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**ON A PLANCHEREL FORMULA FOR CERTAIN DISCRETE,  
FINITELY GENERATED, TORSION-FREE NILPOTENT  
GROUPS**

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**We prove a Plancherel formula for elementarily exponentiable, discrete, finitely generated, torsion-free nilpotent groups.**

**1. Introduction.** Let  $\Gamma$  be a discrete, finitely generated, torsion-free, nilpotent Lie group. If  $\Gamma^{(k)}$  denotes its descending central series, we will call  $\Gamma$   $n$ -step nilpotent if  $\Gamma^{(n)} \neq \{1\}$  but  $\Gamma^{(n+1)} = \{1\}$ . Malcev has shown that any  $\Gamma$  of this type may be embedded as a discrete cocompact subgroup of a simply connected, connected nilpotent Lie group (see [1], Chapter 1); thus we may utilize all that is known about uniform subgroups of these groups, which is summarized beautifully in ([3], Secs. 5.1 and 5.2).

This work extends the pioneering work of R. Howe on the representation theory of groups of this type, and uses much of the machinery he developed (see [5]). The techniques used to prove the Plancherel formula for  $\Gamma$  are essentially those used in [4] to prove a similar Plancherel formula for discrete groups which are the rational points of a nilpotent Lie group. The corresponding result for  $\Gamma$  follows easily once we observe that, for a certain type of character  $\{\tilde{\lambda}\}$  in the Pontryagin dual of the center of  $\Gamma$ , the  $\Gamma$ -orbit of any extension  $\lambda$  of  $\tilde{\lambda}$  to a character on the Lie algebra  $\mathcal{L}$  of  $\Gamma$  is dense in the set  $\lambda + z(\mathcal{L})^\perp \subseteq \widehat{\mathcal{L}}$  (Proposition 2.5).

Let  $\mathcal{L}_{\mathbb{R}}$  be a real finite-dimensional  $r$ -step nilpotent Lie algebra, and let  $\mathcal{L} \subseteq \mathcal{L}_{\mathbb{R}}$  be a discrete additive subgroup of  $\mathcal{L}_{\mathbb{R}}$ . A calculation with the Campbell-Baker-Hausdorff formula shows that if  $\mathcal{L}$  is

1. an additive discrete subgroup of  $\mathcal{L}_{\mathbb{R}}$ , not necessarily of cofinite volume, and
2.  $\mathcal{L}$  satisfies  $[\mathcal{L}, \mathcal{L}] \subseteq r!\mathcal{L}$ ,

then  $\Gamma = \exp \mathcal{L}$  forms a discrete subgroup of the connected, simply connected nilpotent Lie group  $N = \exp \mathcal{L}_{\mathbb{R}}$ . If  $\mathcal{L}$  satisfies condition 1, we will refer to  $\mathcal{L}$  as a *lattice*. If  $\mathcal{L}$  satisfies both conditions we will say that  $\mathcal{L}$ , and  $\Gamma = \exp \mathcal{L}$ , are *elementarily exponentiable*, or *e.e.* for short. If  $\mathcal{L}$  is *e.e.*, and  $i$  is an *e.e.* lattice contained in  $\mathcal{L}$  which is closed under the bracket operation, we will call  $i$  an *ideal* of  $\mathcal{L}$  if  $i$  is  $\text{Ad}^*(\mathcal{L})$ -invariant. Note that for  $i$  to be *e.e.*, we must have  $[i, i] \subseteq r!i$ , where  $r$  is the length of  $\mathcal{L}_{\mathbb{R}}$ . We assume throughout this paper that the  $\Gamma$  under consideration are *e.e.*

We define the rising central series of ideals of  $\mathcal{L}$  as follows; set  $\mathcal{L}_{(0)} = (0)$ , and define  $\pi_n: \mathcal{L} \rightarrow \mathcal{L} \setminus \mathcal{L}_{(n-1)}$  to be the natural quotient map for each  $n \geq 1$ . Then  $\mathcal{L}_{(n)}$  is defined to be the preimage under  $\pi_n$  of the center of  $\mathcal{L} \setminus \mathcal{L}_{(n-1)}$ . Thus  $\mathcal{L}_{(n)}$  is an increasing sequence of ideals of  $\mathcal{L}$ , satisfying  $\mathcal{L}_{(r)} = \mathcal{L}$ , since  $\mathcal{L}$  is  $r$ -step nilpotent.

We show in what follows that the  $\mathcal{L}_{(n)}$  are *e.e.*

LEMMA 1.1.  $\mathcal{L}_{(n)} = \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(n)}$ , for  $n \in 0, \dots, r$ .

*Proof.* For  $n = 0$ , the result is trivial.

Suppose now that  $\mathcal{L}_{(k)} = \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(k)}$ . By definition,  $\mathcal{L}_{(k+1)} = \pi_k^{-1}(z(\mathcal{L} \setminus \mathcal{L}_{(k)}))$ . We show:

(a).  $\mathcal{L}_{(k+1)} \subseteq \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(k+1)}$ .

For if  $X \in \mathcal{L}_{(k+1)}$ ,  $[X, Y] = Z \in \mathcal{L}_{(k)}$  for all  $Y \in \mathcal{L}$ . Let  $\bar{\mathcal{L}}$  be the image of  $\mathcal{L}$  in  $\mathcal{L}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{R}(k)}$ , and let  $\bar{X}$  be the image of  $X$ . It follows from Theorems 5.1.4 and 5.2.3 in [3] that  $\bar{\mathcal{L}}$  is uniform in  $\mathcal{L}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{R}(k)}$ . For  $X \in \mathcal{L}_{(k+1)}$ ,  $\bar{X}$  commutes with all  $\bar{Y} \in \bar{\mathcal{L}}$ . We apply Theorem 5.1.5 from [3] to see that  $\bar{X} \in z(\mathcal{L}_{\mathbb{R}} \setminus \mathcal{L}_{\mathbb{R}(k)})$ . It follows that  $X \in \mathcal{L}_{\mathbb{R}(k+1)}$ .

(b).  $\mathcal{L} \cap \mathcal{L}_{\mathbb{R}(k+1)} \subseteq \mathcal{L}_{(k+1)}$ .

If  $X \in \mathcal{L} \cap \mathcal{L}_{\mathbb{R}(k+1)}$ , then  $[X, Y] = Z \in \mathcal{L} \cap \mathcal{L}_{\mathbb{R}(k)} = \mathcal{L}_{(k)}$ . This completes the proof of the lemma.  $\square$

It now follows easily that  $\mathcal{L}_{(k)}$  is *e.e.* for all  $k$ ; suppose  $Z = [X, Y] \in [\mathcal{L}_{(k)}, \mathcal{L}_{(k)}]$ . Then since  $\mathcal{L}$  is *e.e.*,  $Z = r!X$  for some  $X \in \mathcal{L}$ . By definition,  $[\mathcal{L}_{(k)}, \mathcal{L}_{(k)}] \subseteq [\mathcal{L}_{(k)}, \mathcal{L}] \subseteq \mathcal{L}_{(k-1)} \subseteq \mathcal{L}_{(k)} = \mathcal{L}_{\mathbb{R}(k)} \cap \mathcal{L}$ ; so  $Z \in \mathcal{L}_{\mathbb{R}(k)}$ , hence  $X \in \mathcal{L}_{\mathbb{R}(k)}$ . Thus  $X \in \mathcal{L}_{(k)}$ , and so  $Z \in r!\mathcal{L}_{(k)}$ . Therefore,  $\mathcal{L}_{(k)}$  is *e.e.*

We will refer to a subgroup  $\Gamma' \subseteq \Gamma$  (or a subalgebra  $\mathcal{L}' \subseteq \mathcal{L}$ ) as *saturated* if  $x^n \in \Gamma'$  implies  $x \in \Gamma'$  ( $kX \in \mathcal{L}'$  implies  $X \in \mathcal{L}'$ ).

I am grateful to the referee of this paper for suggestions which greatly improved the presentation of these results.

**2. Generic coadjoint orbits.** Let  $\mathcal{L}$  be an *e.e.* lattice in a real nilpotent Lie algebra  $\mathcal{L}_{\mathbb{R}}$ , so that  $\Gamma = \exp(\mathcal{L}_{\mathbb{R}})$  is uniform in  $N = \exp(\mathcal{L}_{\mathbb{R}})$ . We assume throughout this section that a strong Malcev basis  $\{X_1, X_2, \dots, X_n\}$  for  $\mathcal{L}_{\mathbb{R}}$  through its ascending central series has been chosen so that it satisfies:

1. The  $\mathbb{Z}$ -span of  $\{X_1, \dots, X_n\}$  is the lattice  $\mathcal{L}$  in  $\mathcal{L}_{\mathbb{R}}$ :

2. The real span of  $\{X_1, \dots, X_i\}$  forms an ideal of  $\mathcal{L}_{\mathbb{R}}$ , and in particular the real span of  $\{X_1, \dots, X_k\}$  is the center of  $\mathcal{L}_{\mathbb{R}}$  (we will also regard  $\mathcal{L}_{\mathbb{R}}$  as having the inner product defined by setting  $\langle X_i, X_j \rangle = \delta_{i,j}$ );

3.  $\Gamma = \exp(\mathbb{Z}X_1) \exp(\mathbb{Z}X_2) \dots \exp(\mathbb{Z}X_n)$ . Thus we may coordinatize  $\Gamma$  as follows:  $\gamma \leftrightarrow (x_1, \dots, x_n) \in \mathbb{Z}^n$  whenever  $\gamma = \exp x_1 X_1 \cdot \dots \cdot \exp x_n X_n$ .

That a basis for  $\mathcal{L}_{\mathbb{R}}$  may be chosen satisfying all these conditions is shown in Section 5.1 of [3], and as part of the proof of Proposition 5.4.11 in [3].

We think of  $\mathcal{L}$  as the Lie algebra of  $\Gamma$ ; as an abelian group it is isomorphic to  $\mathbb{Z}^n$  via the map

$$\begin{aligned} \Psi: \mathbb{Z}^n &\longrightarrow \mathcal{L} \\ (a_1, \dots, a_n) &\longmapsto a_1 X_1 + \dots + a_n X_n. \end{aligned}$$

Thus the natural Pontryagin dual of  $\mathcal{L}$  is  $T^n \cong (\mathbb{R} \setminus \mathbb{Z})^n$  via the map

$$\begin{aligned} \Phi: T^n &\longrightarrow \widehat{\mathcal{L}} \\ (\bar{\lambda}_1, \dots, \bar{\lambda}_n) &\longmapsto \lambda \end{aligned}$$

where  $\lambda(a_1, \dots, a_n) = \exp(2\pi i(\lambda_1 a_1 + \dots + \lambda_n a_n))$ , for any choice  $\{\lambda_i\}$  of representatives for the elements  $\bar{\lambda}_i$  of  $\mathbb{R} \setminus \mathbb{Z}$ .

Let  $z(\mathcal{L})$  denote the center of the Lie algebra  $\mathcal{L}$ ; if  $z(\mathcal{L}_{\mathbb{R}})$  is the center of the real Lie algebra  $\mathcal{L}_{\mathbb{R}}$ , then  $z(\mathcal{L}) = \mathcal{L} \cap z(\mathcal{L}_{\mathbb{R}})$  (Lemma 5.1.5 in [3]). Hence  $z(\mathcal{L})$  is a saturated subalgebra of  $\mathcal{L}$ .

For a fixed  $\lambda \in \widehat{\mathcal{L}}$ , let  $\mathfrak{i}_{\lambda}$  be the largest ideal of  $\mathcal{L}$  contained in the subalgebra  $\mathfrak{r}_{\lambda} = \{X \in \mathcal{L} : \lambda[X, Y] = 1 \text{ for all } Y \in \mathcal{L}\}$ . Then  $R_{\lambda} = \exp(\mathfrak{r}_{\lambda})$  is the isotropy subgroup of  $\lambda$  under the  $\text{Ad}^*$ -action

of  $\Gamma$ ; in particular it is shown in Lemma 2 of [5] that  $\mathbf{r}_\lambda$  is an *e.e.* lattice in  $\mathcal{L}$ . Furthermore,  $\mathbf{i}_\lambda = \bigcap_{\gamma \in \Gamma} \text{Ad}(\gamma)\mathbf{r}_\lambda$  is an *e.e.* ideal of  $\mathcal{L}$  (the intersection of *e.e.* subalgebras is *e.e.*, and the  $\text{Ad}(\gamma)\mathbf{r}_\lambda$  are each *e.e.*, being the images of an *e.e.* subalgebra under a Lie algebra automorphism). It follows that  $\exp(\mathbf{i}_\lambda) = I_\lambda \subseteq R_\lambda$  is a normal subgroup of  $\mathcal{L}$  which always contains  $\exp(z(\mathcal{L})) = z(\Gamma)$ . Let  $r: \widehat{\mathcal{L}} \rightarrow z(\widehat{\mathcal{L}})$  send an element  $\lambda \in \widehat{\mathcal{L}}$  to its restriction to  $z(\mathcal{L})$ . In what follows, we will show that for all  $\tilde{\lambda} \in z(\widehat{\mathcal{L}})$  except those in a set of Haar measure zero, the elements of  $r^{-1}(\tilde{\lambda})$  satisfy  $\mathbf{i}_\lambda = z(\mathcal{L})$ .

LEMMA 2.1. *Suppose  $\lambda = \Phi(\bar{\lambda}_1, \dots, \bar{\lambda}_n)$  where  $\bar{\lambda}_i \in \mathbb{R} \setminus \mathbb{Z}$ . Then  $\lambda$  has trivial kernel in  $\mathcal{L}$  if and only if for some choice of representatives  $\lambda_i \in \mathbb{R}$  of  $\bar{\lambda}_i \in \mathbb{R} \setminus \mathbb{Z}$ , the set  $\{\lambda_1, \dots, \lambda_n, 1\}$  is linearly independent over  $\mathbb{Q}$ .*

*Proof.* Suppose  $\lambda$  has trivial kernel, and suppose that for some elements  $q_1, \dots, q_n, q \in \mathbb{Q}$ , we have

$$\lambda_1 q_1 + \dots + \lambda_n q_n + q = 0$$

for some choice of representatives  $\{\lambda_i\}$  for  $\{\bar{\lambda}_i\}$ . After multiplication of this equation by some integer, we have  $\lambda_1 a_1 + \dots + \lambda_n a_n + a = 0$  for some integers  $\{a_i\}$ . Then  $\lambda_1 a_1 + \dots + \lambda_n a_n = -a$ , so  $(a_1, \dots, a_n) \in \text{kernel}(\lambda)$ . Thus  $(a_1, \dots, a_n) = 0$ . It follows immediately that  $q_i = 0$  for each  $i = 1, \dots, n$ , and  $q = 0$ .

Conversely, suppose that  $\lambda$  has a nontrivial kernel, and that  $(a_1, \dots, a_n) \neq 0$  is an element of the kernel of  $\lambda$ . Let  $\{\lambda_i\}$  be any choice of representatives for the elements  $\{\bar{\lambda}_i\}$  which determine  $\lambda$ ; then  $a_1 \lambda_1 + \dots + a_n \lambda_n = k$  for some element  $k \in \mathbb{Z}$ . The set  $\{\lambda_1, \dots, \lambda_n, 1\}$  is thus a linearly dependent set over  $\mathbb{Q}$ .  $\square$

LEMMA 2.2. *Let  $i$  be an *e.e.* ideal of the lattice  $\mathcal{L}$ . Let  $\mathcal{L}_{(n)}$  be the rising central series of  $\mathcal{L}$ . Suppose  $i \supseteq z(\mathcal{L})$ ,  $i \neq z(\mathcal{L})$ . Then  $i \cap (\mathcal{L}_{(2)} - z(\mathcal{L}))$  is nonempty.*

*Proof.* Since everything in sight is *e.e.*, and since  $I = \exp(i)$  is normal, we prove the result on the group level. Let  $\bar{I}$  denote the image of  $I$  in  $\bar{G} = G \setminus z(\Gamma)$ . Then since  $\bar{I}$  is a nontrivial normal subgroup of  $\bar{G}$ , its intersection with the center of  $\bar{G}$  is nontrivial. It follows that  $I \cap (\Gamma_{(2)} - z(\Gamma))$  is nonempty, and so the result on the algebra level follows.  $\square$

LEMMA 2.3. For  $\lambda \in \widehat{\mathcal{L}}$ ,  $\mathfrak{i}_\lambda = z(\mathcal{L})$  if and only if there exists no element  $X \in \mathcal{L}_{(2)}$  such that  $\text{ad}(X)$  maps  $\mathcal{L}$  into the kernel of  $\tilde{\lambda}$ , where  $\tilde{\lambda} = \lambda|_{z(\mathcal{L})}$ .

*Proof.* Suppose  $X \in \mathcal{L}_{(2)} - z(\mathcal{L})$ , and  $\text{ad}(X)$  maps  $\mathcal{L}$  into the kernel of  $\tilde{\lambda}$ . Take  $K$  to be the additive subgroup in  $\mathcal{L}$  generated by  $X$  and the elements of  $z(\mathcal{L})$ . Then  $K$  is an ideal of  $\mathcal{L}$ , since  $[K, \mathcal{L}] \subseteq z(\mathcal{L})$ , and clearly  $K \subseteq \mathfrak{r}_\lambda$ . Therefore  $\mathfrak{i}_\lambda \neq z(\mathcal{L})$ .

Conversely, suppose  $\mathfrak{i}_\lambda \neq z(\mathcal{L})$ . Then  $\mathfrak{i}_\lambda$  properly contains  $z(\mathcal{L})$ . By the result of Lemma 2.2, we may choose  $X \in \mathcal{L}_{(2)} - z(\mathcal{L})$  such that  $X \in \mathfrak{i}_\lambda$ ; then  $\text{ad}(X)$  maps  $\mathcal{L}$  into the kernel of  $\tilde{\lambda}$ . This completes the proof of Lemma 2.3.  $\square$

Throughout the rest of this paper,  $\lambda$  will denote an element of  $\widehat{\mathcal{L}}$ , and  $\tilde{\lambda}$  will denote its restriction to  $z(\mathcal{L})$ .

COROLLARY. If  $\mathfrak{i}_\lambda = z(\mathcal{L})$ , then for all  $\phi \in r^{-1}(\tilde{\lambda})$ ,  $\mathfrak{i}_\phi = z(\mathcal{L})$ . Therefore, if  $\ker \tilde{\lambda} \subseteq z(\mathcal{L})$  is trivial,  $\mathfrak{i}_\lambda = z(\mathcal{L})$ .

PROPOSITION 2.4. For a set  $\{\tilde{\lambda}\} \subseteq \widehat{z(\mathcal{L})}$  of full Haar measure in  $\widehat{z(\mathcal{L})}$ ,  $\ker \tilde{\lambda}$  is trivial.

*Proof.* Suppose  $\tilde{\lambda}$  corresponds to the element  $(\bar{\lambda}_1, \dots, \bar{\lambda}_k) \in T^k$ , and that for some set of representatives in  $\mathbb{R}$  of the  $\bar{\lambda}_i$ , the set  $\{\lambda_1, \dots, \lambda_k, 1\}$  is linearly dependent over  $\mathbb{Q}$ . Then for some set of integers  $a_1, \dots, a_k, a \in \mathbb{Z}$ , not all zero, we have that  $a_1\lambda_1 + \dots + a_k\lambda_k = a$ . If we choose some other set of representatives for the  $\bar{\lambda}_i$ , the previous expression changes only by an integral constant.

It follows that the preimage in  $\mathbb{R}^k$  of the set of elements  $\{\bar{\lambda}_i\}$  satisfying this linear dependence condition consists of the union of hyperplanes of the form

$$a_1x_1 + a_2x_2 + \dots + a_kx_k = a,$$

where  $a_i, a$  vary over the elements of  $\mathbb{Z}$ . These are of measure zero in  $\mathbb{R}^k$  individually, hence their (countable) union is of measure zero. The corresponding set in  $T^n$  is therefore of measure zero, and thus the elements  $(\bar{\lambda}_1, \dots, \bar{\lambda}_k) \in T^n$  which satisfy the linear independence condition of Lemma 2.1 are of full measure in  $T^n$ . This completes the proof of Proposition 2.4.  $\square$

We finish this section by proving

**PROPOSITION 2.5.** *If  $\ker \tilde{\lambda}$  is trivial, then the closure of  $\text{Ad}^*(\Gamma)\lambda$  is  $\lambda + z(\mathcal{L})^\perp$ .*

Suppose that  $\gamma \in \Gamma$ , and  $\lambda$  has coordinates  $(\lambda_1, \dots, \lambda_n) \in T^n$  via the map  $\Phi$ . Then  $\text{Ad}^*(\gamma) = (\lambda_1, \dots, \lambda_k, p_{k+1}(\gamma : \lambda), \dots, p_n(\gamma : \lambda))$ , where each  $p_i(\gamma : \lambda)$  is a polynomial in the coordinates of  $\gamma$  (with respect to the map  $\Psi$ ), with coefficients from the  $\mathbb{Q}$ -span of  $\{\lambda_1, \dots, \lambda_n\}$ .

In  $\tilde{\mathcal{L}}$ ,  $z(\mathcal{L})^\perp$  consists of all elements of the form  $(0, \dots, 0, \lambda_{k+1}, \dots, \lambda_n)$ , for  $\lambda_i \in \mathbb{R} \setminus \mathbb{Z}$ .

We wish to show that if  $\lambda_1, \dots, \lambda_k$  satisfy the equivalent conditions of Lemma 2.1 (where  $z(\mathcal{L})$  plays the role of  $\mathcal{L}$ ), then the set  $\{\text{Ad}^*(\gamma)\lambda - \lambda : \gamma \in \Gamma\}$  is dense in  $z(\mathcal{L})^\perp$ . In what follows,  $\lambda$  of this type will be called “generic”, as will coadjoint orbits of the form  $\mathcal{O}_\lambda = \lambda + z(\mathcal{L})^\perp$  where  $\lambda$  is generic.

We may regard the polynomials  $p_i(\gamma : \lambda)$  as polynomials in  $\{x_1, \dots, x_n\}$ ,  $x_i \in \mathbb{Z}$ , by identifying  $\gamma$  with  $(x_1, \dots, x_n)$  via  $\Psi$ .

**LEMMA 2.6.** (H. Weyl, [2]). *Suppose that  $\{p_i\}_{i=1}^n$  is a set of polynomials in one integer variable, with coefficients in  $\mathbb{R} \setminus \mathbb{Z}$ . If for each set of integers  $a_1, \dots, a_n$ , not all zero, the polynomial  $a_1 p_1 + \dots + a_n p_n$  has an irrational coefficient, then the set of points  $\{(p_1(k), \dots, p_n(k)) : k \in \mathbb{Z}\}$  is dense in  $(\mathbb{R} \setminus \mathbb{Z})^n$ .*

Now let  $T = (t_1, \dots, t_s) \in N^s$ , and let  $X^T = x_1^{t_1} \dots x_s^{t_s}$  be a multinomial in  $s$  integer variables.

**LEMMA 2.7.** *Suppose  $\{X^{T_i}\}_{i=1}^r$  is a set of distinct multinomials in  $s$  integer variables. Then there is  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}^s$  so that  $X^{T_i} \circ \phi$  are monomials and are distinct.*

*Proof.* We put a lexicographic order on  $N^s$  as follows: let  $i \in 1, \dots, s$  be the greatest integer with  $m_i \neq m'_i$ ; then  $(m_1, \dots, m_s) > (m'_1, \dots, m'_s)$  if  $m_i > m'_i$ . A simple induction argument shows that for the finite set  $\{T_i\} \subseteq N^s$  there is  $N \in N^s$ ,  $N = (N_1, \dots, N_s)$ , so that if  $T_i > T_j$  in the ordering on  $N^s$ , then  $N \cdot T_i > N \cdot T_j$  ( $N \cdot T$  denotes the usual dot product).

Now define  $\phi(x) = (x^{N_1}, \dots, x^{N_s})$ ; then we have  $X^{T_i} \circ \phi = x^{T_i \cdot N}$ , and so the monomials  $X^{T_i} \circ \phi$  remain distinct.  $\square$

LEMMA 2.8. *Suppose  $\{p_i\}_{i=1}^n$  is a set of polynomials in  $s$  integer variables, with coefficients in  $\mathbb{R} \setminus \mathbb{Z}$ . If for all integers  $a_1, \dots, a_n$ , not all zero,  $p = a_1 p_1 + \dots + a_n p_n$  has an irrational coefficient, then the image of  $\mathbb{Z}^s$  under  $(p_1, \dots, p_n)$  is dense in  $(\mathbb{R} \setminus \mathbb{Z})^n$ .*

*Proof.* Let  $\{X^{T_i}\}$  be the set of monomials which appear in the polynomials  $p_i$ , and let  $\phi$  be as in Lemma 2.7. We note that the monomials in  $a_1 p_1 + \dots + a_n p_n$  are a subset of the  $\{X^{T_i}\}$ , so that they remain distinct if composed with  $\phi$ ; and so for all  $a_1, \dots, a_n$ , not all zero,  $a_1 p_1 \circ \phi + \dots + a_n p_n \circ \phi$  has an irrational coefficient. We invoke Lemma 2.6, and the result follows.

Thus we will have proven Proposition 2.5 if we can show that whenever  $\lambda$  is generic, the polynomial

$$P(x: \lambda) = a_{k+1} p_{k+1}(x: \lambda) + \dots + a_n p_n(x: \lambda)$$

(where  $a_{k+1}, \dots, a_n$  are integers, not all zero) has an irrational coefficient.

Assume  $\lambda$  is generic. We begin by writing  $\text{Ad}^*(\gamma) = \text{Ad}^*(x_1, \dots, x_n)$  as a matrix with respect to the coordinates  $(\lambda_1, \dots, \lambda_n)$  of  $\lambda$  given by the map  $\Phi$ . The condition  $[\mathcal{L}, \mathcal{L}] \subseteq r! \mathcal{L}$  implies that the entries of the matrix  $\text{Ad}^*(x_1, \dots, x_n)$  are elements of  $\mathbb{Z}[x_1, \dots, x_n]$ .

Therefore  $\text{Ad}^*(\gamma)(\lambda_1, \dots, \lambda_k, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_k, p_{k+1}(\gamma: \lambda), \dots, p_n(\gamma: \lambda))$ .  $\text{Ad}^*(\gamma)$  is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ p_{k+1,1} & p_{k+1,2} & p_{k+1,3} & \cdots & p_{k+1,k} & 1 & 0 & \cdots & 0 \\ p_{k+2,1} & p_{k+2,2} & p_{k+2,3} & \cdots & p_{k+2,k} & p_{k+2,k+1} & 1 & \cdots & 0 \\ p_{k+3,1} & p_{k+3,2} & p_{k+3,3} & \cdots & p_{k+3,k} & p_{k+3,k+1} & p_{k+3,k+2} & & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{n,1} & p_{n,2} & p_{n,3} & \cdots & p_{n,k} & p_{n,k+1} & p_{n,k+2} & \cdots & 1 \end{pmatrix}$$

with each  $p_{i,j} \in \mathbb{Z}[x_1, \dots, x_n]$ .

We break the problem down as follows.

1.  $p_i(\gamma, \lambda) = p_{i,1}(\gamma)\lambda_1 + p_{i,2}(\gamma)\lambda_2 + \dots + p_{i,k}(\gamma)\lambda_k + \dots + p_{i,i-1}(\gamma)\lambda_{i-1} + \lambda_i$ , for  $i=k+1, \dots, n$ .



2. Let  $\{a_{k+1}, \dots, a_n\}$  be any set of integers, not all zero. Define

$$\begin{aligned} p(\gamma, \lambda) &= a_{k+1}p_{k+1}(\gamma, \lambda) + \dots + a_n p_n(\gamma, \lambda) \\ &= \sum_{i=k+1}^n a_i \left\{ \sum_{t=1}^{i-1} \lambda_t p_{i,t}(\gamma) + \lambda_i \right\} \\ &= \sum_{t=1}^k \lambda_t \left\{ \sum_{i=k+1}^n a_i p_{i,t} \right\} + \sum_{t=k+1}^n \lambda_t \left\{ \sum_{i=t}^n a_i p_{i,t} \right\}. \end{aligned}$$

We note that  $p_{i,i} = 1$  for all  $i$ . Let  $A = (0, \dots, 0, a_{k+1}, \dots, a_n) \in \mathcal{L}$ . If we write  $\text{Ad}(\gamma^{-1})A = (\Phi_1, \dots, \Phi_n)$ , then we have  $p(\gamma, \lambda) = \sum_{i=1}^n \lambda_i \Phi_i$ . We write  $p(\gamma, \lambda) = p_c(\gamma, \lambda) + p_n(\gamma, \lambda)$  where  $p_c = \sum_{i=1}^k \lambda_i \Phi_i$  and  $p_n = \sum_{i=k+1}^n \lambda_i \Phi_i$ .

We first show that if  $\{\lambda_1, \dots, \lambda_k\}$  are as in lemma 2.1, then the polynomial  $p_c$  has a nontrivial irrational coefficient. Since the  $\Phi_i$ ,  $i = 1, \dots, k$ , consist of polynomials with integral coefficients and  $\{\lambda_1, \dots, \lambda_k\}$  are linearly independent over  $\mathbb{Q}$ , all the coefficients of the polynomial  $p_c$  are irrational. Therefore, we need only show that  $p_c$  is not a constant polynomial, or equivalently that some  $\Phi_i$ ,  $i = 1, \dots, k$ , is not constant.

These are the central components of the vector giving  $\text{Ad}(\gamma^{-1})A$ , where not all of the entries  $a_i$  are zero. Therefore the orbit described is that of a non-central element of  $\mathcal{L}$ , and so it will suffice to show that for any non-central element  $T$  of  $\mathcal{L}$ ,  $\text{Ad}(\Gamma)(T)$  has nondegenerate orthogonal projection onto the center of  $\mathcal{L}_{\mathbb{R}}$ .

**LEMMA 2.9.** *Let  $\{X_1, \dots, X_n\}$  be as before, and let  $P_z$  be projection onto the real span of  $\{X_1, \dots, X_k\}$  with  $\mathbb{R} - \text{span} \{X_{k+1}, \dots, X_n\}$  as kernel. Then if  $T$  is a non-central element of  $\mathcal{L}_{\mathbb{R}}$ ,  $P_z(\text{Ad}(\Gamma)T - T)$  is not identically zero.*

*Proof.* It suffices to show that  $P_z(\text{Ad}(N)T - T)$  is not identically zero, where  $N = \exp(\mathcal{L}_{\mathbb{R}})$ . Let  $\{t_1, \dots, t_s\}$  be a subset of  $\mathbb{R}$  and  $\{Y_1, \dots, Y_s\}$  be a subset of  $\mathcal{L}_{\mathbb{R}}$ .

If we use the formula  $\text{Ad}(\exp Y)T = \exp(\text{ad}(Y)T)$  to write  $\text{Ad}(\exp t_1 Y_1 \cdot \exp t_2 Y_2 \cdot \dots \cdot \exp t_s Y_s)T - T$  as a polynomial expression in  $\{t_1, \dots, t_s\}$  with coefficients in  $\mathcal{L}_{\mathbb{R}}$ , we see that the coefficient of the monomial  $t_1 t_2 \dots t_s$  is a rational multiple of  $[Y_1, [Y_2, [\dots [Y_s, T]] \dots]]$ . Now suppose that  $P_z(\text{Ad}(N)T - T) = 0$ ; then we

must have  $[Y_1, [Y_2, [\dots[Y_s, T]\dots]]] = 0$ . However, since  $T$  is not central, there exists a sequence of elements  $\{Y_1, \dots, Y_s\} \subseteq \mathcal{L}_{\mathbb{R}}$  such that  $W = [Y_1, [Y_2, [\dots[Y_s, T]\dots]] \in z(\mathcal{L})$ ,  $W \neq 0$ . Then  $P_z(W) = W$  is nonzero, giving a contradiction, and completing the proof of Lemma 2.9.

LEMMA 2.10. *The polynomial  $p(\gamma, \lambda)$  has an irrational coefficient (of a nontrivial monomial) whenever  $\lambda$  is generic.*

*Proof.* Suppose that  $p(\gamma, \lambda)$  has entirely rational coefficients. Then since  $p_c(\gamma, \lambda)$  is nontrivial,  $p_n(\gamma, \lambda) = p_r(\gamma, \lambda) - p_c(\gamma, \lambda)$ , where  $p_r(\gamma, \lambda)$  has rational coefficients. Since we have  $p_n(\gamma, \lambda) = \sum_{i=k+1}^n \lambda_i \Phi_i$ , where the  $\Phi_i$  have integer coefficients, we must have some subset  $\{\lambda_{\sigma(t)}\}_{t=1}^l$  of coefficients which are from the  $\mathbb{Q}$ -span of  $\{\lambda_1, \dots, \lambda_k\}$ , and the rest must be rational. Thus we have

$$\sum_{i=1}^k \lambda_i \Phi_i + \sum_{t=1}^l \lambda_{\sigma(t)} \Phi_{\sigma(t)} = K,$$

where  $K$  is some real constant. However, the  $\lambda_{\sigma(t)}$ ,  $t = 1, \dots, l$ , satisfy  $\lambda_{\sigma(t)} = \sum_{i=1}^k q_{t,i} \lambda_i$ , where the  $q_{t,i}$  are rational. Therefore we may write the equation above as

$$\sum_{i=1}^k \lambda_i \Phi_i + \sum_{t=1}^l \left\{ \sum_{s=1}^k q_{s,t} \lambda_s \right\} \Phi_{\sigma(t)} = K.$$

Since the  $\lambda_i$  are linearly independent over  $\mathbb{Q}$ , we must have, for each  $i$ , that

$$\Phi_i + \sum_{t=1}^l q_{t,i} \Phi_{\sigma(t)} = K_i,$$

for some real constant  $K_i$ . Let  $v_i = X_i + \sum_{t=1}^l q_{t,i} X_{\sigma(t)} \in \mathcal{L}_{\mathbb{R}}$ , for  $i = 1, \dots, k$ . The above implies that the function  $\gamma \mapsto \langle v_i, \text{Ad}(\gamma)A \rangle$  is a constant function, and so that the projection of  $\text{Ad}(N)A$  onto the subspace  $W = \mathbb{R}\text{-span} \{v_i\}$  is degenerate. We will show that this is impossible, giving a contradiction.

LEMMA 2.11. *Let  $\mathcal{O}$  be a non-trivial Ad-orbit of  $N$ . Then  $P_W(\mathcal{O})$  is nondegenerate, i.e.,  $P_W(\text{Ad}(\Gamma)A - A)$  is nonzero.*

*Proof.* Let  $\{Y_1, \dots, Y_s\}$  be as in the proof of Lemma 2.9., such that  $[Y_1, [\dots[Y_{s-1}, [Y_s, A]]\dots]]$  is a nonzero element of  $z(\mathcal{L})$ . Then if

we express  $\text{Ad}(\exp t_1 Y_1 \cdots \exp t_s Y_s)A$  as a polynomial in  $t_1, \dots, t_s$  taking on values in  $\mathcal{L}_{\mathbb{R}}$ , we see that the monomial  $t_1 \dots t_s$  appears as a coefficient only of central elements  $\{X_1, \dots, X_k\}$ . Let  $i \leq k$  be such that  $[Y_1, [\dots [Y_s, A] \dots]]$  has a nontrivial  $X_i$ -component. Then  $\langle v_i, \text{Ad}(\exp t_1 Y_1 \dots \exp t_s Y_s)A \rangle$  is a polynomial with a nonzero coefficient of  $t_1 \dots t_s$ , so it is nontrivial, and so  $P_W(\mathcal{O})$  is nondegenerate.

This completes the proof of Lemma 2.10; taken together with Lemma 2.8, this completes the proof of Proposition 2.5.  $\square$

**3. Traceable factor representations associated with generic coadjoint orbits.** Suppose now that  $\lambda$  is a generic element of  $\widehat{\mathcal{L}}$ , with  $\text{Ad}^*(\Gamma)$ - orbit closure  $\mathcal{O}_\lambda$ . We define  $\tau_\lambda$  to be the representation of  $\Gamma$  induced from the restriction of  $\lambda$  to  $z(\mathcal{L})$ , regarded as a character on  $z(\Gamma)$  (this is possible because  $z(\mathcal{L}) = z(\mathcal{L}_{\mathbb{R}}) \cap \mathcal{L}$  by Theorem 5.1.5 in [3], and because  $\exp z(\mathcal{L}) = z(\Gamma)$ );  $\tau_\lambda$  is defined on the Hilbert space

$$H_\lambda = \left\{ f: \Gamma \longrightarrow \mathbb{C} \mid \int_{\Gamma \setminus z(\Gamma)} |f|^2 dx < \infty, f(z\gamma) = \lambda(z)f(\gamma), \right. \\ \left. z \in z(\Gamma), \gamma \in \Gamma \right\},$$

with inner product  $\langle f, g \rangle = \int_{\Gamma \setminus z(\Gamma)} f \cdot \bar{g} \, dx$ . Since elements in  $\mathcal{O}_\lambda$  agree on the center of  $\mathcal{L}$ ,  $\tau_\lambda$  depends only upon the coadjoint orbit closure  $\mathcal{O}_\lambda$ . Recall that  $\tau_\lambda$  is a factor representation if  $CR(\tau_\lambda) = \tau_\lambda(\Gamma)' \cap \tau_\lambda(\Gamma)'' = \mathbb{C}I$  (in general,  $A'$  denotes the commutator of the set  $A$ ). In what follows, we will show that if  $\mathcal{O}_\lambda$  is a generic coadjoint orbit closure in  $\widehat{\mathcal{L}}$ , then  $\tau_\lambda$  is a factor representation.

LEMMA 3.1. (Lemma 1, [4]). *Let  $U \in \tau_\lambda(\Gamma)'$ , and let  $H_\lambda$  be the Mackey space as defined above for the representation  $\tau_\lambda$ . Then  $U$  is entirely determined by its value on the function  $\delta_1 \in H_\lambda$  defined by*

$$\delta_1(\gamma) = \begin{cases} 0 & \gamma \notin z(\Gamma) \\ \lambda(\gamma) & \gamma \in z(\Gamma) \end{cases}.$$

LEMMA 3.2. (Lemma 2, [4]). *If  $U \in CR(\tau_\lambda)$ , then  $U$  is convolution by an element of  $H_\lambda$  which is constant on conjugacy classes.*

Furthermore, if  $f \in H_\lambda$  is constant on conjugacy classes in  $\Gamma$ , and if convolution by  $f$  is a bounded operator on  $H_\lambda$ , then convolution by  $f$  is in  $CR(\tau_\lambda)$ .

By Lemma 3.2, to show that  $\tau_\lambda$  is a factor it suffices to show that the only element of  $H_\lambda$  which is constant on conjugacy classes of  $\Gamma$  is  $\delta_1$ ; for convolution by  $\delta_1$  is the identity map on  $H_\lambda$ , and so all elements of  $CR(\tau_\lambda)$  are multiples of the identity.

**THEOREM 3.3.** *If  $\lambda$  is a generic element of  $\widehat{\mathcal{L}}$ , then  $\tau_\lambda$  is a factor representation.*

*Proof.* Let  $\Gamma^{(i)}$  denote the  $i$ -th element of the rising central series of  $\Gamma$ , and let  $\gamma \in \Gamma$ ,  $\gamma \notin \Gamma^{(2)}$ . We will show that a function constant on the conjugacy class  $C(\gamma)$  cannot be in  $H_\lambda$  unless it is zero there.

Let  $I_\gamma$  denote the isotropy subgroup in  $\Gamma$  of the element  $\gamma z(\Gamma)$ , where  $\Gamma$  acts on  $\Gamma \backslash z(\Gamma)$  by conjugation. If  $\gamma \notin \Gamma^{(2)}$ , then  $I_\gamma$  is a proper subgroup of  $\Gamma$ . The cosets of  $z(\Gamma)$  intersected by the conjugacy class  $C(\gamma)$  are in bijective correspondence with  $\Gamma \backslash I_\gamma$ ; therefore, if  $I_\gamma$  were also a saturated subgroup, the number of cosets intersected by the conjugacy class of  $\gamma$  would be infinite, and so a function in  $H_\lambda$  which is constant on conjugacy classes would have to be zero on  $C(\gamma)$ .

Therefore we prove

**LEMMA 3.4.** *If  $\gamma \in \Gamma$ , and  $\gamma \notin \Gamma^{(2)}$ , then  $I_\gamma$  is a saturated subgroup of  $\Gamma$ .*

*Proof.* Let  $\gamma = \exp T$ ,  $x = \exp X$ , for  $T, X \in \mathcal{L}$ . If  $x^n \in I_\gamma$  for some  $n$ , then we have  $x^n \gamma x^{-n} \gamma^{-1} \in z(\Gamma)$ . By the Campbell-Baker-Hausdorff formula,

$$\begin{aligned} x^n \gamma x^{-n} \gamma^{-1} &= \exp nX \exp T \exp -nX \exp -T \\ &= \exp \left\{ n[X, T] + \frac{1}{2}(n^2[X, [X, T]] - n[T, [X, T]]) + \dots \right\} \\ &= \exp P(n) \in z(\Gamma), \end{aligned}$$

where successive terms involve brackets of increasing order. Clearly the polynomial  $P(n)$  is in  $z(\mathcal{L})$ , and since  $I_\gamma$  is a subgroup of  $\Gamma$ ,  $P(kn) \in z(\mathcal{L})$  as well for each  $k \in \mathbb{Z}$ . Therefore each individual

term of  $P(nk)$ , viewed as a polynomial in  $k$ , is in  $z(\mathcal{L})$ . Using the Campbell-Baker-Hausdorff formula again, we rewrite the above as

$$\begin{aligned} & \exp nkX(\exp T \exp -nkX \exp -T) \\ &= \exp nkX \exp -(\text{Ad}(\exp T)nkX) \\ &= \exp \left( nkX - \text{Ad}(\exp T)nkX - \frac{1}{2}[nkX, \text{Ad}(\exp T)nkX] - \dots \right), \end{aligned}$$

where successive terms involve powers of  $k$  which are higher than 2. It follows that  $n\{X - \text{Ad}(\exp T)X\} \in z(\mathcal{L})$ , and therefore, since  $z(\mathcal{L})$  is saturated,  $X - \text{Ad}(\exp T)X$  is in  $z(\mathcal{L})$ . We have

$$\text{Ad}(\exp T)X - X = [T, X] + \frac{1}{2}[T, [T, X]] + \frac{1}{6}[T, [T, [T, X]]] + \dots$$

and we wish to see that  $[T, X] \in z(\mathcal{L})$ . We expand  $[T, X]$  in terms of a strong malcev basis  $\{X_i\}$  through the lower central series of  $\mathcal{L}$ ; then we may write  $[T, X] = a_1X_1 + \dots + a_sX_s$ , where  $a_s \neq 0$ . Since  $[T, [T, X]]$  and subsequent terms belong to ideals which are further down in the lower central series,  $X_s$  will be absent from basis expansions of these terms and so we must have  $X_s \in z(\mathcal{L})$ , and therefore  $[T, X] \in z(\mathcal{L})$ . Therefore  $x\gamma x^{-1}\gamma^{-1} \in z(\Gamma)$ , and so  $I_\gamma$  is saturated.

Now we suppose that  $x \in \Gamma^{(2)}$ ,  $x \notin z(\Gamma)$ . The conjugacy class of  $x$  is thus a nontrivial subset of the coset  $xz(\Gamma)$ . Since  $\lambda$  is a generic character, the kernel of  $\tilde{\lambda}$  in  $z(\Gamma)$  is trivial; therefore a function in  $H_\lambda$ , with left  $z(\Gamma)$ -covariance, could not possibly be constant on  $C(x)$  unless it were zero on  $C(x)$ .

Therefore, we see that a function in  $H_\lambda$  which is constant on conjugacy classes is supported only upon  $z(\Gamma)$ , and hence must be a multiple of  $\delta_1$ . This completes the proof of Theorem 3.3.  $\square$

What follows is proved in Section 3 in [4], and applies here with  $I_\lambda = z(\Gamma)$ .

**THEOREM 3.5.** (Theorem 3, [4]). *If  $\lambda$  is generic,  $\tau_\lambda$  is a traceable factor representation, with trace (for  $f \in \mathcal{L}^1(\Gamma)$ )*

$$\Theta_\lambda(f) = \sum_{u \in z(\Gamma)} \lambda(u)f(u) = \langle f, \delta_1 \rangle,$$

where

$$\delta_1(u) = \begin{cases} \lambda(u) & \text{if } u \in z(\Gamma) \\ 0 & \text{if } u \notin z(\Gamma). \end{cases}$$

Furthermore, we have the orbital trace formula

$$\Theta_\lambda(f) = \int_{\widehat{\mathcal{O}}_\lambda} F^\wedge(\chi) d\chi,$$

where  $d\chi$  is the lift of Haar measure on  $z(\mathcal{L})^\perp$  to the closure  $\lambda + z(\mathcal{L})^\perp$  of  $\mathcal{O}_\lambda$ , and  $F = f \circ \exp \in L^1(\mathcal{L})$ .  $F^\wedge(\chi)$  denotes the usual Fourier transform of  $F$ .

These are the same traces R. Howe found as elements of dual cones of primitive ideals in the primitive ideal space of  $\Gamma$  (see Proposition 3 of [5]).

Now let  $F \in C_c(\mathcal{L})$ , so that  $f = F \circ \log \in C_c(\Gamma)$ . If  $\{X_1, \dots, X_n\}$  is our chosen basis, we can define an inclusion  $i: z(\widehat{\mathcal{L}}) \rightarrow \widehat{\mathcal{L}}$  as follows:  $\tilde{\lambda} \mapsto \lambda$  if

$$\lambda(a_1 X_1 + \dots + a_n X_n) = \tilde{\lambda}(a_1 X_1 + \dots + a_k X_k),$$

for all  $\vec{a} \in \mathbb{Z}^n$ . Then by Fourier inversion on the abelian group  $\mathcal{L}$ ,

$$f(e) = F(0) = \int_{\widehat{\mathcal{L}}} F^\wedge(\xi) d\xi = \int_{\widehat{z(\mathcal{L})}} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda,$$

where Haar measures are normalized so that their supports have measure 1.

We let  $d_\lambda(\chi)$  be the lift of normalized Haar measure on  $z(\mathcal{L})^\perp$  to  $\lambda + z(\mathcal{L})^\perp$ ; if  $\lambda$  is generic, then this is the measure on the closure of  $\mathcal{O}_\lambda$  which appears in the orbital trace formula for  $\tau_\lambda$ . Since *a.a.*  $\tilde{\lambda} \in z(\widehat{\mathcal{L}})$  are generic, the above becomes

$$\begin{aligned} \int_{\widehat{z(\mathcal{L})}} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda \\ = \int_{\widehat{z(\mathcal{L})}} \left\{ \int_{\widehat{\mathcal{O}}_\lambda} F^\wedge(\chi) d_\lambda(\chi) \right\} d\lambda = \int_{\widehat{z(\mathcal{L})}} \Theta_\lambda(f) d\lambda. \end{aligned}$$

We have proven

**THEOREM 3.6.** (Plancherel Formula). *Suppose  $f \in C_c(\Gamma)$ , and that for generic  $\lambda \in \widehat{\mathcal{L}}$ ,  $\Theta_\lambda$  is the trace associated with the factor*

representation  $\tau_\lambda$  induced from  $\tilde{\lambda}$  on  $z(\mathcal{L})$ . Then  $\tilde{\lambda} \mapsto \Theta_\lambda(f)$  is defined for a.a.  $\lambda$ , is integrable on  $\widehat{z(\mathcal{L})}$ , and we have

$$f(e) = \int_{\widehat{z(\mathcal{L})}} \Theta_\lambda(f) d\mu(\lambda),$$

where  $\mu$  is Haar measure on  $\widehat{z(\mathcal{L})}$ , normalized so that  $\mu(\widehat{z(\mathcal{L})}) = 1$ .

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