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FACTORIZATION METHOD FOR A BIMEROMORPHIC MORPHISM

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The main goal of this paper is to develop a constructive factorization method for a bimeromorphic morphism between smooth germs of complex analytic spaces of dimension 3 provided is composition of blowing- ups with smooth, closed and irreductible centers. The factorization is determined by the irreductible components of the morphism's exceptional divisor. This result may be used for instance to prove the invariance of the characteristic pairs of quasi-ordinary singularities.

1. Notation and terminology. In the following Z will be a smooth complex analytic space of dimension 3, $\pi = \pi_1 \circ \ldots \circ \pi_n : Z(n) \to Z$ will be a chain of blowing-ups, $E(i,i)$ the exceptional divisor of each π_i; $E(i,j)$ the strict transform of $E(i,i)$ by $\pi_{i+1} \circ \ldots \circ \pi_j$, and $E_i = E(i,n)$.

Let S be a surface in Z, let us call S_i to the strict transform of S_{i-1} by π_i and $m(S,i)$ to the multiplicity of S_{i-1} along the center of π_i. If D is a divisor of Z, then D^t will be the pull back of D by π.

We will call a local blowing-up of a analytic germ $(Z, \mathbf{p})$ to the morphism

$$(\pi) : (Z', \pi^{-1}(\mathbf{p})) \to (Z, \mathbf{p})$$

induced over the germ of Z' along the closed analytic subspace $\pi^{-1}(\mathbf{p})$ by a composition $\pi : Z' \to Z$ of blowing-ups.

A good local blowing-up will be a local blowing-up $(\pi) = (\pi_1 \circ \ldots \circ \pi_n)$ such that for all i the center of π_i is an irreducible smooth and closed subspace with normal crossings with $E(j,i)$, $\forall j < i$.

Two blowing-up's chains $\pi = \pi_1 \circ \ldots \circ \pi_n : Z' \to Z$, $\delta = \delta_1 \circ \ldots \circ \delta_n : T \to Z$ are called equivalent if does exist a permutation σ, and $\beta : Z' \to T$ in such a way that $\pi = \beta \circ \delta$, and $\beta(E_i) = T_{\sigma(i)}$, being $E_1 + \ldots + E_n$, $T_1 + \ldots + T_n$ the exceptional divisors of π and δ. Two local blowing-up (π), (δ) of the germ $(Z, \mathbf{p})$ are equivalent if they have equivalent representatives π, δ.

2. Quadratic transformations.

LEMMA 2.1. *Let* $\pi_1 \circ \ldots \circ \pi_n : Z' \to Z$ *be a chain of blowing-up with irreductible centers. If for each* $j \neq i$ *is* $\pi(E_j) \neq \pi(E_i)$ *and if* $\pi^{-1}(\pi(E_n))$ *is a divisor of* Z'*, then* π *is equivalent to another chain whose first transformation is the blowing-up with center* $\pi(E_n)$.

Proof. Follows easily from the commutativity of the factors as a consequence of the hypothesis. ☐

Let $D = \sum_{i \in I} m_i E_i$ be a divisor of Z ($m_i \neq 0$ and $E_i \neq E_j$ if $i \neq j$), D will be called A-divisor of Z if no non zero divisor $\sum_{i \in I} n_i E_i$ is principal. For instance the exceptional divisor of a chain of blowing-up is an A-divisor.

PROPOSITION 2.2. *Let* S *be a surface of* Z*,* Y *a closed subspace of* Z*,* $D_Z S$ *the associated divisor to* S*, and* m *the multiplicity of* S *along* Y*. If* D_Z *is linearly equivalent to an* A*-divisor* D*,* $\pi : Z(1) \to Z$ *is the blowing-up of* Z *with center* Y*, and* E *is the exceptional divisor of* π*, then* $D^t - mE$ *is an* A*-divisor and is linearly equivalent to* $D_{Z(1)} S_1$.

Proof. If D is an A-divisor then D^t is an A-divisor, because the property of being principal for a divisor is stable by blowing-up's pull back, then being $D - D_Z S$ principal because D and $D_Z S$ are equivalent, $(D - D_Z S)^t$ is principal, and:

$$(D - D_Z S)^t = D^t - (D_Z S)^t = (D^t - mE) - D_{Z(1)} S_1$$

then $D_{Z(1)} S_1$ and $D^t - mE$ are equivalent. ☐

We will call tree of the chain of blowing-up $\pi = \pi_1 \circ \ldots \circ \pi_n$ to the set $\{E_i : 1 \leq i \leq n\}$ together with the arrow (E_i, E_j) for all (i, j) such that the center of π_j is contained in $E(i, j-1)$. We will call ordered tree of π to the tree of π jointly with the weight i in each E_i.

THEOREM 2.3. *Let* $\pi = \pi_1 \circ \ldots \circ \pi_n : Z' \to Z$ *be a chain of blowing-ups,* S *a surface of* Z*, and* D *an* A*-divisor. If* D *and* $D_Z S$ *are equivalent then the* A*-divisor:*

$$D^t - \sum_{1 \leq i \leq n} a_i E_i - \left[m(S, n) + \sum_{1 \leq i \leq n} a_i m(E(i, n-1), n) \right] E_n$$

is equivalent to $D_{Z(n)}S_n$.

Proof. We are going to prove the theorem by induction over n.

If $n = 1$ the theorem is Proposition 2.2.

Let $n > 1$ and let us suppose that the asertion is true until $n-1$, therefore there is an A-divisor $(D_{n-1})^t - \sum_{1\leq i\leq n-1} a_iE(i,n-1)$, equivalent to $D_{Z(n-1)}S_{n-1}$, then by Prop. 2.2 the divisor $[(D_{n-1})^t - \sum_{1\leq i\leq n-1} a_iE(i,n-1)]^t - mE$, and $D_{Z(n)}S_n$ are equivalent. We conclude the proof because

$$[E(i,n-1)]^t = E_i + m[E(i,n-1),n]E_n.$$

□

With the above notation we will say that

$$\sum_{1\leq i\leq n} a_iE_i = D_{Z(n)}\pi^{-1}(S) - D_{Z(n)}S_n$$

is the divisor associated to S by π.

COROLLARY 2.4. *Let* $\pi : Z' \to Z$ *a chain of* n *quadratic transformations whose centers are contained in the sucesive strict transform of a surface* S. *Let* $\sum_{1\leq i\leq n} a_iE_i$, *the A- divisor associated to* S *by* π, *with the* π_i*'s ordered in such a way that* $a_1 \leq \ldots \leq a_n$, *then there is a chain of quadratic transformation* $\delta_1 \circ \ldots \circ \delta_n$ *equivalent to* π *such that* for each $E_i = E(i,n)$, $E(i,i)$ *is the exeptional divisor of* δ_i.

Proof. Being all the transformations quadratic, for each pair of components of π there are two possibilities:

(a) The center of the first one is contained in the center of the second.

(b) We are in the situation of Lemma 2.1.

In case (a) the inequality follows, and in case (b) by Lemma 2.1 the factor's order don't make any difference. □

3. Combinatorial blowing-up. A chain of blowing-up $\pi = \pi_1 \circ \ldots \circ \pi_n : Z(n) \to Z$ is said to be a combinatorial blowing-up if there are smooth and irreductible surfaces $\{X, Y, T\}$ of Z, with normal crossings, in such a way that the center of every component π_{i+1} of π is the intersection of some surfaces of the family

$\{X_i, Y_i, T_i, E(i,j) : 1 \leq j \leq i\}$. In the case under consideration we will say that $\{X, Y, T\}$ are the surfaces associated to π.

A combinatorial blowing-up π will be called special if for every i, $1 \leq i \leq n$, the center of π_{i+1} is contained in $E(i,i)$.

THEOREM 3.1. *Given a special combinatorial blowing-up π there is an algorithm to decide the tree of π.*

Proof. Let $\{X, Y, T\}$ surfaces associated to π, and $\sum_{i\geq 1} a_{i,1}E_i$, $\sum_{i\geq 1} b_{i,1}E_i$, $\sum_{i\geq 1} c_{i,1}E_i$, respectively the associated divisors to X, Y, T by π.

In view of Theorem 2.3 E_i is the exceptional divisor of π_1 if and only if for all j is :

$$a_{i,1} \leq a_{j,1},\ b_{i,1} \leq b_{j,1},\ c_{i,1} \leq c_{j,1}.$$

Let E_1 the exceptional divisor of π_1, then there are two posibilities:

(a) If $a_{1,1} + b_{1,1} + c_{1,1} = 3$ then π_1 is a quadratic transformation.

(b) If $a_{1,1} + b_{1,1} + c_{1,1} = 2$ then π_1 is a monoidal transformation.

Moreover if $a_{1,1} = 1$ (resp. $b_{1,1}$, $c_{1,1} = 1$) the center of π_1 is contained in X (resp. in Y, T).

There is at least one of these surfaces which contained the center of π_1 and the center of π_2 is not contained in the your strict transform (this one is X if $a_{1,1} = 1$ and $b_{1,1} = 1$ (resp. $c_{1,1} = 1$) then for all i is $a_{i,1} \leq b_{i,1}$ (resp. $a_{i,1} \leq c_{i,1}$)) in this situation the associated divisors by $\pi_2 \circ \ldots \circ \pi_n$ to $E(1,1)$, Y_1, T_1 respectively are:

$$\sum_{i\geq 2} a_{i,2}E_i = \sum_{i\geq 2} a_{i,1}E_i$$

$$\sum_{i\geq 2} b_{i,2}E_i = \sum_{i\geq 2} b_{i,1}E_i - b_{1,1}\sum_{i\geq 2} a_{i,1}E_i$$

$$\sum_{i\geq 2} c_{i,2}E_i = \sum_{i\geq 2} c_{i,1}E_i - c_{1,1}\sum_{i\geq 2} a_{i,1}E_i.$$

By repeating this process with these divisors instead of the trivial ones we get the desired construction. □

Let us write $a(i,j) = 1$ when (E_i, E_j) is an arrow of the ordered tree of a chain of blowing-ups π, and $a(i,j) = 0$ if that does not happen.

By Theorem 2.3 the associated divisor to S by π is :

$$D_Z\pi^{-1}(S) - D_Z S_n = \sum_{1\leq i\leq n} c_i E_i$$

with $c_i = m(S,i) + \sum_{j<i} c_j a(j,i)$.

Then we have $m(S,1) =$ multiplicity of $\pi^{-1}(S)$ on E_1, and inductively we may calculate $m(S,i)$.

Let J be a coherent ideal of Z, and $\pi_1 \circ \ldots \circ \pi_n$ be a chain of blowing-up. We define by induction $\pi^{*0}(J) = J$, and for $i > 0$

$$\pi^{*i}(J) = (\pi_i^*(\pi^{*(i-1)}(J)) : (I(Z_i, E(i,i)))^{\delta(i)})$$

where $\delta(i)$ is the multiplicity of $\pi^{*(i-1)}(J)$ along the center of π_i, and $I(Z_i, (E(i,i))$ the ideal sheaf of $E(i,i)$ in Z_i.

Lemma 3.2. *Let $\pi_1 \circ \ldots \circ \pi_n : Z(n) \to Z$ be a good local blowing-up. Provided we know*

(i) *The ordered tree of $\pi_1 \circ \ldots \circ \pi_r$ $(r \leq n)$*

(ii) *$\forall i \leq r$ the ideal J_i such that $\pi^{*(i-1)}(J_i)$ define the center of π_i.*

Then we may explicitly calculate:

(a) *$\forall i \leq r$ the divisor D_i associated to $E(i,i)$ by $\pi_{i+1} \circ \ldots \circ \pi_n$*

(b) *$\forall j \leq i \leq r$ the divisor $D_{j,i}$ associated to $E(j,i)$ by $\pi_{i+1} \circ \ldots \circ \pi_n$*

(c) *For all surface S of Z, $\forall i \leq r$ the divisor $D_i S$ associated to S_i by $\pi_{i+1} \circ \ldots \circ \pi_n$.*

Proof. We assume that $I(a)$ is the ideal sheaf of $E(a,a)$ in $Z(a)$ and $b(i,j) =$ multiplicity of $\pi^{*(j-1)}(J_i)$ along the center of π_j

(a) We may calculate inductively the divisors D_i as follows:

D_1 is the divisor determined by $\pi^*(J_1)$ and the divisor determined by $\pi^*(J_i)$ is $[\sum_{j<i} b(i,j) D_j] + D_i$ because $\forall s,\ 0 \leq s \leq i$

$$(\pi_{s+1} \circ \ldots \circ \pi_n)^*(\pi^{*s}(J_i)) = (\pi_{s+2} \circ \ldots \circ \pi_n)^*(I(s+1)^{b(i,s+1)} \pi^{*s+1}(J_i))$$

then:

$$\begin{aligned}\pi^*(J_i) =& [(\pi_2 \circ \ldots \circ \pi_n)^*(I(1)^{b(i,1)})] \bullet [(\pi_3 \circ \ldots \circ \pi_n)^*(I(2)^{b(i,2)})] \\ & \bullet \ldots \bullet [(\pi_i \circ \ldots \circ \pi_n)^*(I(i-1)^{b(i,i-1)})] \bullet [(\pi_i \circ \ldots \circ \pi_n)^* \\ & (\pi^{*i}(J_i))].\end{aligned}$$

(b) Let us prove these points by induction:

$$\begin{aligned}
D_j^j &= (\pi_{j+1} \circ \ldots \circ \pi_n)^{-1}(E(j,j)) = D_j \\
D_j^{i-1} &= (\pi_i \circ \ldots \circ \pi_n)^{-1}(E(j,i-1)) \\
&= (\pi_{i+1} \circ \ldots \circ \pi_n)^{-1}\pi_i^{-1}(E(j,i)) \\
&= (\pi_{i+1} \circ \ldots \circ \pi_n)^{-1}(E(j,i+1)) + a(j,i)E(i,i) \\
&= D_j^i + a(j,i)D_i.
\end{aligned}$$

(c) It is enough to show inductively that

$$D_iS = D_{Z(n)}\pi^{-1}(S) - \sum_{j \leq i} m(S,j)Dj$$

these formula is true because:

$$\begin{aligned}
&D_0S = \pi^{-1}(S) \\
&D_{i-1}S = (\pi_i \circ \ldots \circ \pi_n)^{-1}(S_{i-1}) = (\pi_{i+1} \circ \ldots \circ \pi_n)^{-1}(\pi_i^{-1}(S_{i-1})) = \\
&\qquad = (\pi_{i+1} \circ \ldots \circ \pi_n)^{-1}(m(S,i)E(i,i) + S_i) = m(S,i)D_i + D_iS
\end{aligned}$$

and the formula follows. ☐

PROPOSITION 3.3. *Let $\pi = \pi_1 \circ \ldots \circ \pi_n : Z(n) \to Z(0) = Z$ be a local blowing-up, if we know the ordered tree of $\pi_1 \circ \ldots \circ \pi_d$, then we have $\forall i \leq d$ an ideal sheaf J_i such that $\pi^{*_{i-1}}(J_i)$ is the ideal sheaf of the center of π_i.*

Proof. By induction over i.

(a) For $i = 1$, J_1 is the ideal of $\pi(E_1)$ in Z.

(b) Let $i > 1$ and let us suppose that the assertion is true for $r < i$.

For all $r < i$ there are a finite set of generators of J_r, $G(r)$, such that $m(S,r) = 1$, $m(S,t) = 0 \; \forall S \in G(r)$, $\forall t > r$.

If we construct a finite set $H(r)$ of surfaces of Z such that:

(i) $\forall t$, $r \leq t \leq i$, $\forall S \in H(r)$, $m(S,t)$ depends only on t.

(ii) The ideal sheaf of $Z(n)$ generated by the local equations of

$$\{D_{Z(n)}S_n + D_iS : S \in H(r)\}$$

is equal to the one generated by the local equations of

$$\{D_{Z(n)}T_n + D_iT : m(T,i) = 1\}.$$

Then one has that:

$$\text{center of } \pi_i = \bigcap_{S \in H(r)} S_{i-1}.$$

To get $H(r)$, $\forall r$, we begin with a finite set $H(i)$ of surfaces of Z which verifies (ii), then inductively $H(r)$ is obtained from $H(r+1)$ as follows: if

$$\delta(r) = \max\{m(S,r) : S \in H(r+1)\}$$

then

$$H(r) = \{S \cdot T^{(\delta(r)-m(S,r))} : S \in H(r+1),\ T \in G(r)\}.$$

With this construction J_i is the ideal generated by $H(0)$. ☐

REMARK 3.4. The assertion of Proposition 2.3 is also true for $\{\pi_i : 1 \leq i \leq d\} \bigcup \{\pi_c\}$ when we know:

(a) The ordered tree of $\pi_1 \circ \ldots \circ \pi_d$

(b) $\forall i,\ i \leq d$ the arrows (E_i, E_c)

With the additional condition $\forall j,\ d < j < c,\ (E_j, E_c)$ is not an arrow. The proof is the same as Proposition 2.3.

THEOREM 3.5. *Given a combinatorial local blowing-up π, we may determine the ordered tree of a local blowing-up π' equivalent to π.*

Proof. Let X, Y, T surfaces associated to π, and let $D_X = \sum_{i\geq 1} a_{i,1}E^i$, (resp. $D_Y = \sum_{i\geq 1} b_{i,1}E_i$, $D_T = \sum_{i\geq 1} c_{i,1}E_i$) be the associated divisor to the surface X (resp. Y, T). Then we may choose the first component E_i by releasing an index i in such a way that

(a) $\max\{a_{i,1},\ b_{i,1},\ c_{i,1}\} = 1$.

(b) $a_{i,1} + b_{i,1} + c_{i,1} = 3$, or $a_{i,1} + b_{i,1} + c_{i,1} = 2$ and $\pi^{-1}(\pi(E_i))$ is a divisor.

In the first case π_1 is a quadratic transformation, in the second one is monoidal.

Let us call $A(1)$ the tree $\{E_1\}$ and let us suppose now that we know the tree $A(r)$ of $\pi_1 \circ \ldots \circ \pi_r$.

By Lemma 3.2 we know the divisors:

$$\begin{aligned} D_rX &= \sum_{i>r} a_{i,r+1}E_i \\ D_rY &= \sum_{i>r} b_{i,r+1}E_i \\ D_rT &= \sum_{i>r} c_{i,r+1}E_i \\ D_j^r &= \sum_{i>r} s(j,r,i)E_i. \end{aligned}$$

Then E_d is the $(r+1)$-th component if d satisfies:

$$\begin{aligned} &\text{(a) } \max\{a_{i,r+1},\ b_{i,r+1},\ c_{i,r+1},\ s(j,r,d) : j \leq r\} = 1 \\ &\text{(b) } a_{i,r+1} + b_{i,r+1} + c_{i,r+1} + \sum_{j\leq r} s(j,r,d) = 3 \end{aligned}$$

(π_d is a quadratic transformation); or

$$a_{i,r+1} + b_{i,r+1} + c_{i,r+1} + \sum_{j\leq r} s(j,r,d) = 2$$

(π_d is a monoidal transformation), and $\pi^*(J_d)$ is inversible, where J_d is the ideal which corresponds by Remark 3.4 to the ordered tree $\{E_i : 1 \leq i \leq r\} \bigcup \{E_d\}$ with the arrow (E_i, E_d) if $s(i,r,d) = 1$. □

4. The general situation.

THEOREM 4.1. *Given a good local blowing-up π we may determine the ordered tree of a local blowing-up π' equivalent to π; and the class (monoidal or quadratic) of each component.*

Proof. Let $\sum_{i\in I(1)} E_i$, be the exceptional divisor of π, and if $\pi_1' \circ \ldots \circ \pi_r'$ is determined, let $D_rS = \sum_{i\in I(r)} a^r(S,i)E_i$ the associated divisor to S_r by π (3.2.c).

Let us call $P(1)$ the quotient set $I(1)/\equiv$ by the relation:

$i \equiv j$ if for every surface S of Z is $a^0(S,i) \neq 0 \Leftrightarrow a^0(S,j) \neq 0$

this is equivalent to $i \equiv j$ if $\pi(E_i) = \pi(E_j)$.

If $j' \in P(1)$ let $A(1,j') = \{i \in I(1) : i \in j'\}$.

Let $j' \in P(1)$ and $t \in A(1,j')$, by Th. 2.3 E_t will be the first element of $\{E_i : i \in A(1,j')\}$ if $\forall i \in A(1,j')$ is $a^0(S,E_t) < a^0(S,E_i)$, for every surface S of Z such that $Y_j \subset S$.

Let, $\forall b' \in P(1)$, $E_{1(b')}$ be the first element of $\{E_i : i \in A(1,b')\}$, then if j' verifies that :

(a) $\pi^{-1}(E_{1(j')})$ is a divisor.

(b) For all k', such that $\pi^{-1}(\pi(E_{1(k')}))$ is a divisor, is $\pi(E_{1(k')}) \not\subset \pi(E_{1(j')})$.

Then $E_{1(j')}$ will be the divisor corresponding to the first factor of π' and we will change this j' by $1'$ and call $E_1 = E_{1(1')}$.

Let us assume that we know the ordered tree of $\pi'_1 \circ \ldots \circ \pi'_v$ and the sets $A(v,j')$, $j' \in P(v)$. Let $Q(v+1)$ the quotient set $(A(v,v') - \{v(v)'\})/\sim$ by the relation:

$i \sim j$ if for every surface S of Z is $a^v(S,i) \neq 0 \Leftrightarrow a^v(S,j) \neq 0$. Then $P(v+1) = (P(v) - \{v'\}) \bigcup Q(v+1)$.

Let $j' \in P(v+1)$, $A(v+1,j') = \{i \in I(v+1) : i \in j'\}$. By Theorem 2.3 $E_{(v+1)(j')}$ is the first element of $\{E_i : i \in A(v+1,j')\}$ if $\forall i \in A(v+1,j')$, and for every surface S of Z is $a^v(S,(v+1)(j')) \leq a^v(S,i)$.

We extend the ordered tree of $\pi'_1 \circ \ldots \circ \pi'_v$, $\{E_i : 1 \leq i \leq v\}$ to $\{E_{(v+1)(j')} : j' \in P(v+1)\} \bigcup \{E_i : 1 \leq i \leq v\}$ by adding the arrows $(E_i, E_{(v+1)(j')})$ if the multiplicity of D_i^v at $E_{(v+1)(j')}$ is 1.

By Remark 3.4 we assign to each $E_{(v+1)(j')}$ the ideal $J_{(v+1)(j')}$. There is at least one $E_{(v+1)(j')}$ such that

(a) $\pi^*(J_{(v+1)(j')})$ is an invertible ideal of Z'.

(b) $J_{(v+1)(k')} \not\subset J_{(v+1)(j')}$, $\forall k' \neq j'$.

Then we may take as π'_{v+1} the blowing-up corresponding to this $E_{(v+1)(j')}$.

The codimension of center of each π'_i is calculated inductively by the following lemma (see [**6**]). ☐

LEMMA 4.2. [**6**]. *Let* $\{x_1, x_2, x_3\}$ *be a local coordinate system on* $(Z, \mathbf{p})$ *centered at* $\mathbf{p}$. *Let* $\pi = \pi_1 \circ \ldots \circ \pi_n : Z_n \to Z$ *be a local blowing-up, and* Y_i *the center of* π_i $(i = 1, \ldots, n)$. *If* $D = \sum_{i<r} \beta_i E_{i(r-1)}$, *is the divisor of* Z_{r-1} *determined by* $(\pi_1 \circ \ldots \circ \pi_{r-1})^*(dx_1 \wedge dx_2 \wedge dx_3)$ *and* d *is the codimension of* Y_r, *then the divisor of* $Z(r)$ *determined by* $(\pi_1 \circ \ldots \circ \pi_r)^*(dx_1 \wedge dx_2 \wedge dx_3)$ *is:*

$$\sum_{i<r}\beta_i E_i(r) + \left[d-1+\sum_{i<r}\beta_i \cdot \mathrm{mult}_D\, E_{i(r-1)}\right] E_{r(r)}.$$

5. Quasi-ordinary singularities. Let S be a surface of Z, and $\mathbf{p}$ an quasi-ordinary singularity of S. We call canonical blowing-up of $(S,\mathbf{p})$ to a local blowing-up $(\pi) = (\pi_1 \circ \ldots \circ \pi_n) : (Z_n, \pi_{-1}(\mathbf{p})) \to (Z,\mathbf{p})$ such that $\forall i$, $1 \leq i \leq n$, π_i is permissible, i.e. if S_{i-1} have permissible curves then π_i is a monoidal transformation with permissible center for S_{i-1}.

We call canonical resolution of $(S,\mathbf{p})$ to a canonical local blowing-up such that:

(a) $(S_n, \pi^{-1}(\mathbf{p}))$ is regular.

(b) $(S_n, \pi^{-1}(\mathbf{p}))$ have normal crossing with the components of exceptional divisor of (π).

The set of normalized characteristic pairs of $(S,\mathbf{p})$ determine the canonical resolutions of $(S,\mathbf{p})$ (see [**3**]). Moreover if (π) is a canonical resolutions of $(S,\mathbf{p})$ we may know the ordered tree of every canonical resolutions equivalent to (π), and the class of their factors. With this information we determine the first pair of $(S_i,\mathbf{p}_i)$, and the set of normalized characteristic pair of $(S,\mathbf{p})$ because:

If $(\pi \circ \pi')$ is a canonical local blowing-up of the quasi-ordinary singularity $(T,\mathbf{p})$, then:

(1) We know the formulaes which establish relation between the set of normalized characteristic pairs of strict transform of $(T,\mathbf{q})$, (see [**3**]).

(2) The transform of the first characteristic pair is eqivalent to $(1/m, 0)$, $(m \in \mathbb{N})$, if and only if π' is monoidal and the multiplicity of T along the center of π is multiple of the multiplicity of strict transform of T along the center of π'.

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