ON A PLANCHEREL FORMULA FOR CERTAIN DISCRETE, FINITELY GENERATED, TORSION-FREE NILPOTENT GROUPS

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We prove a Plancherel formula for elementarily exponential, discrete, finitely generated, torsion-free nilpotent groups.

1. Introduction. Let $\Gamma$ be a discrete, finitely generated, torsion-free, nilpotent Lie group. If $\Gamma^{(k)}$ denotes its descending central series, we will call $\Gamma$ $n$-step nilpotent if $\Gamma^{(n)} \neq \{1\}$ but $\Gamma^{(n+1)} = \{1\}$. Malcev has shown that any $\Gamma$ of this type may be embedded as a discrete cocompact subgroup of a simply connected, connected nilpotent Lie group (see [1], Chapter 1); thus we may utilize all that is known about uniform subgroups of these groups, which is summarized beautifully in ([3], Secs. 5.1 and 5.2).

This work extends the pioneering work of R. Howe on the representation theory of groups of this type, and uses much of the machinery he developed (see [5]). The techniques used to prove the Plancherel formula for $\Gamma$ are essentially those used in [4] to prove a similar Plancherel formula for discrete groups which are the rational points of a nilpotent Lie group. The corresponding result for $\Gamma$ follows easily once we observe that, for a certain type of character $\{\tilde{\lambda}\}$ in the Pontryagin dual of the center of $\Gamma$, the $\Gamma$-orbit of any extension $\lambda$ of $\tilde{\lambda}$ to a character on the Lie algebra $\mathcal{L}$ of $\Gamma$ is dense in the set $\lambda + z(\mathcal{L})^\perp \subseteq \hat{\mathcal{L}}$ (Proposition 2.5).

Let $\mathcal{L}_\mathbb{R}$ be a real finite-dimensional $r$-step nilpotent Lie algebra, and let $\mathcal{L} \subseteq \mathcal{L}_\mathbb{R}$ be a discrete additive subgroup of $\mathcal{L}_\mathbb{R}$. A calculation with the Campbell-Baker-Hausdorff formula shows that if $\mathcal{L}$ is

1. an additive discrete subgroup of $\mathcal{L}_\mathbb{R}$, not necessarily of cofinite volume, and
2. $\mathcal{L}$ satisfies $[\mathcal{L}, \mathcal{L}] \subseteq r!\mathcal{L}$,
then \( \Gamma = \exp L \) forms a discrete subgroup of the connected, simply connected nilpotent Lie group \( N = \exp L_{\mathbb{R}} \). If \( L \) satisfies condition 1, we will refer to \( L \) as a lattice. If \( L \) satisfies both conditions we will say that \( L \), and \( \Gamma = \exp L \), are elementarily exponentiable, or e.e. for short. If \( L \) is e.e., and \( i \) is an e.e. lattice contained in \( L \) which is closed under the bracket operation, we will call \( i \) an ideal of \( L \) if \( i \) is \( \text{Ad}^* (L) \)-invariant. Note that for \( i \) to be e.e., we must have \([i, i] \subseteq r! i\), where \( r \) is the length of \( L_{\mathbb{R}} \). We assume throughout this paper that the \( \Gamma \) under consideration are e.e..

We define the rising central series of ideals of \( L \) as follows; set \( L(0) = (0) \), and define \( \pi_n : L \rightarrow L/L(n-1) \) to be the natural quotient map for each \( n \geq 1 \). Then \( L(n) \) is defined to be the preimage under \( \pi_n \) of the center of \( L/L(n-1) \). Thus \( L(n) \) is an increasing sequence of ideals of \( L \), satisfying \( L(r) = L \), since \( L \) is \( r \)-step nilpotent.

We show in what follows that the \( L(n) \) are e.e..

**Lemma 1.1.** \( L(n) = L \cap (L_{\mathbb{R}})(n) \), for \( n \in 0, \ldots, r \).

**Proof.** For \( n = 0 \), the result is trivial.

Suppose now that \( L(k) = L \cap (L_{\mathbb{R}})(k) \). By definition, \( L(k+1) = \pi_k^{-1}(z(L/L(k))) \). We show:

(a). \( L(k+1) \subseteq L \cap (L_{\mathbb{R}})(k+1) \).

For if \( X \in L(k+1) \), \([X, Y] = Z \in L(k) \) for all \( Y \in L \). Let \( \overline{L} \) be the image of \( L \) in \( L_{\mathbb{R}} \setminus L_{\mathbb{R}}(k) \), and let \( \overline{X} \) be the image of \( X \). It follows from Theorems 5.1.4 and 5.2.3 in [3] that \( \overline{L} \) is uniform in \( L_{\mathbb{R}} \setminus L_{\mathbb{R}}(k) \). For \( X \in L(k+1) \), \( \overline{X} \) commutes with all \( \overline{Y} \in \overline{L} \). We apply Theorem 5.1.5 from [3] to see that \( \overline{X} \in z(L_{\mathbb{R}} \setminus L_{\mathbb{R}}(k)) \). It follows that \( X \in L_{\mathbb{R}}(k+1) \).

(b). \( L \cap L_{\mathbb{R}}(k+1) \subseteq L(k+1) \).

If \( X \in L \cap L_{\mathbb{R}}(k+1) \), then \([X, Y] = Z \in L \cap L_{\mathbb{R}}(k) = L(k) \). This completes the proof of the lemma. \( \square \)

It now follows easily that \( L(k) \) is e.e. for all \( k \); suppose \( Z = [X, Y] \in [L(k), L(k)] \). Then since \( L \) is e.e., \( Z = r! X \) for some \( X \in L \). By definition, \([L(k), L(k)] \subseteq [L(k), L] \subseteq L(k-1) \subseteq L(k) = L_{\mathbb{R}}(k) \cap L \); so \( Z \in L_{\mathbb{R}}(k) \), hence \( X \in L_{\mathbb{R}}(k) \). Thus \( X \in L(k) \), and so \( Z \in r! L(k) \). Therefore, \( L(k) \) is e.e..

We will refer to a subgroup \( \Gamma' \subseteq \Gamma \) (or a subalgebra \( L' \subseteq L \)) as saturated if \( x^n \in \Gamma' \) implies \( x \in \Gamma' \) (\( kX \in L' \) implies \( X \in L' \)).
I am grateful to the referee of this paper for suggestions which greatly improved the presentation of these results.

2. Generic coadjoint orbits. Let $\mathcal{L}$ be an $e.e.$ lattice in a real nilpotent Lie algebra $\mathcal{L}_\mathbb{R}$, so that $\Gamma = \exp(\mathcal{L}_\mathbb{R})$ is uniform in $N = \exp(\mathcal{L}_\mathbb{R})$. We assume throughout this section that a strong Malcev basis $\{X_1, X_2, ..., X_n\}$ for $\mathcal{L}_\mathbb{R}$ through its ascending central series has been chosen so that it satisfies:

1. The $\mathbb{Z}$-span of $\{X_1, ..., X_n\}$ is the lattice $\mathcal{L}$ in $\mathcal{L}_\mathbb{R}$:

2. The real span of $\{X_1, ..., X_i\}$ forms an ideal of $\mathcal{L}_\mathbb{R}$, and in particular the real span of $\{X_1, ..., X_k\}$ is the center of $\mathcal{L}_\mathbb{R}$ (we will also regard $\mathcal{L}_\mathbb{R}$ as having the inner product defined by setting $\langle X_i, X_j \rangle = \delta_{i,j}$);

3. $\Gamma = \exp(ZX_1) \exp(ZX_2) ... \exp(ZX_n)$. Thus we may coordinate $\Gamma$ as follows: $\gamma \leftrightarrow (x_1, ..., x_n) \in \mathbb{Z}^n$ whenever $\gamma = \exp x_1 X_1 \cdot ... \cdot \exp x_n X_n$.

That a basis for $\mathcal{L}_\mathbb{R}$ may be chosen satisfying all these conditions is shown in Section 5.1 of [3], and as part of the proof of Proposition 5.4.11 in [3].

We think of $\mathcal{L}$ as the Lie algebra of $\Gamma$; as an abelian group it is isomorphic to $\mathbb{Z}^n$ via the map

$$\Psi: \mathbb{Z}^n \rightarrow \mathcal{L}$$

$$(a_1, ..., a_n) \mapsto a_1 X_1 + ... + a_n X_n.$$ 

Thus the natural Pontryagin dual of $\mathcal{L}$ is $T^n \cong (\mathbb{R}/\mathbb{Z})^n$ via the map

$$\Phi: T^n \rightarrow \hat{\mathcal{L}}$$

$$(\bar{\lambda}_1, ..., \bar{\lambda}_n) \mapsto \lambda$$

where $\lambda(a_1, ..., a_n) = \exp(2\pi i (\lambda_1 a_1 + ... + \lambda_n a_n))$, for any choice $\{\lambda_i\}$ of representatives for the elements $\bar{\lambda}_i$ of $\mathbb{R}/\mathbb{Z}$.

Let $z(\mathcal{L})$ denote the center of the Lie algebra $\mathcal{L}$; if $z(\mathcal{L}_\mathbb{R})$ is the center of the real Lie algebra $\mathcal{L}_\mathbb{R}$, then $z(\mathcal{L}) = \mathcal{L} \cap z(\mathcal{L}_\mathbb{R})$ (Lemma 5.1.5 in [3]). Hence $z(\mathcal{L})$ is a saturated subalgebra of $\mathcal{L}$.

For a fixed $\lambda \in \hat{\mathcal{L}}$, let $i_\lambda$ be the largest ideal of $\mathcal{L}$ contained in the subalgebra $r_\lambda = \{X \in \mathcal{L}: \lambda[X, Y] = 1 \text{ for all } Y \in \mathcal{L}\}$. Then $R_\lambda = \exp(r_\lambda)$ is the isotropy subgroup of $\lambda$ under the $\Ad^*$-action
of \( \Gamma \); in particular it is shown in Lemma 2 of [5] that \( r_\chi \) is an e.e.lattice in \( \mathcal{L} \). Furthermore, \( i_\lambda = \bigcap_{\gamma \in \Gamma} \text{Ad}(\gamma)r_\chi \) is an e.e.ideal of \( \mathcal{L} \) (the intersection of e.e. subalgebras is e.e., and the \( \text{Ad}(\gamma)r_\chi \) are each e.e., being the images of an e.e. subalgebra under a Lie algebra automorphism). It follows that \( \exp(i_\lambda) = I_\chi \subseteq R_\lambda \) is a normal subgroup of \( \mathcal{L} \) which always contains \( \exp(z(\mathcal{L})) = z(\Gamma) \). Let \( r : \widehat{\mathcal{L}} \to \widehat{z(\mathcal{L})} \) send an element \( \lambda \in \widehat{\mathcal{L}} \) to its restriction to \( z(\mathcal{L}) \). In what follows, we will show that for all \( \lambda \in z(\mathcal{L}) \) except those in a set of Haar measure zero, the elements of \( r^{-1}(\lambda) \) satisfy \( i_\lambda = z(\mathcal{L}) \).

**Lemma 2.1.** Suppose \( \lambda = \Phi(\bar{\lambda}_1, ..., \bar{\lambda}_n) \) where \( \bar{\lambda}_i \in \mathbb{R}\setminus\mathbb{Z} \). Then \( \lambda \) has trivial kernel in \( \mathcal{L} \) if and only if for some choice of representatives \( \lambda_i \in \mathbb{R} \) of \( \bar{\lambda}_i \in \mathbb{R}\setminus\mathbb{Z} \), the set \( \{\lambda_1, ..., \lambda_n, 1\} \) is linearly independent over \( \mathbb{Q} \).

**Proof.** Suppose \( \lambda \) has trivial kernel, and suppose that for some elements \( q_1, ..., q_n, q \in \mathbb{Q} \), we have

\[
\lambda_1 q_1 + ... + \lambda_n q_n + q = 0
\]

for some choice of representatives \( \{\lambda_i\} \) for \( \{\bar{\lambda}_i\} \). After multiplication of this equation by some integer, we have \( \lambda_1 a_1 + ... + \lambda_n a_n + a = 0 \) for some integers \( \{a_i\} \). Then \( \lambda_1 a_1 + ... + \lambda_n a_n = -a \), so \( (a_1, ..., a_n) \in \text{kernel}(\lambda) \). Thus \( (a_1, ..., a_n) = 0 \). It follows immediately that \( q_i = 0 \) for each \( i = 1, ..., n \), and \( q = 0 \).

Conversely, suppose that \( \lambda \) has a nontrivial kernel, and that \( (a_1, ..., a_n) \neq 0 \) is an element of the kernel of \( \lambda \). Let \( \{\lambda_i\} \) be any choice of representatives for the elements \( \{\bar{\lambda}_i\} \) which determine \( \lambda \); then \( a_1 \lambda_1 + ... + a_n \lambda_n = k \) for some element \( k \in \mathbb{Z} \). The set \( \{\lambda_1, ..., \lambda_n, 1\} \) is thus a linearly dependent set over \( \mathbb{Q} \).

**Lemma 2.2.** Let \( i \) be an e.e.ideal of the lattice \( \mathcal{L} \). Let \( \mathcal{L}(n) \) be the rising central series of \( \mathcal{L} \). Suppose \( i \supseteq z(\mathcal{L}) \), \( i \neq z(\mathcal{L}) \). Then \( i \cap (\Gamma(2) - z(\Gamma)) \) is nonempty.

**Proof.** Since everything in sight is e.e., and since \( I = \exp(i) \) is normal, we prove the result on the group level. Let \( \overline{I} \) denote the image of \( I \) in \( \overline{G} = G\setminus z(\Gamma) \). Then since \( \overline{I} \) is a nontrivial normal subgroup of \( \overline{G} \), its intersection with the center of \( \overline{G} \) is nontrivial. It follows that \( \overline{I} \cap (\overline{\Gamma(2)} - z(\Gamma)) \) is nonempty, and so the result on the algebra level follows. \( \square \)
LEMMA 2.3. For $\lambda \in \hat{\mathcal{L}}$, $i_\lambda = z(\mathcal{L})$ if and only if there exists no element $X \in \mathcal{L}(2)$ such that $\text{ad}(X)$ maps $\mathcal{L}$ into the kernel of $\tilde{\lambda}$, where $\tilde{\lambda} = \lambda|_{z(\mathcal{L})}$.

Proof. Suppose $X \in \mathcal{L}(2) - z(\mathcal{L})$, and $\text{ad}(X)$ maps $\mathcal{L}$ into the kernel of $\tilde{\lambda}$. Take $K$ to be the additive subgroup in $\mathcal{L}$ generated by $X$ and the elements of $z(\mathcal{L})$. Then $K$ is an ideal of $\mathcal{L}$, since $[K, \mathcal{L}] \subseteq z(\mathcal{L})$, and clearly $K \subseteq r_\lambda$. Therefore $i_\lambda \neq z(\mathcal{L})$.

Conversely, suppose $i_\lambda \neq z(\mathcal{L})$. Then $i_\lambda$ properly contains $z(\mathcal{L})$. By the result of Lemma 2.2, we may choose $X \in \mathcal{L}(2) - z(\mathcal{L})$ such that $X \in i_\lambda$; then $\text{ad}(X)$ maps $\mathcal{L}$ into the kernel of $\tilde{\lambda}$. This completes the proof of Lemma 2.3. \(\square\)

Throughout the rest of this paper, $\lambda$ will denote an element of $\hat{\mathcal{L}}$, and $\tilde{\lambda}$ will denote its restriction to $z(\mathcal{L})$.

COROLLARY. If $i_\lambda = z(\mathcal{L})$, then for all $\phi \in r^{-1}(\tilde{\lambda})$, $i_\phi = z(\mathcal{L})$. Therefore, if $\ker \tilde{\lambda} \subseteq z(\mathcal{L})$ is trivial, $i_\lambda = z(\mathcal{L})$.

PROPOSITION 2.4. For a set $\{\tilde{\lambda}\} \subseteq z(\mathcal{L})$ of full Haar measure in $z(\mathcal{L})$, $\ker \tilde{\lambda}$ is trivial.

Proof. Suppose $\tilde{\lambda}$ corresponds to the element $(\tilde{\lambda}_1, ..., \tilde{\lambda}_k) \in T^n$, and that for some set of representatives in $\mathbb{R}$ of the $\tilde{\lambda}_i$, the set $\{\lambda_1, ..., \lambda_k, 1\}$ is linearly dependent over $\mathbb{Q}$. Then for some set of integers $a_1, ..., a_k, a \in \mathbb{Z}$, not all zero, we have that $a_1 \lambda_1 + ... + a_k \lambda_k = a$. If we choose some other set of representatives for the $\tilde{\lambda}_i$, the previous expression changes only by an integral constant.

It follows that the preimage in $\mathbb{R}^k$ of the set of elements $\{\tilde{\lambda}_i\}$ satisfying this linear dependence condition consists of the union of hyperplanes of the form

$$a_1 x_1 + a_2 x_2 + ... + a_k x_k = a,$$

where $a_i, a$ vary over the elements of $\mathbb{Z}$. These are of measure zero in $\mathbb{R}^k$ individually, hence their (countable) union is of measure zero.

The corresponding set in $T^n$ is therefore of measure zero, and thus the elements $(\tilde{\lambda}_1, ..., \tilde{\lambda}_k) \in T^n$ which satisfy the linear independence condition of Lemma 2.1 are of full measure in $T^n$. This completes the proof of Proposition 2.4. \(\square\)
We finish this section by proving

**Proposition 2.5.** If \( \ker \tilde{\lambda} \) is trivial, then the closure of \( \text{Ad}^*(\Gamma)\lambda \) is \( \lambda + z(\mathcal{L}) \perp \).

Suppose that \( \gamma \in \Gamma \), and \( \lambda \) has coordinates \( (\lambda_1, ..., \lambda_n) \in T^n \) via the map \( \Phi \). Then \( \text{Ad}^*(\gamma) = (\lambda_1, ..., \lambda_k, p_{k+1}(\gamma : \lambda), ..., p_n(\gamma : \lambda)) \), where each \( p_i(\gamma : \lambda) \) is a polynomial in the coordinates of \( \gamma \) (with respect to the map \( \Psi \)), with coefficients from the \( \mathbb{Q} \)-span of \( \{\lambda_i, ..., \lambda_n\} \).

In \( \mathcal{L} \), \( z(\mathcal{L}) \perp \) consists of all elements of the form \( (0, ..., 0, \lambda^+_{k+1}, ..., \lambda_n) \), for \( \lambda_i \in \mathbb{R} \setminus \mathbb{Z} \).

We wish to show that if \( \lambda_1, ..., \lambda_k \) satisfy the equivalent conditions of Lemma 2.1 (where \( z(\mathcal{L}) \) plays the role of \( \mathcal{L} \)), then the set \( \{\text{Ad}^*(\gamma)\lambda - \lambda : \gamma \in \Gamma\} \) is dense in \( z(\mathcal{L}) \perp \). In what follows, \( \lambda \) of this type will be called “generic”, as will coadjoint orbits of the form \( \mathcal{O}_\lambda = \lambda + z(\mathcal{L}) \perp \) where \( \lambda \) is generic.

We may regard the polynomials \( p_i(\gamma : \lambda) \) as polynomials in \( \{x_1, ..., x_n\} \), \( x_i \in \mathbb{Z} \), by identifying \( \gamma \) with \( (x_1, ..., x_n) \) via \( \Psi \).

**Lemma 2.6.** (H. Weyl, [2]). Suppose that \( \{p_i\}_{i=1}^n \) is a set of polynomials in one integer variable, with coefficients in \( \mathbb{R} \setminus \mathbb{Z} \). If for each set of integers \( a_1, ..., a_n \), not all zero, the polynomial \( a_1p_1 + ... + a_np_n \) has an irrational coefficient, then the set of points \( \{(p_1(k), ..., p_n(k)) : k \in \mathbb{Z}\} \) is dense in \( (\mathbb{R} \setminus \mathbb{Z})^n \).

Now let \( T = (t_1, ..., t_s) \in N^s \), and let \( X^T = x_1^{t_1}...x_s^{t_s} \) be a multinomial in \( s \) integer variables.

**Lemma 2.7.** Suppose \( \{X^{T_i}\}_{i=1}^r \) is a set of distinct multinomials in \( s \) integer variables. Then there is \( \phi: \mathbb{Z} \rightarrow \mathbb{Z}^s \) so that \( X^{T_i} \circ \phi \) are monomials and are distinct.

**Proof.** We put a lexicographic order on \( N^s \) as follows: let \( i \in 1, ..., s \) be the greatest integer with \( m_i \neq m'_i \); then \( (m_1, ..., m_s) > (m'_1, ..., m'_s) \) if \( m_i > m'_i \). A simple induction argument shows that for the finite set \( \{T_i\} \subseteq N^s \) there is \( N \in N^s \), \( N = (N_1, ..., N_s) \), so that if \( T_i > T_j \) in the ordering on \( N^s \), then \( N \cdot T_i > N \cdot T_j \) \( (N \cdot T \) denotes the usual dot product).

Now define \( \phi(x) = (x^{N_1}, ..., x^{N_s}) \); then we have \( X^{T_i} \circ \phi = x^{T_i \cdot N} \), and so the monomials \( X^{T_i} \circ \phi \) remain distinct. \( \square \)
Lemma 2.8. Suppose \( \{ p_i \}_{i=1}^n \) is a set of polynomials in \( s \) integer variables, with coefficients in \( \mathbb{R} \setminus \mathbb{Z} \). If for all integers \( a_1, \ldots, a_n \), not all zero, \( p = a_1 p_1 + \ldots + a_n p_n \) has an irrational coefficient, then the image of \( \mathbb{Z}^s \) under \( (p_1, \ldots, p_n) \) is dense in \( (\mathbb{R} \setminus \mathbb{Z})^n \).

Proof. Let \( \{ X^T_i \} \) be the set of monomials which appear in the polynomials \( p_i \), and let \( \phi \) be as in Lemma 2.7. We note that the monomials in \( a_1 p_1 + \ldots + a_n p_n \) are a subset of the \( \{ X^T_i \} \), so that they remain distinct if composed with \( \phi \); and so for all \( a_1, \ldots, a_n \), not all zero, \( a_1 p_1 \circ \phi + \ldots + a_n p_n \circ \phi \) has an irrational coefficient. We invoke Lemma 2.6, and the result follows.

Thus we will have proven Proposition 2.5 if we can show that whenever \( \lambda \) is generic, the polynomial

\[
P(x: \lambda) = a_{k+1} p_{k+1}(x: \lambda) + \ldots + a_n p_n(x: \lambda)
\]

(where \( a_{k+1}, \ldots, a_n \) are integers, not all zero) has an irrational coefficient.

Assume \( \lambda \) is generic. We begin by writing \( \text{Ad}^*(\gamma) = \text{Ad}^*(x_1, \ldots, x_n) \) as a matrix with respect to the coordinates \( (\lambda_1, \ldots, \lambda_n) \) of \( \lambda \) given by the map \( \Phi \). The condition \( [\mathcal{L}, \mathcal{L}] \subseteq r! \mathcal{L} \) implies that the entries of the matrix \( \text{Ad}^*(x_1, \ldots, x_n) \) are elements of \( \mathbb{Z}[x_1, \ldots, x_n] \).

Therefore \( \text{Ad}^*(\gamma)(\lambda_1, \ldots, \lambda_k, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_k, p_{k+1}(\gamma: \lambda), \ldots, p_n(\gamma: \lambda)) \). \( \text{Ad}^*(\gamma) \) is given by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
p_{k+1,1} & p_{k+1,2} & p_{k+1,3} & \cdots & p_{k+1,k} & 1 & 0 & \ldots & 0 \\
p_{k+2,1} & p_{k+2,2} & p_{k+2,3} & \cdots & p_{k+2,k} & p_{k+2,k+1} & 1 & \ldots & 0 \\
p_{k+3,1} & p_{k+3,2} & p_{k+3,3} & \cdots & p_{k+3,k} & p_{k+3,k+1} & p_{k+3,k+2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_{n,1} & p_{n,2} & p_{n,3} & \cdots & p_{n,k} & p_{n,k=1} & p_{n,k+2} & \cdots & 1
\end{pmatrix}
\]

with each \( p_{i,j} \in \mathbb{Z}[x_1, \ldots, x_n] \).

We break the problem down as follows.

1. \( p_i(\gamma, \lambda) = p_{i,1}(\gamma) \lambda_1 + p_{i,2}(\gamma) \lambda_2 + \ldots + p_{i,k}(\gamma) \lambda_k + \ldots + p_{i,i-1}(\gamma) \lambda_{i-1} + \lambda_i \), for \( i = k+1, \ldots, n \).
2. Let \( \{a_{k+1}, \ldots, a_n\} \) be any set of integers, not all zero. Define

\[
p(\gamma, \lambda) = a_{k+1}p_{k+1}(\gamma, \lambda) + \ldots + a_np_n(\gamma, \lambda)
\]

\[
= \sum_{i=k+1}^{n} a_i \left\{ \sum_{i=1}^{i-1} \lambda_{t}p_{i,t}(\gamma) + \lambda_i \right\}
\]

\[
= \sum_{t=1}^{k} \lambda_t \left\{ \sum_{i=k+1}^{n} a_i p_{i,t} \right\} + \sum_{t=k+1}^{n} \lambda_t \left\{ \sum_{i=t}^{n} a_i p_{i,t} \right\}.
\]

We note that \( p_{i,i} = 1 \) for all \( i \). Let \( A = (0, \ldots, 0, a_{k+1}, \ldots, a_n) \in \mathcal{L} \). If we write \( \text{Ad}(\gamma^{-1})A = (\Phi_1, \ldots, \Phi_n) \), then we have \( p(\gamma, \lambda) = \sum_{i=1}^{n} \lambda_i \Phi_i \). We write \( p(\gamma, \lambda) = p_c(\gamma, \lambda) + p_n(\gamma, \lambda) \) where \( p_c = \sum_{i=1}^{k} \lambda_i \Phi_i \) and \( p_n = \sum_{i=k+1}^{n} \lambda_i \Phi_i \).

We first show that if \( \{\lambda_1, \ldots, \lambda_k\} \) are as in lemma 2.1, then the polynomial \( p_c \) has a nontrivial irrational coefficient. Since the \( \Phi_i \), \( i = 1, \ldots, k \), consist of polynomials with integral coefficients and \( \{\lambda_1, \ldots, \lambda_k\} \) are linearly independent over \( \mathbb{Q} \), all the coefficients of the polynomial \( p_c \) are irrational. Therefore, we need only show that \( p_c \) is not a constant polynomial, or equivalently that some \( \Phi_i \), \( i = 1, \ldots, k \), is not constant.

These are the central components of the vector giving \( \text{Ad}(\gamma^{-1})A \), where not all of the entries \( a_i \) are zero. Therefore the orbit described is that of a non-central element of \( \mathcal{L} \), and so it will suffice to show that for any non-central element \( T \) of \( \mathcal{L} \), \( \text{Ad}(\gamma)(T) \) has nondegenerate orthogonal projection onto the center of \( \mathcal{L}_{\mathbb{R}} \).

**Lemma 2.9.** Let \( \{X_1, \ldots, X_n\} \) be as before, and let \( P_z \) be projection onto the real span of \( \{X_1, \ldots, X_k\} \) with \( \mathbb{R} - \text{span} \{X_{k+1}, \ldots, X_n\} \) as kernel. Then if \( T \) is a non-central element of \( \mathcal{L}_{\mathbb{R}} \), \( P_z(\text{Ad}(\gamma)T - T) \) is not identically zero.

**Proof.** It suffices to show that \( P_z(\text{Ad}(N)T - T) \) is not identically zero, where \( N = \exp(\mathcal{L}_{\mathbb{R}}) \). Let \( \{t_1, \ldots, t_s\} \) be a subset of \( \mathbb{R} \) and \( \{Y_1, \ldots, Y_s\} \) be a subset of \( \mathcal{L}_{\mathbb{R}} \).

If we use the formula \( \text{Ad}(\exp Y)T = \exp(\text{ad}(Y)T) \) to write \( \text{Ad}(\exp t_1 Y_1 \cdot \exp t_2 Y_2 \cdot \ldots \cdot \exp t_s Y_s)T - T \) as a polynomial expression in \( \{t_1, \ldots, t_s\} \) with coefficients in \( \mathcal{L}_{\mathbb{R}} \), we see that the coefficient of the monomial \( t_1 t_2 \ldots t_s \) is a rational multiple of \([Y_1, [Y_2, \ldots, [Y_s, T]] \ldots]\). Now suppose that \( P_z(\text{Ad}(N)T - T) = 0 \); then we
must have \([Y_1, [Y_2, \ldots [Y_s, T]\ldots]] = 0\). However, since \(T\) is not central, there exists a sequence of elements \(\{Y_1, \ldots, Y_s\} \subseteq \mathcal{L}_\mathbb{R}\) such that \(W = [Y_1, [Y_2, \ldots [Y_s, T]\ldots]] \in z(\mathcal{L})\), \(W \neq 0\). Then \(P_z(W) = W\) is nonzero, giving a contradiction, and completing the proof of Lemma 2.9.

**Lemma 2.10.** The polynomial \(p(\gamma, \lambda)\) has an irrational coefficient (of a nontrivial monomial) whenever \(\lambda\) is generic.

**Proof.** Suppose that \(p(\gamma, \lambda)\) has entirely rational coefficients. Then since \(p_c(\gamma, \lambda)\) is nontrivial, \(p_n(\gamma, \lambda) = p_r(\gamma, \lambda) - p_c(\gamma, \lambda)\), where \(p_r(\gamma, \lambda)\) has rational coefficients. Since we have \(p_n(\gamma, \lambda) = \sum_{i=k+1}^n \lambda_i \Phi_i\), where the \(\Phi_i\) have integer coefficients, we must have some subset \(\{\lambda_{\sigma(t)}\}_{t=1}^l\) of coefficients which are from the \(\mathbb{Q}\)-span of \(\{\lambda_1, \ldots, \lambda_k\}\), and the rest must be rational. Thus we have

\[
\sum_{i=1}^k \lambda_i \Phi_i + \sum_{t=1}^l \lambda_{\sigma(t)} \Phi_{\sigma(t)} = K,
\]

where \(K\) is some real constant. However, the \(\lambda_{\sigma(t)}\), \(t = 1, \ldots, l\), satisfy \(\lambda_{\sigma(t)} = \sum_{i=1}^k q_{t,i} \lambda_i\), where the \(q_{t,i}\) are rational. Therefore we may write the equation above as

\[
\sum_{i=1}^k \lambda_i \Phi_i + \sum_{t=1}^l \left\{ \sum_{s=1}^k q_{s,t} \lambda_s \right\} \Phi_{\sigma(t)} = K.
\]

Since the \(\lambda_i\) are linearly independent over \(\mathbb{Q}\), we must have, for each \(i\), that

\[
\Phi_i + \sum_{t=1}^l q_{t,i} \Phi_{\sigma(t)} = K_i,
\]

for some real constant \(K_i\). Let \(v_i = X_i + \sum_{t=1}^l q_{i,t} X_{\sigma(t)} \in \mathcal{L}_\mathbb{R}\), for \(i = 1, \ldots, k\). The above implies that the function \(\gamma \mapsto <v_i, \text{Ad}(\gamma)A>\) is a constant function, and so that the projection of \(\text{Ad}(N)A\) onto the subspace \(W = \mathbb{R}\)-span \(\{v_i\}\) is degenerate. We will show that this is impossible, giving a contradiction.

**Lemma 2.11.** Let \(O\) be a non-trivial \(\text{Ad}\)-orbit of \(N\). Then \(P_W(O)\) is nondegenerate, i.e., \(P_W(\text{Ad}(\Gamma)A - A)\) is nonzero.

**Proof.** Let \(\{Y_1, \ldots, Y_s\}\) be as in the proof of Lemma 2.9., such that \([Y_1, \ldots [Y_{s-1}, [Y_s, A]\ldots]]\) is a nonzero element of \(z(\mathcal{L})\). Then if
we express \( \text{Ad}(\exp t_1 Y_1 \cdot \ldots \cdot \exp t_s Y_s)A \) as a polynomial in \( t_1, \ldots, t_s \) taking on values in \( L_\bb{R} \), we see that the monomial \( t_1 \ldots t_s \) appears as a coefficient only of central elements \( \{X_1, \ldots, X_k\} \). Let \( i \leq k \) be such that \( [Y_1, \ldots, [Y_s, A]] \ldots ] \) has a nontrivial \( X_i \)-component. Then \( < v_i, \text{Ad}(\exp t_1 Y_1 \ldots \exp t_s Y_s)A > \) is a polynomial with a nonzero coefficient of \( t_1 \ldots t_s \), so it is nontrivial, and so \( P_W(O) \) is nondegenerate.

This completes the proof of Lemma 2.10; taken together with Lemma 2.8, this completes the proof of Proposition 2.5. \( \square \)

3. Traceable factor representations associated with generic coadjoint orbits. Suppose now that \( \lambda \) is a generic element of \( \hat{\mathcal{L}} \), with \( \text{Ad}^*(\Gamma) \)-orbit closure \( O_\lambda \). We define \( \tau_\lambda \) to be the representation of \( \Gamma \) induced from the restriction of \( \lambda \) to \( z(\mathcal{L}) \), regarded as a character on \( z(\Gamma) \) (this is possible because \( z(\mathcal{L}) = z(\mathcal{L}_\bb{R}) \cap \mathcal{L} \) by Theorem 5.1.5 in [3], and because \( \exp z(\mathcal{L}) = z(\Gamma) \); \( \tau_\lambda \) is defined on the Hilbert space

\[
H_\lambda = \left\{ f: \Gamma \to \mathbb{C} \mid \int_{\Gamma \setminus z(\Gamma)} |f|^2 dx < \infty \right\},
\]

with inner product \( < f, g > = \int_{\Gamma \setminus z(\Gamma)} f \cdot \overline{g} \ dx \). Since elements in \( O_\lambda \) agree on the center of \( \mathcal{L} \), \( \tau_\lambda \) depends only upon the coadjoint orbit closure \( O_\lambda \). Recall that \( \tau_\lambda \) is a factor representation if \( CR(\tau_\lambda) = \tau_\lambda(\Gamma)' \cap \tau_\lambda(\Gamma)'' = \mathcal{C}I \) (in general, \( A' \) denotes the commutator of the set \( A \)). In what follows, we will show that if \( O_\lambda \) is a generic coadjoint orbit closure in \( \hat{\mathcal{L}} \), then \( \tau_\lambda \) is a factor representation.

**Lemma 3.1.** (Lemma 1, [4]). Let \( U \in \tau_\lambda(\Gamma)' \), and let \( H_\lambda \) be the Mackey space as defined above for the representation \( \tau_\lambda \). Then \( U \) is entirely determined by its value on the function \( \delta_1 \in H_\lambda \) defined by

\[
\delta_1(\gamma) = \begin{cases} 
0 & \gamma \notin z(\Gamma) \\
\lambda(\gamma) & \gamma \in z(\Gamma) 
\end{cases}.
\]

**Lemma 3.2.** (Lemma 2, [4]). If \( U \in CR(\tau_\lambda) \), then \( U \) is convolution by an element of \( H_\lambda \) which is constant on conjugacy classes.
Furthermore, if \( f \in H_\lambda \) is constant on conjugacy classes in \( \Gamma \), and if convolution by \( f \) is a bounded operator on \( H_\lambda \), then convolution by \( f \) is in \( CR(\tau_\lambda) \).

By Lemma 3.2, to show that \( \tau_\lambda \) is a factor it suffices to show that the only element of \( H_\lambda \) which is constant on conjugacy classes of \( \Gamma \) is \( \delta_1 \); for convolution by \( \delta_1 \) is the identity map on \( H_\lambda \), and so all elements of \( CR(\tau_\lambda) \) are multiples of the identity.

**Theorem 3.3.** If \( \lambda \) is a generic element of \( \hat{\mathcal{L}} \), then \( \tau_\lambda \) is a factor representation.

**Proof.** Let \( \Gamma^{(i)} \) denote the \( i \)-th element of the rising central series of \( \Gamma \), and let \( \gamma \in \Gamma, \gamma \notin \Gamma^{(2)} \). We will show that a function constant on the conjugacy class \( C(\gamma) \) cannot be in \( H_\lambda \) unless it is zero there.

Let \( I_\gamma \) denote the isotropy subgroup in \( \Gamma \) of the element \( \gamma z(\Gamma) \), where \( \Gamma \) acts on \( \Gamma \backslash z(\Gamma) \) by conjugation. If \( \gamma \notin \Gamma^{(2)} \), then \( I_\gamma \) is a proper subgroup of \( \Gamma \). The cosets of \( z(\Gamma) \) intersected by the conjugacy class \( C(\gamma) \) are in bijective correspondence with \( \Gamma \backslash I_\gamma \); therefore, if \( I_\gamma \) were also a saturated subgroup, the number of cosets intersected by the conjugacy class of \( \gamma \) would be infinite, and so a function in \( H_\lambda \) which is constant on conjugacy classes would have to be zero on \( C(\gamma) \).

Therefore we prove

**Lemma 3.4.** If \( \gamma \in \Gamma, \text{ and } \gamma \notin \Gamma^{(2)} \), then \( I_\gamma \) is a saturated subgroup of \( \Gamma \).

**Proof.** Let \( \gamma = \exp T, x = \exp X, \text{ for } T, X \in \mathcal{L} \). If \( x^n \in I_\gamma \) for some \( n \), then we have \( x^n \gamma x^{-n} \gamma^{-1} \in z(\Gamma) \). By the Campbell-Baker-Hausdorff formula,

\[
x^n \gamma x^{-n} \gamma^{-1} = \exp nX \exp T \exp -nX \exp -T
= \exp \left\{ n[X, T] + \frac{1}{2} (n^2[X, [X, T]] - n[T, [X, T]]) + \ldots \right\}
= \exp P(n) \in z(\Gamma),
\]

where successive terms involve brackets of increasing order. Clearly the polynomial \( P(n) \) is in \( z(\mathcal{L}) \), and since \( I_\gamma \) is a subgroup of \( \Gamma \), \( P(kn) \in z(\mathcal{L}) \) as well for each \( k \in \mathbb{Z} \). Therefore each individual
term of $P(nk)$, viewed as a polynomial in $k$, is in $z(\mathcal{L})$. Using the Campbell-Baker-Hausdorff formula again, we rewrite the above as

$$
\exp nkX (\exp T \exp -nkX \exp -T) = \exp nkX \exp -(\text{Ad}(\exp T)nkX)
$$

$$
= \exp \left( nkX - \text{Ad}(\exp T)nkX - \frac{1}{2} [nkX, \text{Ad}(\exp T)nkX] - \ldots \right),
$$

where successive terms involve powers of $k$ which are higher than 2. It follows that $n\{X - \text{Ad}(\exp T)X\} \in z(\mathcal{L})$, and therefore, since $z(\mathcal{L})$ is saturated, $X - \text{Ad}(\exp T)X$ is in $z(\mathcal{L})$. We have

$$
\text{Ad}(\exp T)X - X = [T, X] + \frac{1}{2} [T, [T, X]] + \frac{1}{6} [T, [T, [T, X]]] + \ldots
$$

and we wish to see that $[T, X] \in z(\mathcal{L})$. We expand $[T, X]$ in terms of a strong malcev basis $\{X_i\}$ through the lower central series of $\mathcal{L}$; then we may write $[T, X] = a_1X_1 + \ldots + a_sX_s$, where $a_s \neq 0$. Since $[T, [T, X]]$ and subsequent terms belong to ideals which are further down in the lower central series, $X_s$ will be absent from basis expansions of these terms and so we must have $X_s \in z(\mathcal{L})$, and therefore $[T, X] \in z(\mathcal{L})$. Therefore $x\gamma x^{-1}\gamma^{-1} \in z(\Gamma)$, and so $I_\gamma$ is saturated.

Now we suppose that $x \in \Gamma^{(2)}$, $x \notin z(\Gamma)$. The conjugacy class of $x$ is thus a nontrivial subset of the coset $xz(\Gamma)$. Since $\lambda$ is a generic character, the kernel of $\tilde{\lambda}$ in $z(\Gamma)$ is trivial; therefore a function in $H_\lambda$, with left $z(\Gamma)$-covariance, could not possibly be constant on $C(x)$ unless it were zero on $C(x)$.

Therefore, we see that a function in $H_\lambda$ which is constant on conjugacy classes is supported only upon $z(\Gamma)$, and hence must be a multiple of $\delta_1$. This completes the proof of Theorem 3.3. \hfill \Box

What follows is proved in Section 3 in [4], and applies here with $I_\lambda = z(\Gamma)$.

**Theorem 3.5.** (Theorem 3, [4]). If $\lambda$ is generic, $\tau_\lambda$ is a traceable factor representation, with trace (for $f \in \mathcal{L}^1(\Gamma)$)

$$
\Theta_\lambda(f) = \sum_{u \in z(\Gamma)} \lambda(u)f(u) = < f, \delta_1 >,
$$
where
\[ \delta_1(u) = \begin{cases} \lambda(u) & \text{if } u \in z(\Gamma) \\ 0 & \text{if } u \notin z(\Gamma). \end{cases} \]

Furthermore, we have the orbital trace formula
\[ \Theta_\lambda(f) = \int_{\mathcal{O}_\lambda} F^\wedge(\chi) d\chi, \]
where \( d\chi \) is the lift of Haar measure on \( z(\mathcal{L})^\perp \) to the closure \( \lambda + z(\mathcal{L})^\perp \) of \( \mathcal{O}_\lambda \), and \( F = f \circ \exp \in L^1(\mathcal{L}) \). \( F^\wedge(\chi) \) denotes the usual Fourier transform of \( F \).

These are the same traces R. Howe found as elements of dual cones of primitive ideals in the primitive ideal space of \( \Gamma \) (see Proposition 3 of [5]).

Now let \( F \in C_c(\mathcal{L}) \), so that \( f = F \circ \log \in C_c(\Gamma) \). If \( \{X_1, ..., X_n\} \) is our chosen basis, we can define an inclusion \( i: z(\mathcal{L}) \to \hat{\mathcal{L}} \) as follows: \( \tilde{\lambda} \mapsto \lambda \) if
\[ \lambda(a_1 X_1 + ... + a_n X_n) = \tilde{\lambda}(a_1 X_1 + ... + a_k X_k), \]
for all \( \tilde{a} \in \mathbb{Z}^n \). Then by Fourier inversion on the abelian group \( \mathcal{L} \),
\[ f(e) = F(0) = \int_{\hat{\mathcal{L}}} F^\wedge(\xi) d\xi = \int_{z(\mathcal{L})} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda, \]
where Haar measures are normalized so that their supports have measure 1.

We let \( d_\lambda(\chi) \) be the lift of normalized Haar measure on \( z(\mathcal{L})^\perp \) to \( \lambda + z(\mathcal{L})^\perp \); if \( \lambda \) is generic, then this is the measure on the closure of \( \mathcal{O}_\lambda \) which appears in the orbital trace formula for \( \tau_\lambda \). Since a.a. \( \tilde{\lambda} \in z(\mathcal{L}) \) are generic, the above becomes
\[ \int_{z(\mathcal{L})} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda \]
\[ = \int_{z(\mathcal{L})} \left\{ \int_{\mathcal{O}_\lambda} F^\wedge(\chi) d\lambda(\chi) \right\} d\lambda = \int_{z(\mathcal{L})} \Theta_\lambda(f) d\lambda. \]

We have proven

**Theorem 3.6.** (Plancherel Formula). Suppose \( f \in C_c(\Gamma) \), and that for generic \( \lambda \in \hat{\mathcal{L}} \), \( \Theta_\lambda \) is the trace associated with the factor
representation \( \tau_\lambda \) induced from \( \tilde{\lambda} \) on \( z(\mathcal{L}) \). Then \( \tilde{\lambda} \mapsto \Theta_\lambda(f) \) is defined for a.a. \( \lambda \), is integrable on \( z(\mathcal{L}) \), and we have

\[
f(e) = \int_{z(\mathcal{L})} \Theta_\lambda(f) d\mu(\lambda),
\]

where \( \mu \) is Haar measure on \( z(\mathcal{L}) \), normalized so that \( \mu(z(\mathcal{L})) = 1 \).

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