradiction.

(2) a_- does not meet F. In this Case there is a disk D as in 3.11, with a_+ and the negative side of c corresponding to the vertices on ∂D. An application of 3.13 and 3.14 yields a contradiction.

Case 3. It is similar to Case 1, with the roles of d_- and a_+ interchanged.

Case 4. In this case 8 (or 6) edges cross ∂D_+, at most 4 (3) of them can reach the negative side of c, so there are 4 (3) edges incident to the negative side of b which cross ∂D_+, and 4 (3) coming from a simple vertex or the positive side of b which have to meet the negative side of c. Let k_1, k_2, k_3, k_4 be the labels corresponding to the negative side of c, ordered according to their cyclical ocurrence in c. Let γ_1 (γ_4) = (edge incident at c at label k_1 $(k_4))\bigcap E_-$, and let α be the arc in c which goes through the negative side of c, and which joins the labels k_1 and k_4. There is an arc $\beta \subset \partial D_+$ joining the endpoints of γ_1 and γ_4, and so that there is a disk $F \subset E_-$, whose boundary is $\partial F = \gamma_1 \cup \gamma_4 \cup \alpha \cup \beta$. There are two possibilities:

(1) The edges incident to the negative side of b meet F. We have a situation as in 3.11, with the negative side of b and the positive side of c as the vertices on the boundary of a disk D which contains F. An application of 3.13 and 3.14 yields a contradiction.

(2) The edges incident to the negative side of b do not meet F. Then there is a disk D as in 3.11, with the positive side of b and the negative side of c corresponding to the vertices on ∂D. An application of 3.13 and 3.14 yields a contradiction.

Case 5. It is similar to Case 4, in fact simpler, with d_- instead of c.

Case 7. It is similar to Case 4, just changing the roles of b and c.

Case 8. It is similar to Case 5, changing b by c, and d_- by a_+.

Case 6. In this case doing an argument identical to [**6**, 4.8] for the same case, we conclude that the 4 (3) edges incident to the positive side of b cross ∂D_+ and meet a_-, the 4 (3) edges incident to the negative side of b do not cross ∂D_+, and there are 4 (3) edges incident to simple vertices which cross ∂D_+ and meet d_-. Then there is a disk which cuts b and d_-, with the negative side of b and d_- as the vertices on ∂D, and no edge crosses ∂D. An application of 3.13 and 3.14 yields a contradiction.

Case 9. In this case doing an argument as in [**6**, 4.8], we conclude that there are 4 (3) edges crossing ∂D_+, all coming from the negative side of b. Then we can find a disk as in 3.11, where d_+ and the positive side of b correspond to the vertices on ∂D. 3.13 and 3.14 show that this is not possible.

Case 10. Doing an argument as in [**6**, 3.11 subclaims 2-4], we get that no edge joins the positive side of b and a_-, no edge incident to a simple vertex, or to the positive side of b, or to a_+ crosses ∂D_+. Then the 4 (3) edges incident to the negative side of b are the only edges which cross ∂D_+. So there is a disk D which cuts b and a_+, as in 3.11, a_+ and the positive side of b correspond to the vertices on ∂D, and no edge crosses ∂D. An application of 3.15 yields a contradiction.

Case 11. Doing an argument as in [**6**, 3.6 claims 1, **2**] we get that there are simple vertices in D_-, and no edge incident to a_- crosses ∂D_-. So there are 4 (3) edges crossing ∂D_-, all coming from simple vertices and which have to meet a_+ or the negative side of b. An argument as in [**6**, 3.7 claim] shows that none of the edges which cross ∂D_- meet a_+, so all have to meet the negative side of b. Here as in Cases 2 and 4 there are two possibilities, in one we find a disk as in 3.11 with a_- and the negative side of b as the vertices on ∂D, and in the other a_+ and the positive side of b correspond to the vertices on ∂D. In both cases an application of 3.15 yields a contradiction.

For Cases 12, 13 and 14 do an argument similar to Cases 9, 4 and 10 respectively. □

We have proved that (M_n, γ_n) is $\emptyset$-taut, then by 2.11 the Seifert surface S is $\emptyset$-taut, so $\chi(S) = \chi(k)$, $S^3 - k$ is irreducible and β always crosses S in the same direction. This implies Cases (c) and (d) of 1.3. This completes the proof of Theorem 1.3.

5. Proof of Corollary 1.5. In this section we prove Corollary 1.5. Let k be the trivial knot, and assume the hypothesis of Theorem 1.3. We want to prove that conclusions (b), (c) and (d) are not possible. Suppose there is a β-taut disk S with $\partial S = k$. The decomposition $(M, \gamma, \beta) \xrightarrow{S} (M_1, \gamma_1, \beta_1)$ is β-taut, and M_1 is a 3-ball with only one suture on its boundary, i.e. the sutured manifold hierarchy as constructed in 2.5 consist of only one step. ∂M_1 is divided into two disks denoted by D_+ and D_-.

LEMMA 5.1. *Case (b) of Theorem 1.3 does not happen.*

Proof. If it happens then β_α does not intersect S, so the only vertices in $D_\pm$ are $a_\pm$, $d_\pm$. Then there is a loop at a_+ or d_+, or there is an edge ϵ other than ϵ_a incident to a_+, say, which crosses ∂D_+. If the former happens then Q_α is ∂-compressible, a contradiction. If the latter happens then ϵ_a and the edge ϵ divides D_+ in two regions; one of them, call it D, does not contain d_+. Take an outermost edge incident to a_+ which crosses ∂D_+; this determines a region, call it D again, which does not contain edges in its interior. It is not difficult to see that, using D, the planar surface P can be isotoped to intersect W fewer times. □

5.2. Suppose in what follows that either $\Delta = 2$ and $p = 2$, or $\Delta = 3$ and $p = 1$. Keep notation as in §3 and 4.

Note that there are four points of intersection between P and k; this produces four edges, other than ϵ_a, ϵ_d, which cross the suture ∂D_+ $(= k)$. These edges are different, for otherwise there is an edge ϵ, say in D_+, joining two points on ∂D_+. ϵ divides D_+ in two parts; one of them, call it D, contains at most one of a_+ or d_+. If there are vertices in D then $\Gamma \bigcap D$ has $P(x)$, for some label x, which contradict 3.4, 3.5. If there is no vertex in D, this gives a ∂-compression disk for Q_α, which is a contradiction.

LEMMA 5.3. *No edge incident to $a_\pm$, $d_\pm$ crosses the suture ∂D_+. (other than ϵ_a, ϵ_d).*

Proof. Let ϵ be an edge incident to a_+ which crosses ∂D_+. ϵ divides D_+ in two regions, one of them, call it D, does not contain d_+. If there are vertices in the interior of D then $\Gamma \bigcap D$ has $P(x)$ for some label x, which contradicts 3.4, 3.5. If there is no vertex inside D, do an argument as in 5.1. □

LEMMA 5.4. *Any edge that crosses ∂D_+ is incident to the negative side of b or c.*

Proof. This is for index restrictions. Let ϵ be an edge crossing ∂D_+. ϵ is part of the boundary of a disk $q \subset Q_1$, with $I(q) = 0$, and ϵ contributes one to the index. If ϵ does not meet the negative side of a vertex, then it reaches either a simple vertex or the positive side of a special vertex, and the arc of ∂q next to ϵ contributes one to the index, so either q is a cancelling disk, which contradicts 2.7, or ϵ is incident to a_- or d_+, which contradicts 5.1. □

LEMMA 5.5. *Let ϵ be an edge that crosses ∂D_+. Then the ends of ϵ are equally labeled.*

Proof. This is because D_+ and D_- form the boundary of a regular neighborhood of the disk S used in the sutured manifold decomposition, and then they look identical. □

LEMMA 5.6. *There are simple vertices in D_+ and D_-.*

Proof. If there is no simple vertex then there are four edges joining b and c, and because there is no loop in Γ, 2 edges (or 1, depending if $\Delta = 2$ or 3) connect each of a_+ and d_+ to b, so two edges with consecutive labels connect a_+ and d_+.

Suppose first that $\Delta = 2$. a_+ is labeled $1, 2, 2, 1$ and d_+ is labeled $2, 1, 1, 2$. There are two edges connecting a_+ and d_+ with labels $1, 2$ in a_+ and labels $2, 1$ in d_+. These edges form a cycle with label sequence 1, 2, 1, 2, which contradicts 3.5.

Suppose that $\Delta = 3$. a_+ has labels $3, 2, 1$, and d_+ has labels $1, 6, 5$ or $4, 3, 2$ or $5, 4, 3$, or $2, 1, 6$, for a_+ and d_+ are antiparallel. Then there is an edge ϵ_2 connecting a_+ and d_+, with label 2 at a_+ and label 6, 3, 4, or 1 at d_+; note however that if the end of ϵ_2 at d_+ is 1 or 3, we contradict the parity rule. Let ϵ_1 be the other edge connecting a_+ and d_+. If the ends of ϵ_1 and ϵ_2 at a_+ are labeled 3, 2 respectively, and their ends at d_+ are labeled 5, 4 or 1, 6, then there is an innermost cycle with label sequence 2, 3, 5, 4 or 2, 3, 1, 6; note that both cases are in contradiction with 3.5. If the ends of ϵ_1 and ϵ_2 at a_+ are labeled 1, 2, and their ends at d_+ are labeled 5, 6, then there is an innermost cycle with label sequence 1, 2, 6, 5, which contradicts 3.5. Note however that if the ends of ϵ_1 and ϵ_2 at a_+ are labeled 1, 2, and their ends at d_+ are labeled 3, 4, then the innermost cycle with label sequence 1, 2, 4, 3 does not contradict 3.5. In this case there is no edge joining b and c with labels 4, or 5, for none of these labels is in the negative side of b or c, so there are four edges joining b and c with labels 6, 1, 2, 3. This shows that there are two edges connecting a_- and d_-, with labels 6, 5 and 2, 1 respectively; these edges form an innermost cycle with label sequence 6,5,1,2, which does contradict 3.5. □

LEMMA 5.7. *No edge joins b and c.*

Proof. If there is an edge then the argument of 5.5 shows that b and c come from the same point of intersection between β_α and S, i.e. β_α meets D only once. This implies that there is no simple vertex in D_+, contradicting 5.6. □

LEMMA 5.8. *If to a label x in the negative side of c is incident an edge ϵ_1 that crosses ∂D_-, then the edge ϵ_2 incident to d_- at label*

x is level or is incident to the negative side of c.

Proof. The edge ϵ_1 is part of the boundary of a disk $q \subset Q_1$, with $I(q) = 0$. ϵ_2 is part of the same disk q. By 5.7, ϵ_1 is incident to a simple vertex, and then by index restrictions ϵ_2 has to meet a simple vertex or the negative side of c. In the first case ϵ_2 has to be level because ϵ_1 is. □

LEMMA 5.9. *Let $\Delta = 2$. There is an x-cycle in D_+ or D_-, for some label x.*

Proof. Because each edge crossing ∂D_- is incident to the negative side of b or c, we can assume w.l.o.g. that there are at least 2 edges incident to the negative side of c which cross ∂D_-. Suppose the negative (positive) side of c has labels $2_3, 1_3, 1_1^*, 2_3^*$ $(2_1, 1_1, 1_3^*, 2_1^*)$, as in Figure 8. Note that no edge in D_- with an end labeled 1_3^* or 2_1^* can cross ∂D_-, for none of these labels is in the negative side of b or c. If 3 or 4 edges incident to the negative side of c cross ∂D_-, then 3 or 4 edges incident to d_- are level or are also incident to the negative side of c. Then for at least one of the labels 1_3^* or 2_1^* there is no edge connecting d_- and a vertex v, whose label at v is 1_3^* or 2_1^*. This shows that the graph $\Lambda = (\Gamma \bigcap D_-) - d_-$ has $P(1_3^*)$ or $P(2_1^*)$. So suppose exactly two edges incident to the negative side of c cross ∂D_-.

Suppose the edges incident to the negative side of c which cross ∂D_- have labels $2_3, 1_3$ in c. The edges incident to d_- at labels $2_3, 1_3$ are level or are incident to the negative side of c, so none of them has its other end labeled 1_3^* or 2_1^*. Let ϵ_1 be the edge incident to d_- at label 1_3; it is also incident to a vertex v. Let ϵ_2 be the edge incident to d_- at label 1_1^*; if the other endpoint of ϵ_2 is not labeled with 1_3^*, then Λ has $P(1_3^*)$, so suppose ϵ_2 has that label. Start a 1_3^*-path γ at the vertex v. This path will finish with the edge ϵ_2, for otherwise there is a 1_3^*-cycle in D_-. Note that ϵ_1, ϵ_2 and γ enclose a region which has $P(1_1)$.

There are other possibilities for the labels of the edges incident to the negative side of c which cross the suture ∂D_-, but a similar argument can be done. If the negative and positive side of c have the other possible labeling, a similar argument is done. □

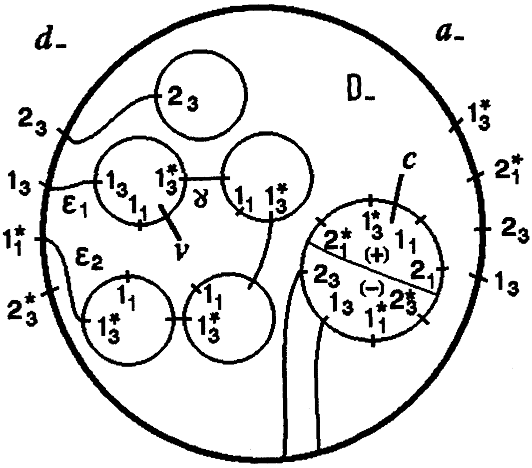

FIGURE 8.

LEMMA 5.10. *Let $\Delta = 3$. There is an x-cycle in D_+ or D_-, for some label x.*

Proof. Because each edge crossing ∂D_- is incident to the negative side of b or c, we can assume w.l.o.g. that there are at least 2 edges incident to the negative side of c which cross ∂D_-. Note that no edge in D_- with an end labeled 4, 5, 6 can cross ∂D_-, unless it is incident to the negative side of c, for none of these labels is in the negative side of b. Let Λ be as in 5.9.

There are several possibilities for the labeling of the negative side of c and d_-, namely 1, 6, 5 or 2, 1, 6 or 4, 3, 2, or 5, 4, 3. Consider the first two cases; in these cases no edge with a label 4 in D_- crosses ∂D_-, so we look for a 4-cycle in Λ. If 3 edges incident to the negative side of c cross ∂D_-, then all the edges incident to d_- are level, by 5.8, and it follows that Λ has $P(4)$. Suppose exactly two edges incident to the negative side of c cross ∂D_-; note that these edges have consecutive labels in c, for otherwise they enclose a region which has $P(4)$ or $P(5)$. If Λ does not have $P(4)$, then there is an edge ϵ joining d_- and a vertex v, whose label at v is 4, and its label at d_- is an even number for d_- is antiparallel to all vertices in D_-. If d_- is labelled 1, 6, 5, then by the previous observations there is no such edge ϵ. So suppose d_- is labeled 2, 1, 6; the endpoint of ϵ at d_- is 2 or 6, but note that in any case Λ has $P(5)$, see Figure 9.

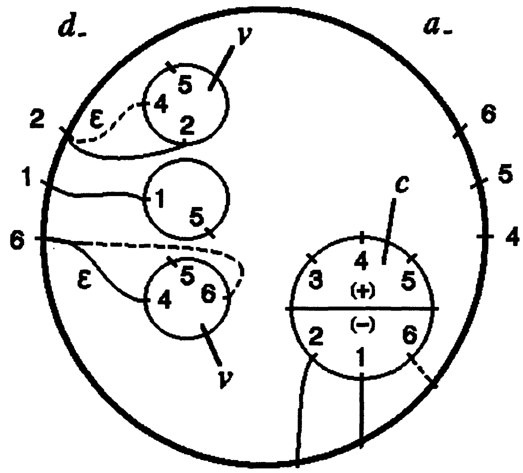

FIGURE 9.

When the negative side of c has the third or fourth possible labeling, do the same argument for the label 6. □

Clearly 5.9 and 5.10 are in contradiction with 3.4 and 3.5. This completes the proof of Corollary 1.5.

6. Surgery on strongly invertible knots.

6.1. Equivariant torus theorem. *Let M be an orientable, irreducible 3-manifold with an involution τ. Suppose M contains an incompressible torus. Then one of the following holds:*

(1) *There is an incompressible torus or Klein bottle T in $int(M)$ transversal to $Fix\tau$ with $T \bigcap \tau T = \emptyset$ or $\tau T = T$.*

(2) *$M = V_{-1} \bigcup V_1 \bigcup U_{-1} \bigcup U_1$, where V_i and U_i are solid tori, $\tau V_i = V_i$ and $\tau U_{-1} = U_1$. There are annuli A_i, $i = \pm 1$, with*

$$A_1 \bigcap A_{-1} = A_i \bigcap \tau A_i = \partial A_i = \partial \tau A_i = V_1 \bigcap V_{-1} = U_1 \bigcap U_{-1},$$

and $V_i \bigcap U_i = A_i$, $V_i \bigcap U_{-i} = \tau A_i$, $\partial V_i = A_i \bigcup \tau A_i$, $\partial U_i = A_i \bigcup \tau A_{-i}$. $A_1 \bigcup A_{-1}$ is an incompressible torus transversal to $Fix\tau$. $\tau | V_i$ is orientation preserving.

This Theorem follows from [**11**, 4.5]. See Figure 4 in [**11**] for an illustration of Case 2. Let $N = M/\tau$. If Case (2) of Theorem 6.1 happens then it is not difficult to see that M is a Seifert fiber space over the 2-sphere with 4 exceptional fibers, and that N is a lens space, $N \neq S^1 \times S^2$, S^3. If Case (1) happens with T being an equivariant Klein bottle, then an Euler characteristic argument shows that N contains either a Klein bottle or a proyective plane. These observations imply the following

COROLLARY 6.2. *Let M be a 3-manifold which is a double cover of S^3 branched along a link k, with deck translation τ. Suppose M contains an incompressible torus. Then there is an incompressible torus T in M which is equivariant, i.e. $\tau T = T$ or $\tau T \bigcap T = \emptyset$.*

THEOREM 6.3. *Let k be strongly invertible knot in S^3, which is not a satellite knot. If $M_k(r)$ contains an incompressible torus then*
(a) $\Delta(r, \mu) \leq 2$.
(b) *If $\Delta(r, \mu) = 2$ then there is an incompressible torus in $M_k(r)$ which intersects the surgery torus in two disks.*

Proof. As k is strongly invertible, there is an involution τ in S^3, with $\tau(k) = k$, and $k \bigcap \text{Fix}\,\tau = \{2 \textit{ points}\}$, $\text{Fix}\,\tau$ is an unknotted simple closed curve. A regular neighborhood $\eta(k)$ of k can be chosen so that it is invariant under τ. Then τ resticts to an involution on $M_k = S^3 - \text{int}\,\eta(k)$. It is not difficult to see that τ can be extended to an involution τ_r on $M_k(r)$ for all r.

It follows from [**15**] that $M_k(r)/\tau_r \cong S^3$, and that the projection $p : M_k(r) \to S^3$ is a double cover branched along a link $k_r = p(Fix\tau)$ of at most two components. The couple (S^3, k_r) can be decomposed as the sum of two tangles (B_2, t_2), (B_1, t_r), where (B_2, t_2) is the projection of the exterior of k, and (B_1, t_r) is the projection of the torus of surgery attached to the exterior of k. (B_2, t_2) is a prime tangle, (B_1, t_r) is a trivial tangle. $(B_1, t_{1/0})$ can be identified with the rational tangle $(B_1, 1/0)$ (cf. 1.1). $(S^3, k_{1/0})$ is the trivial knot; therefore (S^3, k_r) can be seen as obtained from $k_{1/0}$ by replacing the tangle $(B_1, 1/0)$ by the rational tangle $(B_1, r) \cong (B_1, t_r)$, as in 1.2.

Assume in what follows that $M_k(r)$ is irreducible. There is no loss of generality in doing so, because it is known that only integral surgeries can yield a reducible manifold [**9**]; further, by the solution of the cabling conjecture for strongly invertible knots [**6**], if this happened k would be a cable knot, i.e. a satellite knot or a torus knot.

We are assuming that $M_k(r)$ contains an incompressible torus; then by 6.1 there is an incompressible torus T, with $\tau T = T$ or $\tau T \bigcap T = \emptyset$. Suppose first that $\tau T \bigcap T = \emptyset$. In this case $P = p(T)$ is a torus disjoint from k_r. P is incompressible in $S^3 - k_r$, and is not parallel to the boundary of a neighborhood of k_r, for otherwise T would be compressible. So P is a satellite torus. By Corollary 1.6, if $\Delta(1/0, r) > 1$ the torus P can be isotoped to be disjoint from (B_1, r), and then T can be isotoped to be disjoint from $\eta(k)$, i.e. k is a satellite knot, contrary to the hypothesis.

Suppose now that $\tau T = T$. By an Euler characteristic argument $T \bigcap Fix\tau = \{$ 4 points $\}$. Then $P = p(T)$ is a sphere intersecting k_r in 4 points, i.e. P decomposes k_r as the sum of two tangles, which are nontrivial, for otherwise T would be compressible. Both tangles are prime for $M_k(r)$ is irreducible and T is incompressible (see [**13**, Theorem 5]). By Corollary 1.5, if $\Delta(1/0, r) > 2$ the sphere P can be isotoped to be disjoint from (B_1, r), and then T can be isotoped to be disjoint from k, so it is a satellite knot. If $\Delta(1/0, r) = 2$, then P can be isotoped so that $P \bigcap B_1$ consists of an essential disk in $B_1 - t_r$. This implies that $p^{-1}(P \bigcap B) = T \bigcap \eta(k)$ consists of two disks; that is, $T - \operatorname{int} \eta(k)$ is a genus one surface with two boundary components of slope r on $\partial\eta(k)$, where $\Delta(r, \mu) = 2$. □

Added in proof: C. McA. Gordon and J. Luecke have recently announced a proof of Theorem 6.3 for all hyperbolic knots.

References

[1] S. A. Bleiler and C. G. Hodgson, *Spherical space forms and Dehn surgery*, to appear.

[2] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational problems in abstract algebra, Pergamon Press, Oxford and New York (1969), 324-358.

[3] M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, *Dehn Surgery on knots*, Ann. of Math., **125** (1987), 237-300.

[4] C. Ernst and D. W. Sumners, *A calculus for rational tangles: applications to* DNA *recombination*, Math. Proc. Camb. Phil. Soc., **108** (1990), 489-515.

[5] M. Eudave-Muñoz, *Primeness and sum of tangles*, Trans. Amer. Math. Soc., **306** (1988), 773-790.

[6] M. Eudave-Muñoz, *Band sums of links which yield composite links. The cabling conjecture for strongly invertible knots*, Trans. Amer. Math.

Soc., **330** (1992), 463-501.

[7] C. McA. Gordon, *Dehn surgery on knots*, Proc. of the International Congress of Mathematics, Kyoto, Japan 1990, The Japan Mathematical Society, (1991), 631-642.

[8] C. McA. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, to appear.

[9] C. McA. Gordon and J. Luecke, *Only integral Dehn surgeries can yield reducible manifolds*, Math. Proc. Camb. Phil. Soc., **102** (1987), 97-101.

[10] A. Hatcher and U. Oertel, *Boundary slopes for Montesinos knots*, Topology, **28** (1989), 453-480.

[11] W. H. Holzmann, *An equivariant torus Theorem for involutions*, Trans. Amer. Math. Soc., **326** (1991), 887-906.

[12] T. Kobayashi, *Generalized unknotting operations and tangle decompositions*, Proc. Amer. Math. Soc., **105** (1989), 471-478.

[13] W. B. R. Lickorish, *Prime knots and tangles*, Trans. Amer. Math. Soc., **267** (1981), 321-332.

[14] J .M. Montesinos, *Variedades de Seifert que son recubridores cíclicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana (2), **18** (1973), 1-32.

[15] J. M. Montesinos, *Surgery on links and double branched covers of S^3*, Annals of Math. Studies no. 84, Princeton Univ. Press, Princeton, N.J., (1975), 227-260.

[16] M. Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom., **29** (1989), 557-614.

[17] M. Scharlemann, *Producing reducible manifolds by surgery on a knot*, Topology, **29** (1990), 481-500.

[18] M. Scharlemann and A. Thompson, *Unknotting number, genus, and companion tori*, Math. Ann., **280** (1988), 191-205.

[19] M. Scharlemann and A. Thompson, *Link genus and the Conway moves*, Comment. Math. Helvetici, **64** (1989), 527-535.

[20] W. Thurston, *Three dimensional manifolds, Kleininan groups and hyperbolic geometry*, Bull. Amer. Math. Soc., **6** (1982), 357-381.

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