THE NONHOMOGENEOUS MINIMAL SURFACE EQUATION
INVOLVING A MEASURE

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We find existence of a minimum in $BV$ for the variational problem associated with $\text{div } A(Du) + \mu = 0$, where $A$ is a mean curvature type operator and $\mu$ a nonnegative measure satisfying a suitable growth condition. We then show a local $L^\infty$ estimate for the minimum. A similar local $L^\infty$ estimate is shown for sub-solutions that are Sobolev rather than $BV$.

1. Introduction. In this paper we initiate an investigation of weak solutions of the

\begin{equation}
\text{div } A(Du) + \mu = 0
\end{equation}

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Here $A$ is a function for which the mean curvature operator is a prototype and $\mu$ is a nonnegative Radon measure supported in $\Omega$ that satisfies

\begin{equation}
\mu(B(r)) \leq Mr^{q(n-1)} \text{ for all } B(r) \subset \Omega,
\end{equation}

where $M > 0$ and $1 < q \leq \frac{n}{n-1}$.

This paper has its origins in the work of [LS] where it was shown that if $u$ is a weak solution of

$$
\Delta u = \mu,
$$

where $\mu$ is a measure that satisfies the growth condition

$$
\mu(B(r)) \leq Mr^{n-2+\varepsilon}
$$

for some $\varepsilon > 0$ and for all balls $B(r)$ of radius $r$, then $u$ is H"{o}lder continuous. In [RZ] this result was generalized to equations of the form

\begin{equation}
\text{div } A(x, u, \nabla u) + B(x, u, \nabla u) + \mu = 0
\end{equation}
where $\mu$ is a nonnegative Radon measure and $A$ and $B$ are Borel measurable functions satisfying structural conditions that allow, for example, the $p$-Laplacian. It is shown that if $u$ is a Hölder continuous solution of 1.3, then $\mu$ satisfies
\[ \mu(B(r)) \leq Mr^{n-p+\epsilon} \]
for some $\epsilon > 0$. Under further restrictions on the structural conditions, it was shown this growth condition on $\mu$ was sufficient for Hölder continuity of $u$.

Recently, Lieberman [L] improved the results in [RZ] by proving supremum inequalities for solutions of 1.3 without the restrictive structural conditions, thereby establishing necessary and sufficient conditions on the growth of $\mu$ for the Hölder continuity of solutions.

All of this analysis takes place in the framework surrounding the $p$-Laplacian, $p > 1$. It is our purpose to address the situation of $p = 1$. We first consider the question of existence of solutions of 1.5 in the case $A$ is the mean curvature operator. We establish a variational solution by minimizing
\[ \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx + \int_\Omega u \, d\mu \]
in the class $u \in BV(\Omega)$ where $u$ satisfies the Dirichlet condition $u^* = f$ on $\partial\Omega$, with $f$ an integrable function on $\partial\Omega$. In order to ensure the existence of a minimum, it is necessary to assume that the constant $M$ in 1.2 is chosen sufficiently small. This is analogous to the assumption made in [M], in which $\mu$ is taken as a bounded measurable function. We then show that the minimizer $u$ is bounded. In this context, it is not possible to utilize the argument given in [L] to obtain an $L^\infty$ bound since there is no variational equation associated with 1.4. Rather, we employ a technique used in [RZ] modeled on the method of DeGiorgi.

Next, we investigate an equation which contains the formal Euler-Lagrange equation of 1.4. Thus, we consider a weak solution $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ of the equation
\[ \text{div } A(Du) + \mu = 0 \]
where we assume there exist non-negative constants $a_1, a_2$ such that
\[ p \cdot A(p) \geq |p| - a_1 \]
and

(1.7) \[ |A(p)| \leq a_2. \]

It is assumed that \( \mu \) is a nonnegative Radon measure supported in the bounded domain \( \Omega \) and satisfies 1.2. We show that if \( u \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \) is a weak solution of 1.5, then \( |u| \) is bounded by the \( L^1 \)-norm of \( u \) with respect to the measure \( d\nu = dx + d\mu \). Specifically, we show that \( u \) satisfies a supremum inequality, 6.4. The proof of this follows the proof in the corresponding result of [L]. The method of DeGiorgi will still work in this case, however the Moser iteration method used in [L] gives a slightly different result and is included for this reason. It is well known that weak solutions of 1.5 are not necessarily continuous, even under the assumption that \( \mu \) is an absolutely continuous measure with bounded density (c.f. [M]). Therefore, it is not possible to obtain the weak Harnack inequality involving a lower bound for the solution.

The results of this paper are valid for equations with a more general structure. For the sake of simplicity, we employ this simple structure which fully illustrates the method. In a forthcoming paper, we will address the question of regularity of solutions of 1.4 in which almost everywhere continuity is established. The existence of an a priori \( L^\infty \) bound will be essential in this future investigation.

2. Preliminaries. Throughout, we assume that \( \Omega \) is a bounded Lipschitz domain in \( R^n \). The space \( W^{1,1}(\Omega) \) is the space of \( L^1(\Omega) \) functions whose distributional derivatives also lie in \( L^1(\Omega) \).

The class of all functions in \( L^1(\Omega) \) whose distributional partial derivatives are measures with finite total variation in \( \Omega \) comprise the space \( BV(\Omega) \). The notation

\[
\int_{\Omega} |Du| \, dx
\]

will be used to represent the total variation of the vector-valued measure, \( Du \), the gradient of \( u \). Specifically, the total variation of \( Du \) is

\[
\sup \left\{ \int_{\Omega} u \, \text{div} \, v \, dx : v = (v_1, \ldots, v_n) \in C_0^\infty(\Omega; R^n), |v| \leq 1 \right\}.
\]
We also make the notational definition
\[ \int_\Omega \sqrt{1 + |Du|^2} \, dx \]
\[ = \sup \left\{ \int_\Omega (f \, \text{div} \, v + v_0) \, dx : v = (v_1, \ldots, v_n) \in C_0^\infty(\Omega), \right. \]
\[ \left. v_0 \in C_0^\infty(\Omega), \ |v|^2 + |v_0|^2 \leq 1 \right\}. \]

The space \( BV(\Omega) \) is equipped with the norm
\[ \|u\|_{BV} = \int_\Omega |u| \, dx + \int_\Omega |Du| \, dx. \]

The trace of \( u \) on \( \partial \Omega \) is denoted by \( u^* \) (c.f. [Z, Section 5.10]). We will make use of the following lemma on the convergence of the traces of \( BV \) functions.

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) a bounded Lipschitz domain, and let \( \{u_k\}, u \in BV(\Omega) \) with
\[ \lim_{k \to \infty} \int_\Omega |u_k - u| \, dx = 0 \]
\[ \lim_{k \to \infty} \int_\Omega \sqrt{1 + |Du_k|^2} \, dx = \int_\Omega \sqrt{1 + |Du|^2} \, dx. \]

Then
\[ \lim_{k \to \infty} \int_{\partial \Omega} |u_k^* - u^*| \, dH^{n-1} = 0, \]
with \( H^{n-1} \) the \( n-1 \) dimensional Hausdorff measure.

The proof follows directly from the proof in [G, Proposition 2.6; p.34].

We will also have need for the following compactness result for \( BV \) functions [Z, Corollary 5.3.4; p. 227].

**Theorem 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. Then \( BV(\Omega) \cap \{u : \|u\|_{BV(\Omega)} \leq 1\} \) is compact in \( L^1(\Omega) \).

It was shown in [MZ] that if \( \mu \) satisfies the growth condition \( \mu(B(r)) \leq Mr^{n-1} \) on all balls \( B(r) \) (and therefore condition 1.2 in
particular), then $\mu$ can be identified with an element of the dual of $BV(\Omega)$. Furthermore, its norm

$$\tilde{M} = \|\mu\| = \sup \left\{ \int_{\Omega} u \, d\mu : \|u\|_{BV(\Omega)} \leq 1 \right\}$$

is comparable to $M$. Thus,

$$(2.1) \quad \left| \int_{\Omega} u \, d\mu \right| \leq \int_{\Omega} |u| \, d\mu$$

$$\leq \|\mu\| \|u\|_{BV(\Omega)}$$

$$\leq \tilde{M} \|u\|_{BV(\Omega)}$$

The following well known result, [M], will be used in the existence proof below.

$$(2.2) \quad \int_{\Omega} |u| \, dx \leq C \left( \int_{\Omega} |Du| \, dx + \int_{\partial \Omega} u^* \, dH^{n-1} \right)$$

with the constant $C = C(\Omega)$. This yields

$$(2.3) \quad \|u\|_{BV(\Omega)} \leq C \left( \int_{\Omega} |Du| \, dx + \int_{\partial \Omega} u^* \, dH^{n-1} \right)$$

Finally, we state the following Sobolev inequalities which are of critical importance in our development.

**THEOREM 2.3.** Let $\Omega$ be a bounded Lipschitz domain and suppose $\mu$ is a measure supported in $\Omega$ satisfying condition 1.2. Then there exists a constant $C = C(\Omega, q, n)$ such that

$$(2.4) \quad \left( \int_{\Omega} u^q \, d\mu \right)^{1/q} \leq CM^{1/q} \int_{\Omega} |Du| \, dx$$

whenever $u \in BV(\Omega)$ with compact support in $\Omega$.

The proof may be found in [Z, Lemma 4.9.1; p. 209]. Also needed is the standard Sobolev inequality for $W^{1,1}$.

If $u \in W^{1,1}_0(\Omega)$ then there exists a constant $C = C(\Omega, q, n)$ such that

$$(2.5) \quad \left( \int_{\Omega} u^q \, dx \right)^{1/q} \leq C \|Du\|_1.$$  

This is simply the above lemma in the special case that $\mu$ is Lebesgue measure.
3. Existence of a Minimum. With \( \Omega \) a bounded Lipschitz domain and \( f \in L^1(\partial \Omega) \), we define \( I(u; \Omega) \) as follows,

\[
I(u; \Omega) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx + \int_{\Omega} u \, d\mu + \int_{\partial \Omega} |u^* - f| \, dH^{n-1}.
\]

We wish to minimize \( I \) over all \( u \in BV(\Omega) \). That is, we wish to find a function \( u \in BV(\Omega) \) such that

\[
I(u; \text{supp } \varphi) \leq I(u + \varphi; \text{supp } \varphi), \quad \forall \varphi \in C_c^\infty(\Omega).
\]

**Theorem 3.1.** Let \( \Omega \) be a bounded Lipschitz domain. With \( I \) defined as above, there exists \( u \in BV(\Omega) \) such that

\[
I(u; \Omega) = \min_{v \in BV(\Omega)} I(v; \Omega).
\]

**Proof.** Following [G, Section 14.4], the first step is to consider a slightly different Dirichlet problem in the complement of \( \Omega \). For this purpose, let \( B \) be a ball that contains \( \overline{\Omega}, \) the closure of \( \Omega \). Use Theorem 2.16 of [G] to extend \( f \) to a \( W^{1,1} \) function in \( B - \overline{\Omega} \) that will still be denoted by \( f \). Let

\[
J(u; B) = \int_B \sqrt{1 + |Du|^2} \, dx + \int_B u \, d\mu.
\]

Note that since \( \text{supp } \mu \subset \Omega \), the second integral could have been taken over \( \Omega \). We wish to show that there exists \( u \in BV(B) \), coinciding with \( f \) in \( B - \overline{\Omega} \), that minimizes \( J(u; B) \). We proceed by showing that \( J \) is bounded below if the constant \( M \) in 1.2 is sufficiently small.

\[
J(u; B) \geq \int_B |Du| \, dx + \int_{\Omega} u \, d\mu \\
\text{(by 2.1)} \geq \int_B |Du| \, dx - \tilde{M} \|u\|_{BV(\Omega)} \\
\geq \int_B |Du| \, dx - \tilde{M} \left( C \int_{\partial \Omega} u^*_\Omega \, dH^{n-1} \right) \\
\text{(by 2.3)} \geq \frac{1}{2} \int_B |Du| \, dx - \tilde{M} C \int_{\partial \Omega} f \, dH^{n-1}.
\]
The last inequality is obtained when $M$ is small enough to insure $1 - M(C + 1) \geq \frac{1}{2}$.

Let $J(u_k) \to \lambda$ a minimum of $J$. We wish to find $u \in BV(B)$ such that $J(u; B) = \lambda$. For sufficiently large $k$ we obtain from the above inequality that

$$
\lambda + 1 \geq \frac{1}{2} \int_B |Du_k| \, dx - MC \int_\Omega f \, dH^{n-1}.
$$

Thus the terms $\int_B |Du_k| \, dx$ are uniformly bounded, which implies by 2.3 and Theorem 2.2 that there exists $u \in BV(B)$ with $u_k \to u$ in $L^1(B)$. The gradient is lower semi-continuous with respect to $L^1(B)$ convergence so that

$$
\liminf_{k \to \infty} \int_B \sqrt{1 + |Du_k|^2} \, dx \geq \int_B \sqrt{1 + |Du|^2} \, dx.
$$

From Theorem 2.3, the uniform bound on $\int_B |Du_k| \, dx$ also implies that the terms

$$
\left( \int_\Omega \frac{q}{k} \, d\mu \right)^{1/q}
$$

are uniformly bounded. Thus there exists a subsequence, denote it by $\{u_k\}$, that converges weakly in $L^q(\Omega; \mu)$ to some $w \in L^q(\Omega; \mu)$. The Banach-Saks Theorem implies that there exists a subsequence of $\{u_k\}$, again denote it by $\{u_k\}$, such that the sequence of Césaro sums, $\{v_k\}$, defined by

$$
v_k = \frac{u_1 + \cdots + u_k}{k}
$$

converges strongly to $w$ in $L^q(\Omega; \mu)$. Moreover, the sequence $v_k$ also converges strongly to $u$ in $L^1(\Omega)$. This can be seen as follows: choose $\varepsilon > 0$ and let $N$ denote an integer for which $\|u_j - u\|_{L^1(\Omega)} < \varepsilon$ for $j, k \geq N$. Then for $j \leq k,$

$$
\|v_k - u\| = \left\| \frac{(u_1 - u) + \cdots + (u_k - u)}{k} \right\|
\leq \frac{\|u_1 - u\| + \cdots + \|u_{j-1} - u\|}{k} + \frac{\|u_j - u\| + \cdots + \|u_k - u\|}{k}
\leq \frac{\|u_1 - u\| + \cdots + \|u_{j-1} - u\|}{k} + \frac{(k - j + 1)\varepsilon}{k}.
$$
Thus,

$$\limsup_{k \to \infty} \|v_k - u\| \leq \epsilon,$$

which yields the desired result since $\epsilon$ is arbitrary. To show that $w = u$ almost everywhere in $\Omega$ note that the strong convergence of $\{v_k\}$ to $w$ in $L^q(\Omega; \mu)$ implies the existence of a subsequence that converges pointwise to $w$ $\mu$-almost everywhere and therefore (Lebesgue) almost everywhere, since Lebesgue measure is absolutely continuous with respect to $\mu$ in $\Omega$. But the strong convergence of $\{v_k\}$ to $u$ in $L^1(\Omega)$ implies the almost everywhere pointwise convergence of a further subsequence to $u$ in $\Omega$. Hence, $u = w$ almost everywhere in $\Omega$.

Since $u_k$ converges weakly to $u$ in $L^q(\Omega; \mu)$, the lower semicontinuity of the gradient with respect to $L^1(\Omega)$ convergence implies

$$\lambda = \liminf_{k \to \infty} J(u_k; B) \geq J(u; B).$$

Since $u_k$ agrees almost everywhere with $f$ in $B - \overline{\Omega}$, it follows that $u = f$ a.e. in $B - \overline{\Omega}$, thus showing that $J(u; B) \geq \lambda$. This completes the first step.

We now proceed with the second and final step of the proof. For each function $v \in BV(\Omega)$, define

$$v_f(x) = \begin{cases} 
  v(x) & x \in \Omega \\
  f(x) & x \in B - \Omega 
\end{cases}$$

Then $v_f \in BV(B)$ and by (2.15) of [G],

$$\int_B \sqrt{1 + |Dv_f|^2} \, dx + \int_B v_f \, d\mu = \int_B \sqrt{1 + |Dv|^2} \, dx + \int_{B - \overline{\Omega}} \sqrt{1 + |Df|^2} \, dx$$

$$+ \int_B v_f \, d\mu + \int_{\partial \Omega} |v_f^* - f| \, dH^{n-1}$$

$$= I(v; \Omega) + \int_{B - \overline{\Omega}} \sqrt{1 + |Df|^2} \, dx$$

That is,

$$J(v_f; B) = I(v; \Omega) + \int_{B - \overline{\Omega}} \sqrt{1 + |Df|^2} \, dx.$$

Thus, a minimizer of $J(v; B)$ with $v = f$ on $B - \overline{\Omega}$ produces a minimizer of $I(v; \Omega).$
4. An energy inequality. Now that we have obtained existence of a solution \( u \in BV(\Omega) \) to 1.4, we will show that \( u \) is bounded. Before doing this we will obtain an energy estimate to be used in the DeGiorgi type argument of section 5.

Let \( B_R \) denote the ball of radius \( R \) in \( \mathbb{R}^n \). Let \( \eta \) be a cutoff function, \( \eta = 1 \) on \( B_r \), \( 0 < r < r^* \leq R \), \( \eta = 0 \) on \( \partial B_{r^*} \) with \( 0 \leq \eta \leq 1 \) on \( B_{r^*} \) and \( |D\eta| \leq \frac{2}{r^* - r} \). Let \( \varphi = -\eta(u - k)^+ \), then \( \text{supp } \varphi = A_k = \{ u > k \} \cap B_{r^*} \) and

\[
I(u; A_k) \leq I(u + \varphi; A_k)
\]

Using

\[
(4.2) \quad \int_{A_k} |Du| \, dx \leq \int_{A_k} \sqrt{1 + |Du|^2} \, dx \leq \int_{A_k} |Du| + 1 \, dx
\]

and that on \( A_k \)

\[
D(u + \varphi) = (1 - \eta)D(u - k)^+ - D\eta(u - k)^+,
\]

we obtain from 4.1

\[
\int_{A_k} |D(u - k)^+| \, dx \leq \int_{A_k} (1 - \eta) |D(u - k)^+| \, dx
\]

\[
+ \frac{2}{r^* - r} \int_{A_k} (u - k)^+ \, dx
\]

\[
+ \int_{A_k} \eta (u - k)^+ \, d\mu + |A_k|
\]

where \( |A_k| \) is the Lebesgue measure of \( A_k \). This immediately implies

\[
(4.3) \quad \int_{B_r} |D(u - k)^+| \, dx \leq \int_{B_{r^*}} \eta |D(u - k)^+| \, dx
\]

\[
\leq \frac{2}{r^* - r} \int_{B_{r^*}} (u - k)^+ \, dx
\]

\[
+ \int_{B_{r^*}} (u - k)^+ \, d\mu + |A_k|.
\]

5. Supremum estimate for variational solutions.

**Theorem 5.1.** Let \( \sigma \in (0,1) \), \( \Omega \) a bounded Lipschitz domain, and \( B_R \subset \Omega \) with \( R < 1 \). Then for \( u \in BV(\Omega) \) a minimum of \( I \) there exists a constant \( C = C(\sigma, M) \) such that

\[
\sup_{B_{\sigma R}} u \leq C \left( R^{-n} \int_{B_R} u^+ \, dx + R^{-q(n-1)} \int_{B_R} u^+ \, d\mu \right)
\]

where \( q \) is the constant from 1.2 and \( u^+ \) is the positive part of \( u \).

**Proof.** Let \( k \) be a positive constant to be specified later. Set

\[
k_i = k(1 - 2^{-i}), \quad r_i = \sigma R + 2^{-i} R(1 - \sigma), \quad \text{and} \quad \tilde{r}_i = \frac{1}{2}(r_i + r_{i+1}).
\]

For notational convenience, denote by \( B_i \) the ball of radius \( r_i \), \( \tilde{B}_i \) the ball of radius \( \tilde{r}_i \), and let

\[
A_i = B_i \cap \{(u - k_{i+1})^+ > 0\}.
\]

Note that \( B_{i+1} \subset \tilde{B}_i \subset B_i \). Also, for all \( j \) we will use the notation

\[
\int_{B_j} dx = R^{-n} \int_{B_j} dx \quad \text{and} \quad \int_{B_j} d\mu = R^{-q(n-1)} \int_{B_j} d\mu.
\]

Let \( \varphi_i \) be the cutoff functions on \( \tilde{B}_i \) so that \( \varphi_i \equiv 1 \) on \( B_{i+1} \) and

\[
|D \varphi_i| \leq \frac{2}{\tilde{r}_i - r_{i+1}} = \frac{2^{i+3}}{R(1 - \sigma)}.
\]

Then 4.3 implies

\[
\int_{B_{i+1}} |D(u - k_{i+1})^+| \, dx \leq \frac{2^{i+3}}{R(1 - \sigma)} \int_{B_i} (u - k_{i+1})^+ \, dx
\]

\[
+ R^{-n+q(n-1)} \int_{\tilde{B}_i} (u - k_{i+1})^+ \, d\mu + R^{-n} |A_i|.
\]
Now, by 2.4 and 5.1,
\[
\int_{B_{i+1}} (u - k_{i+1})^+ \, d\mu \\
\leq \int_{B_i} \varphi_i (u - k_{i+1})^+ \, d\mu \\
\leq \left( \int_{B_i} \left( \varphi_i (u - k_{i+1})^+ \right)^q \, d\mu \right)^{1/q} (R^{-q(n-1)} \mu(A_i))^{1-1/q} \\
\leq C M^{1/q} \int_{B_i} |D \left( \varphi_i (u - k_{i+1})^+ \right)| \, dx (R^{-q(n-1)} \mu(A_i))^{1-1/q} \\
\leq C R M^{1/q} \left( \int_{B_i} |D (u - k_{i+1})^+| \varphi_i \, dx \\
+ \int_{B_i} (u - k_{i+1})^+ |D \varphi_i| \, dx \right) (R^{-q(n-1)} \mu(A_i))^{1-1/q} \\
\leq C R M^{1/q} \left( \int_{B_i} |D (u - k_{i+1})^+| \, dx \\
+ \frac{2^{i+3}}{R(1 - \sigma)} \int_{B_i} (u - k_{i+1})^+ \, dx \right) (R^{-q(n-1)} \mu(A_i))^{1-1/q}.
\]

Applying 5.2 we have
\[
\int_{B_{i+1}} (u - k_{i+1})^+ \, d\mu \\
\leq C R M^{1/q} \left( \frac{2^{i+4}}{R(1 - \sigma)} \int_{B_i} (u - k_{i+1})^+ \, dx \\
+ R^{-n+q(n-1)} \int_{B_i} (u - k_{i+1})^+ \, d\mu \\
+ R^{-n} |A_i| \right) (R^{-q(n-1)} \mu(A_i))^{1-1/q}.
\]

Thus we have the following iteration inequality,
\[
(5.3) \int_{B_{i+1}} (u - k_{i+1})^+ \, d\mu \\
\leq C M^{1/q} \frac{2^{i+4}}{(1 - \sigma)} \left( \int_{B_i} (u - k_i)^+ \, dx \\
+ \int_{B_i} (u - k_i)^+ \, d\mu + R^{-n} |A_i| \right) (R^{-q(n-1)} \mu(A_i))^{1-1/q}.
\]
To estimate the quantity \( \mu(A_i) \) recall that
\[
A_i = \{ u > k_{i+1} \} \cap B_i,
\]
and note that
\[
J_f <i - h_i = k (l - 2^i - 1) - k (l - 2^{-i+1}) = 2^{-i} k.
\]
which implies
\[
2^{-i} k < u - k_i \text{ on } A_i.
\]
Thus
\[
(5.4) \quad R^{-q(n-1)} \mu(A_i) \leq 2^{i+1} k^{-1} \int_{B_i} (u - k_i)^+ d\mu
\leq 2^{i+1} Y_i.
\]
where
\[
Y_i = k^{-1} \int_{B_i} (u - k_i)^+ dx + k^{-1} \int_{B_i} (u - k_i)^+ d\mu.
\]
We estimate \( |A_i| \) in the same manner, obtaining
\[
(5.5) \quad R^{-n} |A_i| \leq 2^{i+1} Y_i.
\]
Using 5.4 and 5.5 in 5.3 we obtain
\[
(5.6) \quad k^{-1} \int_{B_i+1} (u - k_{i+1})^+ d\mu
\leq CM^{1/q} \frac{2^{i+4}}{(1 - \sigma)} \left( k^{-1} \int_{B_i} (u - k_i)^+ dx ight.
\]
\[
+ k^{-1} \int_{B_i} (u - k_i)^+ d\mu + k^{-1} 2^{i+1} Y_i \left) \left( 2^{i+1} Y_i \right)^{1-1/q}
\leq CM^{1/q} \frac{2^{i+4}}{(1 - \sigma)} \left( (1 + k^{-1} 2^{i+1}) Y_i \right) \left( 2^{i+1} Y_i \right)^{1-1/q}
\leq CM^{1/q} \frac{2^{i+4}}{(1 - \sigma)(k^{-1} + 2^{-i-1}) \left( 2^{i+1} Y_i \right)^{1+\alpha}}.
\]
where \( \alpha = 1 - 1/q > 0 \). Following the same analysis for \( dx \) instead of \( d\mu \) we obtain

\[
(5.7) \quad k^{-1} \int_{B_{i+1}} (u - k_{i+1})^+ \, dx 
\leq CM^{1/q} \left( \frac{2^{i+4}}{(1 - \sigma)} \right) (k^{-1} + 2^{-i-1}) (2^{i+1}Y_i)^{1+\alpha}.
\]

Combining 5.6 and 5.7, we have

\[
(5.8) \quad Y_{i+1} \leq CM^{1/q} \left( \frac{2^{i+4}}{(1 - \sigma)} \right) (k^{-1} + 2^{-i-1}) (2^{i+1}Y_i)^{1+\alpha} 
\leq CM^{1/q} \left( \frac{2^{i+4}}{\kappa(1 - \sigma)} \right) (2^{i+1}Y_i)^{1+\alpha}
\]

where \( \kappa = \min(1, 1/(k^{-1} + 2^{-1})) \). The recursion lemma of [LU, lemma 4.7; p. 66] then implies that \( Y_i \to 0 \), and thus

\[
\sup_{B_{\sigma R}} u \leq k,
\]

provided that

\[
Y_0 = k^{-1} \int_{B_R} u^+ \, dx + k^{-1} \int_{B_R} u^+ \, d\mu
\leq \left( CM^{1/q} \frac{2^{5+\alpha}}{\kappa(1 - \sigma)} \right)^{-1/\alpha} \left( 2^{2+\alpha} \right)^{-1/\alpha^2}.
\]

This is true if

\[
k^{1/\alpha} \geq \left( \frac{CM^{1/q}2^{\alpha+6+2/\alpha}}{(1 - \sigma)} \right)^{1/\alpha} \left( \int_{B_R} u^+ \, dx + \int_{B_R} u^+ \, d\mu \right).
\]

Since \( k^{1/\alpha} \leq 1 \), the result follows. \( \square \)

6. A supremum estimate for weak solutions. We will use a different version of the Sobolev inequalities 2.4 and 2.5.

**Corollary 6.1.** Let \( B_R \) a ball of radius \( R \) in \( \mathbb{R}^n \). Suppose \( u \in W^{1,1}_0(B_R) \) and \( \mu \) is a measure satisfying 1.2, then there exists a constant \( C = C(q, n) \) such that

\[
(6.1) \quad \left( R^{-q(n-1)} \int_{B_R} u^q \, d\mu \right)^{1/q} \leq M^{1/q} CR^{1-n} \int_{B_R} |Du| \, dx
\]
and

\[(6.2) \quad \left( R^{-n} \int_{B_R} u^q \, dx \right)^{1/q} \leq CR^{1-n} \int_{B_R} |Du| \, dx. \]

Let \( u^+ \) denote the positive part of \( u \).

**Theorem 6.2.** Let \( B_R \subset \mathbb{R}^n \) a ball of radius \( R < 1 \). Suppose that \( u \in W^{1,1}(B_R) \cap L^\infty(B_R) \) satisfies the inequality

\[(6.3) \quad \text{div} \ A(Du) + \mu \geq 0 \quad \text{in} \ B_R \]

with \( A \) satisfying 1.6 and 1.7, and \( \mu \) a Radon measure satisfying 1.2. Then for any \( \varepsilon > 0 \) there exists a constant \( C = C(q, n, (a_1 + a_2)/\varepsilon) \) such that

\[(6.4) \quad \sup_{B_{R/2}} |u| \leq C \left( R^{-n} \int_{B_R} u^+ \, dx + R^{-q(n-1)} \int_{B_R} u^+ \, d\mu \right) + \varepsilon \]

**Proof.** Let \( \varepsilon > 0 \) and \( R < 1 \). Fix a cutoff function \( \eta \in C_0^\infty(B_R) \) such that \( \eta = 1 \) in \( B_{R/2} \), \( \eta = 0 \) on \( \partial B_R \), and \( 0 \leq \eta \leq 1 \) in \( B_R \) with \( |D\eta| \leq 4/R \). Set \( \zeta = \eta (1 - \frac{\varepsilon}{u})^+ \) and \( A_\varepsilon = \{ \zeta > 0 \} = \{ u > \varepsilon \} \subset B_R \). Consider the weak formulation of 6.3 with test function \( \zeta^{ks-t}u^s \), for constants \( k, s \) and \( t \) to be chosen later.

\[
(ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1}sD\zeta \cdot A(Du) \, dx \\
+ s \int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}Du \cdot A(Du) \, dx 
\leq \int_{A_\varepsilon} \zeta^{ks-t}u^s \, d\mu.
\]

Use that \( D\zeta = D\eta(1 - \frac{\varepsilon}{u}) + \eta \varepsilon u^{-2}Du \) and 1.6 to obtain

\[
(ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1}u^s(1 - \frac{\varepsilon}{u})D\eta \cdot A(Du) \, dx \\
+ (ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1}u^s \eta \varepsilon u^{-2}(|Du| - a_1) \, dx \\
+ s \int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}(|Du| - a_1) \, dx 
\leq \int_{A_\varepsilon} \zeta^{ks-t}u^s \, d\mu.
\]

Which implies that

\[ s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1} |Du| \, dx \leq \int_{A_\varepsilon} \zeta^{ks-t} u^s \, d\mu \]

\[ + (ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^s (1 - \frac{\varepsilon}{u}) D\eta \cdot A(Du) \, dx \]

\[ + (ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \eta \varepsilon u^{-2}(a_1) \, dx \]

\[ + s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1}(a_1) \, dx. \]

Use 1.7 and that \( \varepsilon/u < 1 \) in \( A_\varepsilon \) to obtain

\[ (6.5) \]

\[ s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1} |Du| \, dx \]

\[ \leq \int_{A_\varepsilon} \zeta^{ks-t} u^s \, d\mu + \frac{a_2 A(k_s - t)}{R} \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, dx \]

\[ + (ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^s (a_1 u^{-1}) \, dx \]

\[ + s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1}(a_1) \, dx \]

\[ \leq \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, d\mu + \frac{a_2 A(k_s - t)}{R} \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, dx \]

\[ + \int_{A_\varepsilon} \zeta^{ks-t-1} u^s (a_1 \varepsilon^{-1}(ks - t + s)) \, dx \]

\[ \leq \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, d\mu \]

\[ + \frac{a_2 A(k_s - t) + a_1 (k_s - t + s)}{\varepsilon R} \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, dx. \]

Set \( w = \zeta^{ks-t} u^s \) and consider

\[ \int_{A_\varepsilon} |Dw| \, dx \leq s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1} |Du| \, dx \]

\[ + (ks - t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^s |D\zeta| \, dx \]
\[ \leq s \int_{A_{e}} \zeta^{ks-t} u^{s-1} |Du| \, dx \]
\[ + (ks - t) \int_{A_{e}} \zeta^{ks-t-1} u^{s} \left( \frac{1}{R} + u^{-1} |Du| \right) \, dx \]
\[ \leq (s + ks - t) \int_{A_{e}} \zeta^{ks-t-1} u^{s-1} |Du| \, dx \]
\[ + \frac{(ks - t)}{R} \int_{A_{e}} \zeta^{ks-t-1} u^{s} \, dx. \]

Then use 6.5 to obtain the energy type estimate

\[ (6.6) \]
\[ \int_{A_{e}} |Dw| \, dx \]
\[ \leq \frac{s + ks - t}{s} \left( \int_{A_{e}} \zeta^{ks-t-2} u^{s} \, d\mu \right) \]
\[ + \frac{a_{2}4(ks - t - 1) + a_{1}(ks - t - 1 + s)}{\varepsilon R} \int_{A_{e}} \zeta^{ks-t-2} u^{s} \, dx \]
\[ + \frac{(ks - t)}{R} \int_{A_{e}} \zeta^{ks-t-2} u^{s} \, dx \]
\[ \leq s(1 + k) \left( \int_{A_{e}} \zeta^{ks-t-2} u^{s} \, d\mu + \left( 4k \frac{a_{1} + a_{2}}{\varepsilon} + 1 \right) \right) \]
\[ \cdot \frac{1}{R} \int_{A_{e}} \zeta^{ks-t-2} u^{s} \, dx, \text{ for } s \geq 1, t \geq 0, \text{ and } k \geq 1/5. \]

Sobolev inequalities 6.1 and 6.2 imply

\[ (6.7) \quad \left( R^{-n} \int_{A_{e}} w^{q} \, dx \right)^{1/q} + \left( M^{-1} R^{-q(n-1)} \int_{A_{e}} w^{q} \, d\mu \right)^{1/q} \]
\[ \leq CR^{-(n-1)} \int_{A_{e}} |Dw| \, dx \]

with \( C = C(n, q) \). Define \( v = \zeta^{k} u \) and set \( t = \frac{2}{q-1} \), so that \( tq = t+2 \).

Also, define a measure \( \nu \) by

\[ d\nu = \frac{dx}{R^{n} \zeta^{t+2}} + \frac{d\mu}{R^{q(n-1)} \zeta^{t+2}}. \]
which is supported on \( A_\varepsilon = \{u > \varepsilon\} \cap B_R \). We combine inequalities 6.6 and 6.7 to yield

\[
(6.8) \quad \left( \int_{A_\varepsilon} v^s d\nu \right)^{1/q} \leq Cs \int_{A_\varepsilon} v^s d\nu.
\]

where \( C = C(q, n, (a_1 + a_2)/\varepsilon) \), since \( k \) will be chosen later to be \( \frac{2}{q-1} + 2 \) and \( s \geq 1 \) will be used.

We now iterate on the inequality 6.8. Take \( s = 1 \) in the first iteration,

\[
\frac{1}{C} \left( \int_{A_\varepsilon} v^q d\nu \right)^{1/q} \leq \int_{A_\varepsilon} v d\nu.
\]

Take \( s = q \) in the second iteration,

\[
\frac{1}{C} \left( \frac{1}{Cq} \left( \int_{A_\varepsilon} v^{q^2} d\nu \right)^{1/q} \right)^{1/q} \leq \int_{A_\varepsilon} v d\nu.
\]

Proceeding with \( s = q^{m-1} \) in the \( m \)th iteration will yield

\[
(6.9) \quad K_m \left( \frac{1}{C} \right)^{S_m} \left( \int_{A_\varepsilon} v^m d\nu \right)^{1/m} \leq \int_{A_\varepsilon} v d\nu.
\]

with the constants \( K_m \) and \( S_m \) given by

\[
K_m = \prod_{j=0}^{m-1} \left( \frac{1}{q^j} \right)^{1/q^j}, \quad S_m = \sum_{j=0}^{m-1} 1/q^j.
\]

As \( m \to \infty \) the constants \( S_m \to \frac{q}{q-1} \) and \( K_m \to K, \, 0 < K < \infty \).

Since \( K_1 > K_2 > \ldots > K \) we have, for all \( m \), from 6.9

\[
\left( \int_{A_\varepsilon} v^m d\nu \right)^{1/m} \leq C^{S_m} \frac{1}{K} \int_{A_\varepsilon} v d\nu
\]

\[
\leq \frac{C^{\frac{q}{q-1}}}{K} \int_{A_\varepsilon} v d\nu.
\]

This then implies (with \( C \) replacing \( \frac{C^{\frac{q}{q-1}}}{K} \))

\[
(6.10) \quad \sup_{A_\varepsilon} v \leq C \int_{A_\varepsilon} v d\nu.
\]
On $B_{R/2}$ we have that $\zeta = (1 - \frac{\epsilon}{u})^+$. Thus when $u > 2\epsilon$, we have $\zeta \geq \frac{1}{2}$. Set $k = t + 2$, and 6.10 implies

$$\sup_{B_{R/2}} u \leq 2^k \sup_{A_\epsilon} u + 2\epsilon$$

$$\leq C \left( R^{-n} \int_{A_\epsilon} u \, dx + R^{-q(n-1)} \int_{A_\epsilon} u \, d\mu \right) + 2\epsilon$$

and the result follows, noting that $\int_{A_\epsilon} u \, dx \leq \int_{B_{R}} u^+ \, dx$. \hfill \Box

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