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CHULL PARK AND DAVID LEE SKOUG

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In this paper we establish various results involving conditional Wiener integrals, $E(F|X)$, for very general conditioning functions X . Most related results in the literature, including the case when the conditioning function X is vector-valued, then follow as corollaries of this more general theory. A simple formula is given for converting these generalized conditional Wiener integrals into ordinary Wiener integrals and then this formula is used to evaluate $E(F|X)$ for various classes of functionals F . Finally these results are used to obtain a generalized conditional form of the Cameron-Martin translation theorem.

1. Introduction. Let $(C[0, T], \mathcal{F}^*, m_w)$ denote Wiener space, where $C[0, T]$ is the space of all continuous functions x on $[0, T]$ vanishing at the origin. Let $F(x)$ be a Wiener integrable function on $C[0, T]$ (i.e., $E[|F(x)|] < \infty$) and let $X(x)$ be a Wiener measurable function on $C[0, T]$. In [13], Yeh introduced the concept of conditional Wiener integrals. He defined the conditional Wiener integral of F given X as a function on the value space of X and derived a Fourier transform inversion formula for computing conditional Wiener integrals. Using this formula for the case $X(x) = x(T)$, Yeh [13, 14] obtained some very useful results including a Kac-Feynman integral equation and a conditional Cameron-Martin translation theorem.

In [4], for certain functions F , Chang and Chang, using Yeh's inversion formula, evaluated the conditional Wiener integral of F given $X(x) = (x(t_1), \dots, x(t_n))$ where $0 < t_1 < t_2 < \dots < t_n = T$. In [8], the current authors obtained a very simple formula for the conditional Wiener integral of F given $X(x) = (x(t_1), \dots, x(t_n))$. In particular we expressed the conditional Wiener integral directly in terms of an ordinary (i.e., nonconditional) Wiener integral. Using this formula it was relatively simple to generalize the Kac-Feynman formula and to obtain a conditional Cameron-Martin translation theorem involving vector-valued conditioning functions.

In this paper we consider much more general conditioning functions. In particular they need not depend upon the values of x at only finitely many points in $(0, T]$. A major thrust of this paper is to develop a useful formula to convert these generalized conditional Wiener integrals into ordinary (i.e., nonconditional) Wiener integrals and then to obtain the corresponding Cameron-Martin translation theorem for these generalized conditional Wiener integrals. We also use this simple formula to compute the generalized conditional Wiener integral for various functions $F(x)$ on $C[0, T]$. Most of the results in [4, 8, 13, and 14] then follow as special cases of the results obtained in this paper.

2. Preliminaries and definitions. Let \mathcal{H} be an infinite dimensional subspace of $L_2[0, T]$ with a complete orthonormal basis $\{\alpha_j\}$. Then the corresponding stochastic integrals

$$(2.1) \quad \gamma_j(x) = \int_0^T \alpha_j(t) dx(t), \quad j = 1, 2, \dots$$

form a set of independent standard Gaussian variables on $C[0, T]$ with

$$(2.2) \quad E[x(t)\gamma_j(x)] = \int_0^t \alpha_j(s) ds \equiv \beta_j(t).$$

For each $n \in \mathbb{N}$ let \mathcal{H}_n be the subspace of \mathcal{H} spanned by $\{\alpha_1, \dots, \alpha_n\}$, and let $X_n : C[0, T] \rightarrow \mathbb{R}^n$ and $X_\infty : C[0, T] \rightarrow \mathbb{R}^\infty$ be defined by

$$(2.3) \quad X_n(x) = (\gamma_1(x), \dots, \gamma_n(x)), \quad X_\infty(x) = (\gamma_1(x), \gamma_2(x), \dots).$$

If \mathcal{B}^n denotes the σ -algebra of Borel sets in \mathbb{R}^n , then a set of the type

$$I = \{x \in C[0, T] : X_n(x) \in B\} \equiv X_n^{-1}(B), \quad B \in \mathcal{B}^n$$

is called a quasi-Wiener interval (or a Borel cylinder). It is well known that

$$(2.4) \quad m_w(I) = \int_B K_n(\vec{\xi}) d\vec{\xi},$$

where

$$(2.5) \quad K_n(\vec{\xi}) = (2\pi)^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \xi_j^2 \right\}.$$

Let \mathcal{F}_n be the σ -algebra formed by the sets $\{X_n^{-1}(B) : B \in \mathcal{B}^n\}$, and let \mathcal{F} be the σ -algebra generated by $\cup_{n=1}^\infty \mathcal{F}_n$. Then, by the definition of conditional expectations (see Doob [5], Tucker [10] and Yeh [12]) for each $F \in L_1(C[0, T], m_w)$,

$$\begin{aligned}
 (2.6) \quad \mu(B) &\equiv \int_{X_n^{-1}(B)} F(x)m_w(dx) = \int_{X_n^{-1}(B)} E(F|\mathcal{F}_n)m_w(dx) \\
 &= \int_B E(F(x)|X_n(x) = \vec{\xi})P_{X_n}(d\vec{\xi}) \\
 &= \int_B E(F(x)|\gamma_j(x) = \xi_j, j = 1, \dots, n)P_{X_n}(d\vec{\xi}), B \in \mathcal{B}^n,
 \end{aligned}$$

where $P_{X_n}(B) = m_w(X_n^{-1}(B))$, and $E(F(x)|X_n(x) = \vec{\xi})$ is a Lebesgue measurable function for $\vec{\xi}$ which is unique up to null sets in \mathbb{R}^n .

Since $\{\mathcal{F}_n\}$ is an increasing sequence of σ -algebras of Weiner measurable sets, for $F \in L_1(C[0, T], m_w)$, $\{E(F|\mathcal{F}_n)\}$ is a martingale sequence. Thus, $E|E(F|\mathcal{F}_n)| \leq E|F|$ for every n , and so by the martingale convergence theorem, $\lim E(F|\mathcal{F}_n) = E(F|\mathcal{F})$ almost surely and for each $A \in \cup_{n=1}^\infty \mathcal{F}_n$,

$$(2.7) \quad \int_A E(F(x)|\mathcal{F})m_w(dx) = \lim \int_A E(F(x)|\mathcal{F}_n)m_w(dx).$$

From this and (2.6), it follows that for every $B \in \cup_{n=1}^\infty \mathcal{B}^n$,

$$\begin{aligned}
 (2.8) \quad &\int_B E(F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots)P_{X_\infty}(d\vec{\xi}) \\
 &= \lim \int_B E(F(x)|\gamma_j(x) = \xi_j, j = 1, \dots, n)P_{X_n}(d\vec{\xi}),
 \end{aligned}$$

where

$$\begin{aligned}
 (2.9) \quad P_{X_n}(d\vec{\xi}) &= \prod_{j=1}^n \left\{ (2\pi)^{-\frac{1}{2}} \exp(-\xi_j^2/2) d\xi_j \right\}, \\
 P_{X_\infty}(d\vec{\xi}) &= \prod_{j=1}^\infty \left\{ (2\pi)^{-\frac{1}{2}} \exp(-\xi_j^2/2) d\xi_j \right\}.
 \end{aligned}$$

In (2.8) we used the convention that if $B \in \mathcal{B}^n$, then $B \in \mathcal{B}^{n+k}$ by identifying B and $B \times \mathbb{R}^k$ in \mathcal{B}^{n+k} for $k = 1, 2, \dots$. Thus if

$B \in \cup_{n=1}^{\infty} \mathcal{B}^n$, then there exists $N \in \mathbb{N}$ such that $B \in \mathcal{B}^n$ for all $n \geq N$, and hence by the martingale property

$$(2.10) \quad \int_B E(F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots) P_{X_{\infty}}(d\vec{\xi}) \\ = \int_B E(F(x)|\gamma_j(x) = \xi_j, j = 1, \dots, n) P_{X_n}(d\vec{\xi}), \text{ for all } n \geq N,$$

from which (2.8) follows.

In the next section we develop quite simple formulas for converting the generalized conditional Wiener integrals of the types $E(F(x)|X_n(x) = \vec{\xi}) = E(F(x)|\gamma_j(x) = \xi_j, j = 1, \dots, n)$ and $E(F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots)$ into ordinary Wiener integrals which can often be computed explicitly. It then turns out that all the conditional Wiener integrals that occur in [4, 8, 13, and 14] are special cases of conditional expectations given in this paper.

3. Useful formulas for conditional Wiener integrals. Let \mathcal{H} , $\{\alpha_j\}$, \mathcal{H}_n and $\{\gamma_j(x)\}$ be as in Section 2. Define projection maps \mathcal{P} and \mathcal{P}_n from $L_2[0, T]$ into \mathcal{H} and \mathcal{H}_n , respectively, by

$$(3.1) \quad \mathcal{P}h(t) = \sum_{j=1}^{\infty} (h, \alpha_j) \alpha_j(t), \\ \mathcal{P}_n h(t) = \sum_{j=1}^n (h, \alpha_j) \alpha_j(t).$$

For $x \in C[0, T]$ and $\vec{\xi} = (\xi_1, \xi_2, \dots)$, let

$$(3.2) \quad x_n(t) = \int_0^T \mathcal{P}_n I_{[0,t]}(s) dx(s) = \sum_{j=1}^n \gamma_j(x) \int_0^t \alpha_j(s) ds, \\ \vec{\xi}_n(t) = \sum_{j=1}^n \xi_j(\alpha_j, I_{[0,t]}),$$

where $I_{[0,t]}$ is the indicator function of the interval $[0, t]$. Similarly, define

$$(3.3) \quad x_{\infty}(t) = \int_0^T \mathcal{P} I_{[0,t]}(s) dx(s) = \sum_{j=1}^{\infty} \gamma_j(x) \int_0^t \alpha_j(s) ds, \\ \vec{\xi}_{\infty}(t) = \sum_{j=1}^{\infty} \xi_j(\alpha_j, I_{[0,t]}).$$

We note here that since $\{\gamma_j(x)\}$ is a sequence of i.i.d. standard Gaussian random variables, the series $x_\infty(t)$ converges m_w -a.e. x (see Shepp [9, p.324]). Since $\vec{\xi}_\infty(t)$ is the evaluation of the random variable $x_\infty(t)$ for $\gamma_j(x) = \xi_j, j = 1, 2, \dots, \vec{\xi}_\infty(t)$ converges P_{x_∞} - a.e. $\vec{\xi}$.

Our first theorem plays a key role throughout this paper.

THEOREM 1. *If $\{x(t), 0 \leq t \leq T\}$ is the standard Wiener process, then the processes $\{x(t) - x_\infty(t), 0 \leq t \leq T\}$ and $\gamma_j(x)$ are (stochastically) independent for $j = 1, 2, \dots$. Also, $\{x(t) - x_n(t), 0 \leq t \leq T\}$ and $\gamma_j(x)$ are independent for $j = 1, \dots, n$.*

Proof. For each j , using (2.2), (3.1) and (3.2)

$$E[\gamma_j(x)\{x(t) - x_\infty(t)\}] = \int_0^t \alpha_j(s)ds - \sum_{i=1}^\infty \delta_{ij} \int_0^t \alpha_j(s)ds = 0.$$

Since both $\gamma_j(x)$ and $x(t) - x_\infty(t)$ are Gaussian and uncorrelated, it follows that they are independent. The second claim follows in similar manner. □

COROLLARY 1. *The processes $\{x(t) - x_\infty(t), 0 \leq t \leq T\}$ and $\{x_\infty(t), 0 \leq t \leq T\}$ are independent, and so are $\{x(t) - x_n(t), 0 \leq t \leq T\}$ and $\{x_n(t), 0 \leq t \leq T\}$.*

The following theorem is one of our main results.

THEOREM 2. *Let $F \in L_1(C[0, T], m_w)$. Then*

(3.4)

$$E[F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots] = E[F(x - x_\infty + \vec{\xi}_\infty)], \text{ and}$$

$$E[F(x)|\gamma_j(x) = \xi_j, j = 1, \dots, n] = E[F(x - x_n + \vec{\xi}_n)].$$

Proof. Since $x - x_\infty$ and x_∞ are independent processes, and $\gamma_j(x)$ and $x - x_\infty$ are independent by Theorem 1, we may write

$$E[F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots]$$

$$= E[F((x - x_\infty) + x_\infty)|\gamma_j(x) = \xi_j, j = 1, 2, \dots]$$

$$= E_y\{E_x[F((y - y_\infty) + x_\infty)|\gamma_j(x) = \xi_j, j = 1, 2, \dots]\},$$

where y is a standart Wiener process independent of x . Thus, we have

$$\begin{aligned} E[F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots] \\ = E_y\{F((y - y_\infty) + \vec{\xi}_\infty)\} = E[F(x - x_\infty + \vec{\xi}_\infty)], \end{aligned}$$

as $x_\infty = \vec{\xi}_\infty$ under the condition $\gamma_j = \xi_j, j = 1, 2, \dots$. The second formula of (3.4) follows by the same reasoning. \square

COROLLARY 2. *Let $F \in L_1(C[0, T], m_w)$. If $\mathcal{H} = L_2[0, T]$, then $E[F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots] = F(\vec{\xi}_\infty)$.*

Proof. This follows from (3.4) by the fact that if $\mathcal{H} = L_2[0, T]$, then $x(t) = \int_0^T I_{[0,t]}(s)ds(s) = \sum_{j=1}^\infty (\alpha_j, I_{[0,t]})\gamma_j(x) = x_\infty(t)$ for m_w - a.e. x . \square

COROLLARY 3. *Let $F \in L_1(C[0, T], m_w)$. Then for every $B \in \mathcal{B}^n$,*

$$\int_{X_n^{-1}(B)} F(x)m_w(dx) = \int_B E[F(x - x_n + \vec{\xi}_n)P_{X_n}(d\vec{\xi})].$$

The above corollary is a simple consequence of the second formula in (3.4). In addition Theorem 4 on page 114 of [2] is a special case of Corollary 3 above with $B = \mathbb{R}^n$.

REMARKS.

(i) For each partition $\tau \equiv \tau_n = \{t_1, \dots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \dots < t_n = T$, let $X_\tau : C[0, T] \rightarrow \mathbb{R}^n$ be defined by $X_\tau(x) = (x(t_1), \dots, x(t_n))$. In [8], the current authors considered vector-valued conditional Wiener integrals of the type $E(F(x)|X_\tau(x) = \vec{\xi})$ for $F \in L_1(C[0, T], m_w)$. We note that these can be rewritten in the form

$$\begin{aligned} (3.5) \quad E(F(x)|X_\tau(x) = \vec{\xi}) &= E(F(x)|x(t_j) = \xi_j, j = 1, \dots, n) \\ &= E(F(x)|x(t_j) - x(t_{j-1}) = \xi_j - \xi_{j-1}, j = 1, \dots, n) \\ &= E\left(F(x) \middle| \int_0^T \alpha_j(t)dx(t) = \frac{\xi_j - \xi_{j-1}}{\sqrt{t_j - t_{j-1}}}, j = 1, \dots, n\right) \end{aligned}$$

where $\xi_0 = t_0 = 0$ and

$$(3.6) \quad \alpha_j(t) = I_{[t_{j-1}, t_j]}(t) / \sqrt{t_j - t_{j-1}}, j = 1, \dots, n.$$

Since $\{\alpha_1(t), \dots, \alpha_n(t)\}$ is obviously an orthonormal set of functions in $L_2[0, T]$, the vector-valued conditional Wiener integral $E(F(x)|X_\tau(x) = \vec{\xi})$ is a special case of the general conditional Wiener integrals of the type $E(F(x)|X_n(x) = \vec{\xi})$ considered in this paper. Thus the conditional Wiener integrals that occur in [4], [8], [13] and [14] are all special cases of those of the type $E(F(x)|X_n(x) = \vec{\xi})$ for appropriate n and $\alpha_1, \dots, \alpha_n$.

It is also interesting to note that for each $x \in C[0, T]$ the polygonal function $[x]$ defined by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})),$$

$$t_{j-1} \leq t \leq t_j, \quad j = 1, \dots, n$$

has another representation, namely

$$[x](t) = x_n(t), \quad 0 \leq t \leq T$$

where the α_j 's are given by (3.6) and $x_n(t)$ is given by (3.2). The formula in [8], p.385, corresponding to (3.4) above is

$$E(F(x)|X_\tau(x) = \vec{\xi}) = E[F(x - [x] + [\vec{\xi}])]$$

where for $\vec{\xi} \in \mathbb{R}^n$, $[\vec{\xi}](t)$ is the polygonal function

$$[\vec{\xi}](t) = \xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \dots, n$$

$$= \vec{\xi}_n(t)$$

where the α_j 's are given by (3.6) and $\vec{\xi}_n(t)$ is given by (3.2).

(ii) Thanks to the referee's suggestions, this paper has gone through a number of improvements. The expressions given by (3.2) and (3.3) were suggested by the referee. This in turn, strengthened Theorems 1 and 2. Another suggestion made by the referee was the possibility of generalizing Theorem 2 to other Gaussian processes. This question is perhaps best handled by using the representation of a Gaussian process using Wiener processes; see [7] and example 3 below.

We close this section with some examples which illustrate that formulas (3.4) are indeed very useful and easy to apply. In particular, the third example deals with the Ornstein-Uhlenbeck process to show that our formulas can be applied to other useful Gaussian processes.

EXAMPLE 1. For $x \in C[0, T]$ let $F(x) = \int_0^T x^2(t) dt$. Then using (3.4) we obtain

$$\begin{aligned} E \left[\int_0^T x^2(t) dt \mid X_\alpha(x) = \vec{\xi} \right] \\ &= E \left[\int_0^T (x(t) - x_n(t) + \vec{\xi}_n(t))^2 dt \right] \\ &= \int_0^T E \left[(x(t) - x_n(t))^2 + (\vec{\xi}_n(t))^2 + 2\vec{\xi}_n(t)(x(t) - x_n(t)) \right] dt. \end{aligned}$$

Since $x - x_n$ and x_n are independent by Corollary 1, $E[x_n(t)(x(t) - x_n(t))] = 0$, and using (2.2) and the fact that $E[x(s)x(t)] = \min\{s, t\}$, we obtain

$$E \left[\int_0^T x^2(t) dt \mid X_n(x) = \vec{\xi} \right] = \int_0^T \left\{ t + (\vec{\xi}_n(t))^2 - \sum_{j=1}^n \beta_j^2(t) \right\} dt.$$

In particular, if $n = 1$ and $\alpha(s) = 1/\sqrt{T}$, we see that

$$\begin{aligned} E \left[\int_0^T x^2(t) dt \mid X_1(x) = \xi \right] &= E \left[\int_0^T x^2(t) dt \mid x(T) = \xi \right] \\ &= \int_0^T \left\{ t + \frac{\xi^2 t^2}{T^2} - \frac{t^2}{T} \right\} dt = \frac{T^2}{6} + \frac{\xi^2 T}{3} \end{aligned}$$

which agrees with the results in [4], [8] and [13].

EXAMPLE 2. For $x \in C[0, T]$ let $F(x) = \exp \left\{ \int_0^T x(t) dt \right\}$. Then

$$\begin{aligned} E \left[\exp \left\{ \int_0^T x(t) dt \right\} \mid X_n(x) = \vec{\xi} \right] \\ &= E \left[\exp \left\{ \int_0^T (x(t) - x_n(t) + \vec{\xi}_n(t)) dt \right\} \right] \\ &= \exp \left\{ \int_0^T \vec{\xi}_n(t) dt \right\} E \left[\exp \left\{ \int_0^T (x(t) - x_n(t)) dt \right\} \right]. \end{aligned}$$

In particular, if we choose the complete orthonormal cosine sequence $\alpha_j(t) = \sqrt{2/T} \cos[(j - 1/2)\pi t/T]$, $j = 1, 2, \dots$, on $[0, T]$, then it is well known (see Shepp [9], p.325) that the corresponding $x_n(t)$ converges to $x(t)$ uniformly in t with probability one, and for each $u \in C[0, T]$,

$$\sum_{j=1}^{\infty} \int_0^T \left\{ \int_0^t \alpha_j(s) ds \int_0^T \alpha_j(s) du(s) \right\} dt = \int_0^T u(t) dt.$$

Thus

$$\lim_{n \rightarrow \infty} E \left[\exp \left\{ \int_0^T x(t) dt \right\} \mid X_n(x) = X_n(u) \right] = \exp \left\{ \int_0^T u(t) dt \right\}$$

as expected. Since the orthonormal cosine sequence given above is complete on $[0, T]$, Corollary 2 can be applied to get

$$E \left[\exp \left\{ \int_0^T x(t) dt \right\} \mid \gamma_j(x) = \gamma_j(u), j = 1, 2, \dots \right] = \exp \left\{ \int_0^T u(t) dt \right\}$$

for a.e. $u \in C[0, T]$.

EXAMPLE 3. Consider the Ornstein-Uhlenbeck process $y(t)$ with mean zero and covariance function $R(s, t) = \sigma^2 \exp\{-\beta|t-s|\}$ where $\beta > 0$. If we take $\sigma = \beta = 1$ for convenience, then $y(t)$ can be expressed in terms of the standart Wiener process $x(t)$ (see p.414 of [7]),

$$(3.7) \quad y(t) = e^{-t} x(e^{2t}), \quad 0 \leq t \leq T.$$

Suppose $F(y)$ is an integrable function of y . Let $\tau = \{0 = t_0, t_1, \dots, t_n = T\}$ be a partition of $[0, T]$. Then, the conditional expectation

$$E[F(y) \mid y(t_j) = \xi_j, j = 0, 1, \dots, n]$$

can be expressed as a non-conditional expectation by utilizing (3.7). Since $e^t y(t) = x(e^{2t})$ and $x(\cdot)$ has independent increments, we write

$$\begin{aligned} E[F(y) \mid y(t_j) = \xi_j, j = 0, 1, \dots, n] \\ = E[F(y) \mid e^{t_j} y(t_j) - e^{t_{j-1}} y(t_{j-1}) = e^{t_j} \xi_j - e^{t_{j-1}} \xi_{j-1}, j = 0, \dots, n] \end{aligned}$$

where $y(t_{-1}) = \xi_{-1} = 0$.

Define $(y_n)(t)$ by

$$(y_n)(t) = e^{-t} \left[e^{t_{j-1}} y(t_{j-1}) + \frac{e^{2t} - e^{2t_{j-1}}}{e^{2t_j} - e^{2t_{j-1}}} (e^{t_j} y(t_j) - e^{t_{j-1}} y(t_{j-1})) \right]$$

for $t_{j-1} \leq t \leq t_j, j = 1, \dots, n$.

Similarly, define $(\vec{\xi}_n)(t)$ by

$$(\vec{\xi}_n)(t) = e^{-t} \left[e^{t_{j-1}} \xi_{j-1} + \frac{e^{2t} - e^{2t_{j-1}}}{e^{2t_j} - e^{2t_{j-1}}} (e^{t_j} \xi_j - e^{t_{j-1}} \xi_{j-1}) \right]$$

for $t_{j-1} \leq t \leq t_j, j = 1, \dots, n$.

Then, $(y_n)(t_j) = y(t_j)$ and $(\vec{\xi}_n)(t_j) = \xi_j$ at each $t_j \in \tau$. Furthermore, (y_n) and $y - (y_n)$ are independent processes as one can easily check using the covariance function of y . Thus, we conclude that

$$E[F(y)|y(t_j) = \xi_j, j = 0, 1, \dots, n] = E[F(y - (y_n) + (\vec{\xi}_n))].$$

4. Conditional expectation of functions involving stochastic integrals. Using the same notation as in section 3 above, for $h \in L_2[0, T]$ let

$$(4.1) \quad \begin{aligned} h_{(n)}(t) &= \mathcal{P}_n h(t) = \sum_{j=1}^n (h, \alpha_j) \alpha_j(t) \text{ and} \\ h_{(\infty)}(t) &= \mathcal{P}_\infty h(t) = \sum_{j=1}^\infty (h, \alpha_j) \alpha_j(t) \end{aligned}$$

Then, we have the following:

LEMMA 1. *Let $h \in L_2[0, T]$. Then*

$$(4.2) \quad \int_0^T h(t) h_{(n)}(t) dt = \int_0^T h_{(n)}^2(t) dt = \|h_{(n)}\|^2 = \sum_{j=1}^n (h, \alpha_j)^2,$$

and

$$(4.3) \quad \|h - h_{(n)}\|^2 = \|h\|^2 - \|h_{(n)}\|^2.$$

Obviously, the above formulas hold when $n = \infty$, and $\|h - h_{(\infty)}\| = 0$ if $\mathcal{H} = L_2[0, T]$.

Our next theorem gives an interesting relationship involving h , $h_{(n)}$, x and x_n that is very useful in computing conditional and ordinary expectations of functions involving the stochastic integral $\int_0^T h(t)dx_n(t)$.

THEOREM 3. *Let $h \in L_2[0, T]$. Then for each $x \in C[0, T]$*

$$(4.4) \quad \int_0^T h(t)dx_n(t) = \int_0^T h_{(n)}(t)dx(t) = \int_0^T h_{(n)}(t)dx_n(t)$$

The formula also holds for $n = \infty$ if we consider $\int_0^T h(t)dx_\infty(t) = \sum_{j=1}^\infty \gamma_j(x)(h, \alpha_j)$.

Proof. Using 3.1, 3.2, 4.1 and the fact that the α_j 's are orthonormal, it is quite easy to show that for each $x \in C[0, T]$, each of the stochastic integrals in 4.4 equals the expression

$$\sum_{j=1}^n (h, \alpha_j) \int_0^T \alpha_j(t)dx(t).$$

□

COROLLARY 4. *Let $h \in L_2[0, T]$. Then*

$$(4.5) \quad E \left[\exp \left\{ - \int_0^T h(t)dx_n(t) \right\} \right] = \exp \left\{ \frac{1}{2} \|h_{(n)}\|^2 \right\}.$$

Proof. By 4.4 and a well known Wiener integration formula

$$\begin{aligned} & E \left[\exp \left\{ - \int_0^T h(t)dx_n(t) \right\} \right] \\ &= E \left[\exp \left\{ - \int_0^T h_{(n)}(t)dx(t) \right\} \right] \\ &= (2\pi)^{-1/2} \int_{-\infty}^\infty \exp \left\{ - \|h_{(n)}\|u \right\} \exp \left\{ - \frac{u^2}{2} \right\} du \\ &= \exp \left\{ \frac{1}{2} \|h_{(n)}\|^2 \right\}. \end{aligned}$$

□

THEOREM 4. *Let $h \in L_2[0, T]$ and assume that*

$$F(x) = f \left[\int_0^T h(t) dx(t) \right]$$

is in $L_1(C[0, T], m_w)$.

a). *If h is a linear combination of $\{\alpha_1, \dots, \alpha_n\}$, say $h(t) = c_1\alpha_1(t) + \dots + c_n\alpha_n(t)$ on $[0, T]$, then*

$$(4.6) \quad E \left[f \left[\int_0^T h(t) dx(t) \right] \mid X_n(x) = \vec{\xi} \right] = f(c_1\xi_1 + \dots + c_n\xi_n).$$

b). *If $\{h, \alpha_1, \dots, \alpha_n\}$ is a linearly independent set of functions in $L_2[0, T]$, then*

$$(4.7) \quad E \left[f \left[\int_0^T h(t) dx(t) \right] \mid X_n(x) = \vec{\xi} \right] \\ = [2\pi(\|h\|^2 - \|h_{(n)}\|^2)]^{-1/2} \\ \cdot \int_{-\infty}^{\infty} f(u) \exp \left\{ -\frac{\left(u - \int_0^T h(t) d\vec{\xi}_n(t) \right)^2}{2\|h - h_{(n)}\|^2} \right\} du.$$

Proof. a). In this case $h_{(n)}(t) \equiv h(t)$ and so by 3.4, 4.4 and 3.2,

$$E \left[f \left[\int_0^T h(t) dx(t) \right] \mid X_n(x) = \vec{\xi} \right] \\ = E \left[f \left[\int_0^T h(t) d\{x(t) - x_n(t) + \vec{\xi}_n(t)\} \right] \right] \\ = E \left[f \left[\int_0^T (h(t) - h_{(n)}(t)) dx(t) + \int_0^T h(t) d\vec{\xi}_n(t) \right] \right] \\ = E \left[f \left[\int_0^T h(t) d\vec{\xi}_n(t) \right] \right] \\ = f \left[\int_0^T h(t) d\vec{\xi}_n(t) \right] \\ = f(c_1\xi_1 + \dots + c_n\xi_n).$$

b). In this case we use 3.4, 4.4, and a well known Wiener integration formula to obtain

$$\begin{aligned} E \left[f \left[\int_0^T h(t) dx(t) \right] \middle| X_n(x) = \vec{\xi} \right] &= E \left[f \left[\int_0^T h(t) d\{x(t) - x_n(t) + \vec{\xi}_n(t)\} \right] \right] \\ &= E \left[f \left[\int_0^T (h(t) - h_{(n)}(t)) dx(t) + \int_0^T h(t) d\vec{\xi}_n(t) \right] \right] \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} f \left(\|h - h_{(n)}\|u + \int_0^T h(t) d\vec{\xi}_n(t) \right) \exp\{-u^2/2\} du. \end{aligned}$$

□

In Theorem 4 above the two extreme cases occur when $h \equiv \alpha_j$ for some j or when h is orthogonal to all the α_j 's.

COROLLARY 5. *Let h, F and f be as in Theorem 4. Then*

$$(4.8) \quad E \left[f \left[\int_0^T \alpha_j(t) dx(t) \right] \middle| X_n(x) = \vec{\xi} \right] = f(\xi_j),$$

while if $\{h, \alpha_1, \dots, \alpha_n\}$ is an orthogonal set of functions in $L_2[0, T]$,

$$(4.9) \quad \begin{aligned} E \left[f \left[\int_0^T h(t) dx(t) \right] \middle| X_n(x) = \vec{\xi} \right] &= E \left[f \left[\int_0^T h(t) dx(t) \right] \right] \\ &= [2\pi\|h\|^2]^{-1/2} \int_{-\infty}^{\infty} f(u) \exp \left\{ -\frac{u^2}{2\|h\|^2} \right\} du. \end{aligned}$$

Proceeding as above we obtain the following generalization of formula 4.9.

COROLLARY 6. *If $\{\phi_1, \dots, \phi_m, \alpha_1, \dots, \alpha_n\}$ is an orthonormal set of functions in $L_2[0, T]$ and if*

$$F(x) = f \left[\int_0^T \phi_1(t) dx(t), \dots, \int_0^T \phi_m(t) dx(t) \right]$$

is in $L_1(C[0, T], m_w)$, then

$$\begin{aligned} E \left(f \left[\int_0^T \phi_1(t) dx(t), \dots, \int_0^T \phi_m(t) dx(t) \right] \middle| X_n(x) = \vec{\xi} \right) \\ = E \left[f \left[\int_0^T \phi_1(t) dx(t), \dots, \int_0^T \phi_m(t) dx(t) \right] \right] \\ = \left[\prod_{j=1}^m [2\pi]^{-1/2} \right] \int_{\mathbb{R}^m} f(u_1, \dots, u_m) \exp \left\{ - \sum_{j=1}^m \frac{u_j^2}{2} \right\} d\vec{u}. \end{aligned}$$

Our next corollary follows from the observations that $\int_0^T (h(t) - h_{(n)}(t)) d\vec{\xi}_n(t) = 0$, and $(h - h_{(n)})_{(n)}(t) = 0$.

COROLLARY 7. *Let h, F and f be as in Theorem 4. Then*

$$\begin{aligned} E \left[f \left[\int_0^T h(t) d\{x(t) - x_n(t)\} \right] \middle| X_n(x) = \vec{\xi} \right] \\ = E \left[f \left[\int_0^T \{h(t) - h_{(n)}(t)\} dx(t) \right] \middle| X_n(x) = \vec{\xi} \right] \\ = E \left[f \left[\int_0^T \{h(t) - h_{(n)}(t)\} dx(t) \right] \right] \\ = [2\pi \| |h - h_{(n)}| \|^2]^{-1/2} \int_{-\infty}^{\infty} f(u) \exp \left\{ - \frac{u^2}{2 \| |h - h_{(n)}| \|^2} \right\} du. \end{aligned}$$

Many interesting examples of conditional Wiener integrals can be obtained as special cases of the following theorem.

THEOREM 5. *Let $g \in L_2[0, T]$. Then*

$$\begin{aligned} (4.10) \quad E \left[\exp \left\{ \int_0^T g(s) x(s) ds \right\} \middle| X_n(x) = \vec{\xi} \right] \\ = \exp \left\{ \sum_{j=1}^n \xi_j(g, \beta_j) + \frac{1}{2} \int_0^T \left[\int_s^T g(t) dt \right]^2 ds - \frac{1}{2} \sum_{j=1}^n (g, \beta_j)^2 \right\}. \end{aligned}$$

Proof. Using integration by parts it follows that

$$\int_0^T g(s)x(s)ds = \int_0^T \left[\int_s^T g(t)dt \right] dx(s)$$

and that

$$\int_0^T \left[\int_s^T g(t)dt \right] \alpha_j(s)ds = \int_0^T g(s)\beta_j(s)ds = (g, \beta_j).$$

Hence using (3.4) we obtain

$$\begin{aligned} & E \left[\exp \left\{ \int_0^T g(s)x(s)ds \right\} \mid X_n(x) = \vec{\xi} \right] \\ &= E \left[\exp \left\{ \int_0^T \left[\int_s^T g(t)dt \right] d(x(s) - x_n(s) + \vec{\xi}_n(s)) \right\} \right] \\ &= \exp \left\{ \sum_{j=1}^n \xi_j \int_0^T \left[\int_s^T g(t)dt \right] \alpha_j(s)ds \right\} \\ &\quad \cdot E \left[\exp \left\{ \int_0^T \left[\int_s^T g(t)dt \right] dx(s) \right. \right. \\ &\quad \left. \left. - \sum_{j=1}^n \gamma_j(x) \int_0^T \left[\int_s^T g(t)dt \right] \alpha_j(s)ds \right\} \right] \\ &= \exp \left\{ \sum_{j=1}^n \xi_j(g, \beta_j) \right\} \\ &\quad \cdot E \left[\exp \left\{ \int_0^T \left[\int_s^T g(t)dt - \sum_{j=1}^n (g, \beta_j)\alpha_j(s) \right] dx(s) \right\} \right] \\ &= \exp \left\{ \sum_{j=1}^n \xi_j(g, \beta_j) + \frac{1}{2} \int_0^T \left[\int_s^T g(t)dt - \sum_{j=1}^n (g, \beta_j)\alpha_j(s) \right]^2 ds \right\} \end{aligned}$$

from which 4.10 follows. □

COROLLARY 8. *Let $g(s) \equiv 1$ and let the α_j 's be given by 3.6.*

Then

$$\begin{aligned} E \left[\exp \left\{ \int_0^T x(s) ds \right\} \mid X_n(x) = \vec{\xi} \right] \\ = \exp \left\{ \frac{T^3}{6} + \frac{1}{2} \sum_{j=1}^n (\xi_j + \xi_{j-1})(t_j - t_{j-1}) \right. \\ \left. - \frac{1}{8} \sum_{j=1}^n (t_j - t_{j-1})(t_j + t_{j-1})^2 \right\}. \end{aligned}$$

COROLLARY 9. Let $n = 1$ and $\alpha_1(s) \equiv 1/\sqrt{T}$. Then

$$\begin{aligned} E \left[\exp \left\{ \int_0^T g(s)x(s) ds \right\} \mid x(T) = \xi \right] \\ = \exp \left\{ \frac{\xi}{T} \int_0^T tg(t) dt + \frac{1}{2} \int_0^T \left[\int_s^T g(t) dt \right]^2 ds - \frac{1}{2T} \left[\int_0^T tg(t) dt \right]^2 \right\}, \\ E \left[\exp \left\{ \int_0^T sx(s) ds \right\} \mid x(T) = \xi \right] = \exp \left\{ \frac{\xi T^2}{3} + \frac{T^5}{90} \right\}, \end{aligned}$$

and

$$E \left[\exp \left\{ \int_0^T x(s) ds \right\} \mid x(T) = \xi \right] = \exp \left\{ \frac{\xi T}{2} + \frac{T^3}{24} \right\}.$$

5. Translation of generalized conditional Wiener integrals.

The Cameron-Martin Theorem [3], [11] states that if $x_0(t) = \int_0^t h(s) ds$ for all $t \in [0, T]$ with $h \in L_2[0, T]$, and if T_1 is the transformation from $C[0, T]$ into itself defined by

$$T_1(x) = x + x_0 \text{ for } x \in C[0, T],$$

then for any Wiener integrable function F on $C[0, T]$ and any Wiener measurable set Γ

$$(5.1) \quad \int_{\Gamma} F(y) m_w(dy) = \int_{T_1^{-1}(\Gamma)} F(x + x_0) J(x_0, x) m_w(dx)$$

where

$$(5.2) \quad J(x_0, x) = \exp \left\{ -\frac{1}{2} \|h\|^2 - \int_0^T h(t) dx(t) \right\}.$$

In particular, if $\Gamma = C[0, T]$, then 5.1 becomes:

$$(5.3) \quad E[F(y)] = E[F(x + x_0)J(x_0, x)].$$

In [14], Yeh gives a conditional version of 5.3 which states that

$$\begin{aligned} E[F(y)|y(T) = \xi] &= E \left[F(y) \mid \int_0^T dy(t) = \xi \right] \\ &= E[F(x+x_0)J(x_0, x)|x(T) = \xi - x_0(T)] \exp \left\{ -\frac{x_0^2(T)}{2T} + \frac{\xi x_0(T)}{T} \right\}. \end{aligned}$$

Our next theorem is a generalized conditional version of 5.3.

THEOREM 6. *Let $h \in L_2[0, T]$ and let $x_0(t) = \int_0^t h(s)ds$ for $t \in [0, T]$. Let $F \in L_1(C[0, T], m_w)$ and let the α_j 's be as in Section 2. Then*

$$(5.4) \quad \begin{aligned} E[F(y)|X_\alpha(y) = \vec{\xi}] \\ &= E[F(x + x_0)J(x_0, x)|X_n(x + x_0) = \vec{\xi}] \\ &\quad \cdot \exp \left\{ \int_0^T h(t)d\vec{\xi}_n(t) - \frac{1}{2}||h_{(n)}||^2 \right\} \end{aligned}$$

where $J(x_0, x)$ is given by 5.2 and $h_{(n)}(t)$ is given by 4.1. The result holds for $n = \infty$ as well.

Proof. By 3.4 we see that

$$(5.5) \quad E[F(y)|X_n(y) = \vec{\xi}] = E[F(y - y_n + \vec{\xi}_n)].$$

Using 5.3 and noting that $(x + x_0)_n = x_n + (x_0)_n$, we have

$$(5.6) \quad E[F(y - y_n + \vec{\xi}_n)] = E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)J(x_0, x)].$$

Next we rewrite $J(x_0, x)$ in the form

$$(5.7) \quad \begin{aligned} J(x_0, x) = & \exp \left\{ -\frac{1}{2} \|h\|^2 \right\} \\ & \cdot \exp \left\{ -\int_0^T h(t) d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t)) \right\} \\ & \cdot \exp \left\{ -\int_0^T h(t) dx_n(t) \right\} \\ & \cdot \exp \left\{ \int_0^T h(t) d(\vec{\xi}_n(t) - (x_0)_n(t)) \right\}. \end{aligned}$$

Using 4.1 we see that

$$(5.8) \quad \int_0^T h(t) d(x_0)_n(t) = \int_0^T h_{(n)}^2(t) dt = \|h_{(n)}\|^2.$$

Since $x_n(t)$ and $x(t) - x_n(t)$ are independent processes on $[0, T]$ by Corollary 1, $\exp \left\{ -\int_0^T h(t) dx_n(t) \right\}$ and

$$\begin{aligned} & F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n) \\ & \cdot \exp \left\{ -\int_0^T h(t) d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t)) \right\} \end{aligned}$$

are also independent. Thus using 5.7, 4.5 and 5.8,

$$(5.9) \quad \begin{aligned} & E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n) J(x_0, x)] \\ & = E \left[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n) \right. \\ & \quad \cdot \exp \left\{ -\int_0^T h(t) d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t)) \right\} \left. \right] \\ & \quad \cdot \exp \left\{ -\frac{1}{2} \|h\|^2 + \frac{1}{2} \|h_{(n)}\|^2 + \int_0^T h(t) d\vec{\xi}_n(t) - \|h_{(n)}\|^2 \right\}. \end{aligned}$$

Therefore, by using 5.9 and 3.4 we obtain

$$\begin{aligned} & E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)J(x_0, x)] \\ &= E \left(\left[F(x + x_0) \exp \left\{ - \int_0^T h(t) d(x(t)) \right\} \right] \middle| X_\alpha(x + x_0) = \vec{\xi} \right) \\ &\quad \cdot \exp \left\{ - \frac{1}{2} \|h\|^2 + \int_0^T h(t) d\vec{\xi}_n(t) - \frac{1}{2} \|h_{(n)}\|^2 \right\} \\ &= E \left([F(x + x_0)J(x_0, x)] \middle| X_\alpha(x + x_0) = \vec{\xi} \right) \\ &\quad \cdot \exp \left\{ \int_0^T h(t) d\vec{\xi}_n(t) - \frac{1}{2} \|h_{(n)}\|^2 \right\}. \end{aligned}$$

This together with 5.6 and 5.5 yields 5.4. The case $n = \infty$ follows by the martingale convergence theorem. \square

REMARK. By choosing the α_j 's as in 3.6, we see that Theorem 4 on page 391 of [8] is a Corollary of Theorem 6 above.

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MIAMI UNIVERSITY

OXFORD, OH 45056-1641

E-mail address: cpark@miavx1.acs.muohio.edu

AND

UNIVERSITY OF NEBRASKA

LINCOLN, NE 68588-0323

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