

*Pacific  
Journal of  
Mathematics*

**GENERATION OF INTEGRAL ORTHOGONAL GROUPS OVER  
DYADIC LOCAL FIELDS**

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Volume 167 No. 2

February 1995

## GENERATION OF INTEGRAL ORTHOGONAL GROUPS OVER DYADIC LOCAL FIELDS

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**In this paper, we introduce the minimal norm Jordan splittings of quadratic lattices over dyadic local fields. By using these splittings, we prove that orthogonal groups over dyadic local fields are generated by the symmetries and the Eichler transformations of the lattices unless the spinor norms of these groups are entire multiplicative groups of underlying fields.**

The generation problem of integral orthogonal groups over local fields was first studied by Kneser (see references in [K]). He obtained that orthogonal groups of lattices over nondyadic local fields are generated by the symmetries of the lattices. This can be regarded as an analogy of Cartan- Dieudonne's theorem about generation of orthogonal groups on spaces (see [L] or [O]). In [OP1] and [OP2], O'Meara and Pollak studied these integral orthogonal groups over dyadic local fields and obtained that these groups are generated by the symmetries and the Eichler transformations of the lattices when the lattices are modular or 2 is unramified. One of the applications of these results is to study the spinor genus theory of integral quadratic forms over number fields, which essentially depends on the knowledge of the spinor norms of these integral orthogonal groups at each local completion. By using these good generators, Kneser [K] was able to determine the spinor norms of integral orthogonal groups over nondyadic local fields explicitly and Hsia [H] determined the spinor norms of integral orthogonal groups over dyadic local fields explicitly when the lattices are modular, and Earnest and Hsia [EH] computed the spinor norms of integral orthogonal groups explicitly over the dyadic fields in which 2 is unramified.

In this paper we will extend O'Meara-Pollak's results to arbitrary dyadic local fields. More precisely, our main result (Theorem 2.1) shows that orthogonal groups of the lattices are still generated by the symmetries and the Eichler transformations of the lattices

unless the integral spinor norms of these groups are the entire multiplicative groups of underlying fields. Therefore, for the purpose of determining the integral spinor norms over arbitrary dyadic local fields, we have solved this generation problem. Some results will also be used in [HSX] which gives a full answer to representations by spinor genera over number fields. Our approach is first to modify the local structures by introducing the notion of “minimal norm Jordan splittings” over a dyadic local field and then to combine the techniques from [OP2] and [X] to obtain the desired results.

NOTATION AND TERMINOLOGY. All unexplained notation and terminology will be from [O], [X] and [OP2]. In particular,  $F$  denotes a dyadic local field,  $\mathfrak{o}$  the ring of integers in  $F$ ,  $\mathfrak{p}$  the maximal ideal of  $\mathfrak{o}$ ,  $U$  the group of units in  $\mathfrak{o}$ ,  $e = \text{ord } 2$  the ramification index of 2 in  $F$ .  $\pi$  a fixed prime element in  $F$ ,  $D(\cdot, \cdot)$  the quadratic defect function,  $\Delta$  a fixed unit of quadratic defect  $4\mathfrak{o}$ ,  $V$  a regular quadratic space over  $F$  associated symmetric bilinear form  $B(x, y)$ ,  $L$  a lattice on  $V$ ,  $dL$  the determinant of  $L$ ,  $sL$  the scale of  $L$ ,  $nL$  the norm of  $L$ ,  $O(L)$  the integral orthogonal group of  $L$ ,  $X(L)$  the subgroup generated by the symmetries and Eichler transformations of  $L$ , and  $\theta(\cdot, \cdot)$  the spinor norm function. We use  $[a, b, \dots]$  to denote spaces.

**1. Minimal norm Jordan splittings.** Since the Jordan splittings of lattices in dyadic local fields are not unique, O. T. O’Meara in [O1] obtained a saturated Jordan splitting of which the norm of every Jordan component is maximal. In this section, we establish a Jordan splitting of which the norm of every Jordan component is minimal and hyperbolic components are as much as possible. This kind of splitting plays important role in solving the generation problem of  $O(L)$ . We call  $\pi^r A(0, 0)$  a hyperbolic plane and  $H$  is denoted as an orthogonal sum of hyperbolic planes (which may sometimes have different scales).

LEMMA 1.1. *Suppose  $L = L_1 \perp L_2$  where  $L_1$  is unimodular with  $\text{ord } nL_1 = u_1$  and  $L_2$  is  $p^r$ -modular with  $\text{ord } nL_2 = u_2$ , and  $r \geq 1$ .*

(1) *If there is a vector  $x_2 \in L_2$  such that  $\text{ord } Q(x_2) \equiv u_1 \pmod{2}$  and  $\text{ord } Q(x_2) \leq u_1$ , then  $L = \overline{L}_1 \perp \overline{L}_2$  where  $\overline{L}_2$  is  $p^r$ -modular with  $n\overline{L}_2 = nL_2$  and  $\overline{L}_1$  is unimodular with  $n\overline{L}_1 \subset nL_1$  or  $\overline{L}_1 \cong H$ .*

(2) *If there is a vector  $z_1 \in L_1$  such that  $\text{ord } Q(z_1) \equiv u_2 \pmod{2}$  and*

$\text{ord } Q(z_1) \leq (u_2 - 2r)$ , then  $L = \overline{L}_1 \perp \overline{L}_2$  where  $\overline{L}_1$  is unimodular with  $n\overline{L}_1 = nL_1$  and  $\overline{L}_2$  is  $p^r$ -modular with  $n\overline{L}_2 \subset nL_2$  or  $\overline{L}_2 \cong H$ .

*Proof.* (1) Without loss of generality, we assume  $\text{rank } L_1 = 2$ . Write  $L_1 \cong A(a, -a^{-1}\delta)$ , adapted to a basis  $\{x_1, y_1\}$  where  $a$  is a norm generator of  $L_1$ ,  $D(1 + \delta) = \delta\vartheta$  and  $-a^{-1}\delta \in \omega L_1$ . Let  $k = (u_1 - \text{ord } Q(x_2))/2$ , so  $-Q(x_1)/Q(\pi^k x_2) \in U$ . Put  $-Q(x_1)/Q(\pi^k x_2) = \xi^2 + \sigma\pi^d$  with  $\xi$  and  $\sigma \in U$ ,  $d \geq 1$ . Consider a unimodular lattice  $\overline{L}_1 = \vartheta(x_1 + \xi\pi^k x_2) + \vartheta y_1$  which splits  $L$ , we obtain  $L = \overline{L}_1 \perp \overline{L}_2$ .

- (i) When  $u_1 < e$ , then  $\text{ord}(-a^{-1}\delta) > u_1$  and  $n\overline{L}_1 \subset nL_1$ .
- (ii) When  $u_1 = e$  and  $L_1 \cong A(2, 2\rho)$ , then  $-d\overline{L}_1 \in U^2$  and  $\overline{L}_1 \cong A(0, 0)$ .

Since  $n\overline{L}_1 \subseteq nL_1 \subseteq nL_2$  and  $nL_1 + nL_2 = n\overline{L}_1 + n\overline{L}_2 = nL$ , we have  $n\overline{L}_2 = nL_2 = nL$ .

(2) It follows from applying (1) to  $(L^\#)^{\pi^r}$ . □

The following proposition strengthen [O, 91:9 Th.(2)].

**PROPOSITION 1.1.** *Suppose  $L = L_1 \perp L_2 \perp \dots \perp L_t$  is a Jordan splitting of  $L$  with  $sL_i = s_i$ ;  $i = 1, \dots, t$ . If  $nL^{s_{i_0}} \supset nL^{s_{i_0}+1}$  and  $nL^{s_{i_0}} \supset (nL)(s_{i_0}s_{i_0-1}^{-1})^2$  for some  $1 \leq i_0 \leq t$ , then for any Jordan splitting of  $L$ ,  $L = K_1 \perp K_2 \perp \dots \perp K_t$ , we have  $nK_{i_0} = nL_{i_0}$ .*

*Proof.* It is obvious that  $L_j \subseteq L^{s_j}$ , so  $nL_j \subseteq nL^{s_j}$  for any  $1 \leq j \leq t$ . Since  $nL^{s_{i_0}} \supset nL^{s_{i_0}+1} \supseteq \dots \supseteq nL^{s_t}$ , we have  $nL^{s_{i_0}} \supset nL_j$  for all  $j > i_0$ . Consider

$$nL_j^{s_{i_0}} = (s_{i_0}s_{i_0-1}^{-1})^2 nL_j \subseteq (s_{i_0}s_{i_0-1}^{-1})^2 (nL) \subset nL^{s_{i_0}}$$

for all  $j < i_0$ . Note

$$L^{s_{i_0}} = L_1^{s_{i_0}} \perp \dots \perp L_{i_0-1}^{s_{i_0}} \perp L_{i_0} \perp L_{i_0+1} \perp \dots \perp L_t.$$

So

$$nL^{s_{i_0}} = \sum_{j < i_0} nL_j^{s_{i_0}} + nL_{i_0} + \sum_{j > i_0} nL_j = nL_{i_0},$$

and  $L^{s_{i_0}}$  is independent of the Jordan splitting of  $L$ . □

**REMARK 1.1.** When  $sL_{i_0} = nL_{i_0}$ , it can be easily checked that  $nL^{s_{i_0}} \supset nL^{s_{i_0}+1}$  and  $nL^{s_{i_0}} \supset (nL^{s_{i_0-1}})(s_{i_0}s_{i_0-1}^{-1})^2$ . The converse statement is usually not true.

LEMMA 1.2. *Suppose  $L$  is a unimodular lattice with  $nL \supset 2sL$ . Then there exist two sublattices  $J$  and  $M$  such that  $L = J \perp M$  with  $nL = nL \supset nM$ . Furthermore  $\text{rank } J = 1$  when  $\text{rank } L$  is odd, and  $\text{rank } J = 2$  when  $\text{rank } L$  is even.*

*Proof.* It follows from [O, 93:18]. □

THEOREM 1.1. *There exists a Jordan splitting  $L = L_1 \perp \cdots \perp L_t$  such that for any Jordan splitting  $L = K_1 \perp \cdots \perp K_t$ , we have  $nK_i \supseteq nL_i$  for all  $1 \leq i \leq t$ , and if  $K_i \cong H$ , then  $L_i \cong H$ .*

*Proof.* Put  $A_1 = \{(K_1, \dots, K_t) \mid L = K_1 \perp \cdots \perp K_t \text{ be a Jordan splitting of } L, \text{ and } K_1 \cong H\}$ .

If this set is empty, we put  $A_1 = \{(K_1, \dots, K_t) \mid L = K_1 \perp \cdots \perp K_t \text{ be a Jordan splitting of } L, \text{ and } nK_1 \text{ is the smallest}\}$ .

Put  $A_2 = \{(K_1, \dots, K_t) \in A_1 \mid K_2 \cong H\} \subseteq A_1$ .

If this set is empty, we put

$$A_2 = \{(K_1, \dots, K_t) \in A_1 \mid nK_2 \text{ is the smallest} \subseteq A_1.$$

...

By induction, put  $A_t = \{(K_1, \dots, K_t) \in A_{t-1} \mid K_t \cong H\}$ .

If this set is empty, we put

$$A_t = \{(K_1, \dots, K_t) \in A_{t-1} \mid nK_t \text{ is the smallest}\}.$$

Let  $(L_1, \dots, L_t) \in A_t$ , so  $L = L_1 \perp \cdots \perp L_t$  is a Jordan splitting. By Lemma 1.1, we have if  $\text{ord } nL_i \equiv \text{ord } nL_j \pmod 2$  for some  $i < j$ , then  $\text{ord } nL_i < \text{ord } nL_j < 2(r_j - r_i) + \text{ord } nL_i$  or  $L_i \cong H$  when  $\text{ord } nL_i \geq \text{ord } nL_j$  or  $L_j \cong H$  when  $\text{ord } nL_j \geq \text{ord } nL_i + 2(r_j - r_i)$ . Here  $r_k = \text{ord } s_k$  and  $s_k = sL_k$  for  $1 \leq k \leq t$ .

Suppose there is a Jordan splitting of  $L = K_1 \perp \cdots \perp K_t$  with  $nK_{i_0} \subset nL_{i_0}$  for some  $1 \leq i_0 \leq t$ . By [O, 91:9 Th.(2)],  $sL_{i_0} \supset nL_{i_0} \supset 2sL_{i_0}$  and  $\text{rank } L_{i_0}$  is even. By Lemma 1.2,  $L_{i_0} = J \perp M$  with  $nL_{i_0} = nJ \supset nM$  and  $\text{rank } J = 2$ . Write  $J = \vartheta x + \vartheta \bar{x}$  where  $Q(x)$  is a norm generator of  $J$ ,  $\text{ord } Q(x) < \text{ord } Q(\bar{x})$  and  $\text{ord } B(x, \bar{x}) = r_{i_0}$ . Put  $x = \sum_{i=1}^t a_i y_i$  where  $y_i$  is a maximal vector of  $K_i$  and  $a_i \in \vartheta$  for  $i = 1, \dots, t$ . Note

$$s_{i_0} = B(x, L) = \sum_{i=1}^t B(a_i y_i, K_i) \supseteq B(a_i y_i, K_i)$$

for all  $1 \leq i \leq t$ , so  $\text{ord } a_i \geq r_{i_0} - r_i$  when  $i_0 \geq i$  by [O, 82:17]. Put  $\bar{x} = \sum_{i=1}^t \bar{a}_i \bar{y}_i$  where  $\bar{y}_i$  is a maximal vector of  $K_i$  and  $\bar{a}_i \in \vartheta$ . So  $\text{ord } \bar{a}_i \geq r_{i_0} - r_i$  for all  $i \leq i_0$  by the same reason. Note

$$\text{ord } B(a_i y_i, \bar{a}_i \bar{y}_i) \geq r_{i_0} + (r_{i_0} - r_i) > r_{i_0}$$

for all  $i < i_0$ , and  $\text{ord } B(a_i y_i, \bar{a}_i \bar{y}_i) \geq r_i > r_{i_0}$  for all  $i > i_0$ . Consider  $r_{i_0} = \text{ord } B(x, \bar{x}) = \text{ord } B(a_{i_0} y_{i_0}, \bar{a}_{i_0} \bar{y}_{i_0}) \geq \text{ord } a_{i_0} + \text{ord } \bar{a}_{i_0} + r_{i_0}$ . Therefore  $\text{ord } a_{i_0} = \text{ord } \bar{a}_{i_0} = 0$ ,  $\text{ord } B(y_{i_0}, \bar{y}_{i_0}) = r_{i_0}$  and  $\text{ord } B(y_{i_0}, \bar{x}) = r_{i_0}$ . Put  $y_{i_0} = \sum_{i=1}^t b_i z_i$  where  $z_i$  is a maximal vector of  $L_i$ , and  $b_i \in \vartheta$ . So  $\text{ord } b_i \geq r_{i_0} - r_i$  for all  $i < i_0$  and  $\text{ord } b_{i_0} = 0$  by the same argument as above. Let  $b_{i_0} z_{i_0} = cx + d\bar{x} + w$  with  $c, d \in \vartheta$  and  $w \in M$ ; note

$$\begin{aligned} r_{i_0} &= \text{ord } B(y_{i_0}, \bar{x}) = \text{ord } B(b_{i_0} z_{i_0}, \bar{x}) \\ &= \text{ord } B(cx + d\bar{x}, \bar{x}) = \text{ord } (cB(x, \bar{x}) + dQ(\bar{x})) \end{aligned}$$

and  $\text{ord } Q(\bar{x}) > \text{ord } Q(x) > r_{i_0}$ . So  $r_{i_0} = \text{ord } (cB(x, \bar{x})) = \text{ord } (c) + r_{i_0}$ . Therefore  $\text{ord } (c) = 0$  and  $\text{ord } Q(b_{i_0} z_{i_0}) = \text{ord } (Q(cx + d\bar{x}) + Q(w)) = \text{ord } Q(x)$ . Suppose all the vectors in  $\{b_i z_i \mid i \neq i_0, 1 \leq i \leq t\}$  which satisfy  $\text{ord } Q(b_i z_i) \leq \text{ord } Q(b_{i_0} z_{i_0})$  are  $b_{i_1} z_{i_1}, b_{i_2} z_{i_2}, \dots, b_{i_l} z_{i_l}$ .

When  $i_k > i_0$  then  $nL_{i_k} \supseteq nL_{i_0} \supset 2sL_{i_0} \supset 2sL_{i_k}$ . So  $L_{i_k} \neq H$  and  $\text{ord } nL_{i_k} + \text{ord } nL_{i_0} \equiv 1 \pmod 2$  by Lemma 1.1.

When  $i_k < i_0$ , then

$$\begin{aligned} \text{ord } nL_{i_k} + 2(r_{i_0} - r_{i_k}) &\leq \text{ord } Q(b_{i_k} z_{i_k}) \\ &\leq \text{ord } Q(b_{i_0} z_{i_0}) = \text{ord } Q(x) = \text{ord } nL_{i_0} < \text{ord } 2sL_{i_0}. \end{aligned}$$

That is  $\text{ord } nL_{i_k} < \text{ord } 2 + (r_{i_k} - r_{i_0}) + r_{i_k} < \text{ord } 2sL_{i_k}$ . So  $L_{i_k} \neq H$  and  $\text{ord } nL_{i_k} + \text{ord } nL_{i_0} \equiv 1 \pmod 2$  by Lemma 1.1. Put  $N = L_{i_1} \perp \dots \perp J \perp \dots \perp L_{i_l}$ , for any  $k_1 < k_2$ ; we have  $\text{ord } nL_{i_{k_1}} \equiv \text{ord } nL_{i_{k_2}} \equiv \text{ord } nJ + 1 \pmod 2$ , and  $\text{ord } nL_{i_{k_1}} < \text{ord } nL_{i_{k_2}} < 2(r_{i_{k_2}} - r_{i_{k_1}}) + \text{ord } nL_{i_{k_1}}$  and  $\text{ord } nJ > \text{ord } nL_{i_k}$  for all  $1 \leq k \leq l$ . Since  $nK_{i_0} \subset nL_{i_0}$ ,  $\text{ord } Q(y_{i_0}) \geq \text{ord } nK_{i_0} > \text{ord } nL_{i_0} = \text{ord } Q(x)$ . Note  $Q(y_{i_0}) = \sum_{i=1}^t Q(b_i z_i)$ . So  $\text{ord } Q(x) = \text{ord } Q(b_{i_0} z_{i_0}) = \text{ord } (\sum_{k=1}^l Q(b_{i_k} z_{i_k}))$ . Write  $-Q(x) / \sum_{k=1}^l Q(b_{i_k} z_{i_k}) = \xi^2 + \sigma\pi^d$  with  $\xi, \sigma \in U$  and  $d \geq 1$ , then

$$Q\left(x + \xi \sum_{k=1}^l b_{i_k} z_{i_k}\right) = -Q\left(\sum_{k=1}^l b_{i_k} z_{i_k}\right) \sigma\pi^d$$

and

$$\text{ord } Q \left( x + \xi \sum_{k=1}^l b_{i_k} z_{i_k} \right) = \text{ord } Q(x) + d > \text{ord } Q(x).$$

Put  $\bar{J} = \vartheta(x + \xi \sum_{k=1}^l b_{i_k} z_{i_k}) + \vartheta\bar{x}$  which is  $s_{i_0}$ -modular. Since  $\text{ord } b_{i_k} = \text{ord } \xi b_{i_k} \geq r_{i_0} - r_{i_k}$  for all  $i_k < i_0$ ,  $\bar{J}$  splits  $N$ . So we obtain another Jordan splitting of  $N$ ,  $N = \overline{L_{i_1}} \perp \cdots \perp \bar{J} \perp \cdots \perp \overline{L_{i_l}}$ .

Since we can check  $nN^{s_{i_k}} = nL_{i_k}$  for all  $1 \leq k \leq l$ ,  $nN^{s_{i_k}} \supset nN^{s_{i_{k+1}}}$  and  $nN^{s_{i_k}} \supset (nN^{s_{i_{k-1}}})(s_{i_k} s_{i_{k-1}}^{-1})^2$  for all  $1 \leq k \leq l$ . We have  $nL_{i_k} = n\overline{L_{i_k}}$  for all  $1 \leq k \leq l$  by the above proposition, but  $n\bar{J} \subset nJ$ . Corresponding to this Jordan splitting of  $N$ , we obtain another Jordan splitting of  $L$  which contradicts our choice of the Jordan splitting of  $L$ .

If  $K_{i_0} \cong H$  but  $L_{i_0} \neq H$ , then  $nL_{i_0} = 2sL_{i_0}$  by the above argument. By [O, 93:14] we can assume  $L_{i_0} \cong \pi^{r_0}A(2, 2\rho)$  adapted to a basis  $\{u, \bar{u}\}$  and  $K_{i_0} \cong \pi^{r_0}A(0, 0)$  adapted to a basis  $\{v, \bar{v}\}$ . Write  $v = \sum_{i=1}^t c_i q_i$  where  $q_i$  is a maximal vector of  $L_i$  and  $c_i \in \vartheta$ , so  $\text{ord } c_i \geq (r_{i_0} - r_i)$  for all  $i < i_0$  and  $\text{ord } (c_{i_0}) = 0$ . Thus  $\text{ord } Q(c_{i_0} q_{i_0}) = r_{i_0} + e = \text{ord } Q(u)$  by Riehm Domination Principle [R]. By the similar arguments as above, we can obtain a new Jordan splitting of  $L$  which contradicts our choice of the Jordan splitting of  $L$ . □

The Jordan splittings which enjoy the property of Theorem 1.1 are called minimal norm Jordan splittings.

**COROLLARY 1.1.**  *$L$  can be splitted as  $L = L_0 \perp H$  such that  $L_0$  cannot be splitted by any hyperbolic plane and  $L_0$  is determined uniquely by  $L$  up to isometry.*

*Proof.* Suppose  $L$  has another splitting  $L = \overline{L_0} \perp \bar{H}$  where  $\overline{L_0}$  cannot be splitted by any hyperbolic plane, and the type of Jordan splitting of  $\overline{L_0}$  is different from that of  $L_0$ . Without loss of generality, we assume that the rank of  $i_0$  Jordan component of  $\overline{L_0}$  is greater than that of  $L_0$  for some  $1 \leq i_0 \leq t$  and  $\bar{H}$  does not contain any  $i_0$  hyperbolic component by Cancellation Theorem [O, 93:14]. So we can choose a Jordan splitting  $\overline{L_0} = J_1 \perp \cdots \perp J_t$  such that  $J_{i_0} = M \perp N$  with  $nJ_{i_0} = nM \supset nN$ ,  $\text{rank } M \leq 2$ ,  $\text{rank } N = 2$ , and  $nN$  is the smallest. Furthermore we assume the Jordan splitting

$R = J_1 \perp \cdots \perp J_{i_0-1} \perp M \perp J_{i_0+1} \perp \cdots \perp J_t$ . is the minimal norm Jordan splitting.

Comparing the Jordan splitting of  $\overline{L_0}$  with that of  $L_0$ , there is a hyperbolic plane  $H_{i_0} = \vartheta u + \vartheta \bar{u} \subseteq L$  with  $sH_{i_0} = sJ_{i_0}$  and  $Q(u) = Q(\bar{u}) = 0$  and  $B(u, \bar{u})\vartheta = sH_{i_0}$ . So  $u = \sum_{i=1}^t b_i z_i$ ,  $\bar{u} = \sum_{i=1}^t \bar{b}_i \bar{z}_i$  where  $z_i$  and  $\bar{z}_i$  are the maximal vectors of each Jordan component for the Jordan splitting  $L = \overline{L_0} \perp H$  with  $1 \leq i \leq t$ . So  $\text{ord } b_i \geq (r_{i_0} - r_i)$ ,  $\text{ord } \bar{b}_i \geq (r_{i_0} - r_i)$  for all  $i < i_0$ , and  $\text{ord } b_{i_0} = \text{ord } \bar{b}_{i_0} = 0$ . Here  $r_i = \text{ord } sJ_i$  for all  $1 \leq i \leq t$ . Write  $z_i = p_i + q_i$  with  $p_i \in J_i$  and  $q_i \in H_i$  where  $H_i$  is a suitable hyperbolic component with  $sH_i = sJ_i$  or 0 for all  $i$ , then  $\text{ord } Q(b_i q_i) \geq 2 \text{ord } b_i + e + r_i > e + r_{i_0}$  for all  $i \neq i_0$ . Consider  $z_{i_0} = v_{i_0} + w_{i_0}$  and  $\bar{z}_{i_0} = \bar{v}_{i_0} + \bar{w}_{i_0}$  where  $v_{i_0}$  and  $\bar{v}_{i_0} \in M$ ,  $w_{i_0}$  and  $\bar{w}_{i_0} \in N$ , then at least one of  $Q(v_{i_0})$ ,  $Q(\bar{v}_{i_0})$ ; or  $Q(w_{i_0})$ ,  $Q(\bar{w}_{i_0})$  is a norm generator.

If  $Q(v_{i_0})$  is a norm generator of  $M$ , then  $\text{ord } Q(b_{i_0} z_{i_0}) = \text{ord } Q(v_{i_0}) < \text{ord } Q(w_{i_0}) \leq e + r_{i_0}$ , and  $\text{ord } Q(v_{i_0}) = \text{ord } Q(\sum_{i \neq i_0} b_i p_i)$  by  $Q(u) = 0$ . So we can get the new splitting  $R = \overline{J_1} \perp \cdots \perp \overline{M} \perp \cdots \perp \overline{J_t}$  with  $n\overline{M} \subset nM$ . That is a contradiction.

If  $Q(w_{i_0})$  is a norm generator of  $M$ , then  $\text{ord } Q(w_{i_0}) \leq e + r_{i_0}$  and  $\text{ord } Q(w_{i_0}) = \text{ord } Q(\sum_{i \neq i_0} b_i p_i + v_{i_0})$ , we can get the new splitting  $\overline{L_0} = \overline{J_1} \perp \cdots \perp \overline{J_t}$  with  $\overline{J_{i_0}} = \overline{M} \perp \overline{N}$  such that  $n\overline{N} \subset nN$ . This contradicts our choice.

Therefore  $L_0$  and  $\overline{L_0}$  have the same type of Jordan splitting and  $L_0 \cong \overline{L_0}$  by Cancellation Theorem [O, 93:14]. □

**2. Generation and spinor norms of  $O(L)$ .** Suppose  $L = L_1 \perp L_2 \perp \cdots \perp L_t$  is a minimal norm Jordan splitting over a dyadic local field  $F$  with  $r_i = \text{ord } sL_i$ ,  $u_i = \text{ord } nL_i$ , for  $i = 1, \dots, t$ .  $Q(x_i) = \varepsilon_i \pi^{u_i}$  is a norm generator of  $L_i$  where  $\varepsilon_i \in U$  and  $x_i \in L_i$ , for  $1 \leq i \leq t$ .

LEMMA 2.1. *Suppose all the Jordan components are one dimension and there exists  $i$  and  $j$  with  $1 \leq i < j \leq n$  such that  $r_j - r_i \leq 2e$  and  $D(-\varepsilon_i \varepsilon_j) = p^s$  with  $1 \leq s \leq e - (r_j - r_i)/2$ . If  $0 < |r_k - r_i| \leq 2e$  or  $0 < |r_k - r_j| \leq 2e$  for some  $1 \leq k \leq n$ , then  $\theta(O^+(L)) = F$ .*

*Proof.* Because of [X, Theorem 3.1] we can assume that  $r_j - r_i$ ,  $r_k - r_j$  and  $r_k - r_i$  are even. Suppose  $r_i < r_j < r_k$ . The other cases can be done by taking the same arguments.



So  $0 < r_k - r_j \leq 2e$  and  $\theta(O^+(\vartheta x_i \perp \vartheta x_j)) = Q[1, \varepsilon_i \varepsilon_j]$  by [X, Prop. 2.3]. By [H, Lemma 3] there exists  $\eta$  in  $U$  such that  $(\eta, -\varepsilon_i \varepsilon_j) = -1$  and  $D(\eta) = p^{2e-s}$ . Note  $D(-1) = p^h$  with  $h \geq e$  and  $(2e - s) + h \geq e + (r_i - r_j)/2 + e > 2e$ . So  $(\eta, -1) = 1$  by [X, Remark 1]. Therefore  $(\eta, -\varepsilon_i \varepsilon_k) = 1$  or  $(\eta, -\varepsilon_j \varepsilon_k) = 1$ .

When  $(\eta, -\varepsilon_j \varepsilon_k) = 1$ , write  $D(-\varepsilon_j \varepsilon_k) = p^t$ .

If  $1 \leq t \leq e - (r_k - r_j)/2$ , then  $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$  by [X, Prop. 2.3]. If  $(3e - (r_k - r_j)/2)/2 \geq t > e - (r_k - r_j)/2$ , note  $2e - s \geq e + (r_j - r_i)/2 \geq t - e + (r_k - r_j)/2$ . Then  $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$ .

If  $t > (3e - (r_k - r_j)/2)/2$ , note  $2e - s \geq e + (r_j - r_i)/2 \geq e - [e/2 - (r_k - r_j)/4]$ . Then  $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$ . Therefore  $\theta(O^+(L)) = \theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_j \perp \vartheta x_k)) = \dot{F}$ .

When  $(\eta, -\varepsilon_i \varepsilon_k) = 1$ , the result follows from the same arguments as above if  $r_k - r_i \leq 2e$ . So we assume  $4e \geq r_k - r_i > 2e$ . Write  $D(-\varepsilon_k \varepsilon_i) = p^d$ .

If  $1 \leq d \leq 2e - (r_k - r_i)/2$ , then

$$2e - s \geq e + (r_j - r_i)/2 \geq (r_k - r_i)/2 \geq (r_k - r_i) - 2e + d.$$

So  $\eta \in \theta(O^+(\vartheta x_i \perp \vartheta x_k))$ .

If  $d > 2e - (r_k - r_i)/2$ , note  $2e - s \geq (r_k - r_i)/2$ ; then  $\eta \in \theta(O^+(\vartheta x_i \perp \vartheta x_k))$ . Therefore  $\theta(O^+(L)) = \theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_i \perp \vartheta x_k)) = \dot{F}$ . □

**LEMMA 2.2.** *If  $L_{i_0} \cong \pi^{r_{i_0}} A(\varepsilon_{i_0} \pi^{u_{i_0} - r_{i_0}}, -\varepsilon_{i_0}^{-1} \pi^{-u_{i_0} + r_{i_0}} \delta_{i_0})$  adapted to a basis  $\{x_{i_0}, y_{i_0}\}$  with  $D(1 + \delta_{i_0}) = \delta_{i_0} \vartheta$  for some  $1 \leq i_0 \leq t$ .*

(1) *When  $\text{ord } \delta_{i_0} < u_{i_0} + e - r_{i_0}$ , and  $u_k + u_{i_0} \equiv 0 \pmod{2}$ , and  $u_k + \text{ord } Q(y_{i_0}) - 2r_k \leq 2e + 1$  with some  $k < i_0$  or  $u_k + \text{ord } Q(y_{i_0}) - 2r_{i_0} \leq 2e + 1$  with some  $k > i_0$ , then  $\theta(O^+(L)) = \dot{F}$ .*

(2) *When  $u_{i_0} + u_k \equiv 1 \pmod{2}$ ,  $u_k + u_{i_0} - 2r_{i_0} \leq 2e + 1$  with some  $k > i_0$  or  $u_k + u_{i_0} - 2r_k \leq 2e + 1$  with some  $k < i_0$ , then  $\theta(O^+(L)) = \dot{F}$ .*

(3) *When  $u_{i_0} + u_k \equiv 0 \pmod{2}$ ,  $L_{i_0} \neq \pi^{r_{i_0}} A(0, 0)$ ,  $D(-\varepsilon_{i_0} \varepsilon_k) = p^t$ ,  $t \leq e - (u_k + u_{i_0} - 2r_{i_0})/2$  with some  $k > i_0$  or  $t \leq e - (u_k + u_{i_0} - 2r_k)/2$  with some  $k < i_0$ , then  $\theta(O^+(L)) = \dot{F}$ .*

*Proof.* (1) Put  $K = \vartheta y_{i_0} \perp \vartheta x_k$ . Since  $\text{ord } Q(x_k) + \text{ord } Q(y_{i_0}) \equiv 1$ , it can be checked that  $\tau_z \in O(L_k \perp L_{i_0}) \subseteq O(L)$  for any maximal vector  $z$  of  $K$ . Therefore  $\theta(O^+(L)) \supseteq Q[1, \dot{\varepsilon}_{i_0} \varepsilon_k \pi]$  which does not contain  $\Delta$ , but  $\Delta$  is in  $\theta(O^+(L_{i_0}))$  by [H]. Thus  $\theta(O^+(L)) = \dot{F}$ .

(2) It follows from the same arguments as the above case (1).

(3) Without loss of generality, we assume  $k > i_0$ . By Lemma 1.1, we know  $u_k - 2r_k + 2r_{i_0} < u_{i_0} < u_k$ . Put  $K = \vartheta x_{i_0} \perp \vartheta x_k$ . Since  $1 \leq t < e$ , we have  $D(-\varepsilon_{i_0}\varepsilon_k) = D(\varepsilon_{i_0}\varepsilon_k)$  and  $u_k + u_{i_0} - 2r_{i_0} \leq 2e$ . It can be checked that  $\tau_z \in O(L_{i_0} \perp L_k) \subseteq O(L)$  for any maximal vector  $z$  of  $K$ . Therefore  $\theta(O^+(L)) \supseteq Q[1, \dot{\varepsilon}_{i_0}\varepsilon_k]$ . By [H, Lemma 3] there exists  $\eta$  in  $U$  such that  $(\eta, -\varepsilon_{i_0}\varepsilon_k) = -1$  with  $D(\eta) = p^{2e-t}$ .

- (i) If  $\text{ord } \delta_{i_0} \geq u_{i_0} + e - r_{i_0}$ , then  $2e - t \geq e + (u_k + u_{i_0} - 2r_{i_0})/2$ . So  $\eta \in \theta(O^+(L_{i_0}))$  by [H, Prop. B], [X, Remark 1] and [H, Lemma 2]. Thus  $\theta(O^+(L)) = \dot{F}$ .
- (ii) If  $\text{ord } \delta_{i_0} < u_{i_0} + e - r_{i_0}$ , we only need to consider  $u_k + \text{ord } Q(y_{i_0}) - 2r_{i_0} > 2e + 1$  with  $k > i_0$ . Note

$$\begin{aligned} &2e - t + u_{i_0} - r_{i_0} + \text{ord } Q(y_{i_0}) - r_{i_0} \\ &\geq e + (u_k + \text{ord } Q(y_{i_0}) - 2r_{i_0})/2 \\ &\quad + u_{i_0}/2 + u_{i_0} + \text{ord } Q(y_{i_0})/2 - 2r_{i_0} \\ &> e + e + u_{i_0}/2 + u_{i_0} + u_{i_0}/2 - 2r_{i_0} \geq 2e. \end{aligned}$$

Then  $\eta \in \theta(O^+(L_{i_0}))$  by [X, Remark 1]. Thus  $\theta(O^+(L)) = \dot{F}$ . □

LEMMA 2.3. *If  $\text{rank } L_i \geq 3$  and  $\text{ord } nL_i + \text{ord } wL_i \equiv 1 \pmod{2}$  for some  $1 \leq i \leq t$ , then  $\theta(O^+(L)) = \dot{F}$ .*

*Proof.* It follows from [H, Prop. A]. □

LEMMA 2.4. *If  $\text{rank } L_i = \text{rank } L_j = 1$  and  $\text{rank } L_k = 2$  for some  $i > j$  and  $k$ ,  $0 < u_i - u_j \leq 2e + 1$  and  $u_i - u_j$  is odd, then  $\theta(O^+(L)) = \dot{F}$ .*

*Proof.* Since  $\Delta$  is not in  $\theta(O^+(L_i \perp L_j)) = Q[1, \dot{\varepsilon}_i\varepsilon_j\pi]$  by [X, Prop. 2.2 i] and [X, Prop. 2.3 i] and  $\Delta \in \theta(O^+(L_k))$  by [H]; therefore  $\theta(O^+(L)) = \dot{F}$ . □

LEMMA 2.5. *Suppose  $\text{rank } L_i = \text{rank } L_j = 1$  and for some  $i > j$  with  $0 < u_i - u_j \leq 2e$  and  $u_i - u_j$  is even and  $D(-\varepsilon_i\varepsilon_j) = p^t$  with  $t \leq e - (u_i - u_j)/2$ . If there is  $L_k \cong \pi^{r_k}A(\varepsilon_k\pi^{u_k-r_k}, -\varepsilon_k^{-1}\pi^{-u_k+r_k}\delta_k)$  with  $\text{ord } \delta_k \geq u_k - r_k + e$  for some  $1 \leq k \leq t$ , then  $\theta(O^+(L)) = \dot{F}$ .*

*Proof.* By [H, Lemma 3] and [X, Prop. 2.3], there exists  $\eta \in \theta(O^+(L_i \perp L_j)) = Q[1, \dot{\varepsilon}_i\varepsilon_j]$  with  $D(\eta) = p^{2e-t}$  and  $2e - t \geq e + (u_i -$

$u_j)/2 \geq e + 1$ . So  $\eta \in \theta(O^+(L_k))$  by [H, Prop. B], [X, Remark 1] and [H, Lemma 2]. Therefore  $\theta(O^+(L)) = \dot{F}$ . □

**LEMMA 2.6.** *If  $\text{rank } L_i = \text{rank } L_j = 1$  and  $\text{rank } L_k = 2$ ,  $0 < u_j - u_i \leq 2e$  and  $u_j - u_i$  is even for some  $k > j > i$ ,  $D(-\varepsilon_i \varepsilon_j) = p^t$  with  $t \leq e - (u_j - u_i)/2$ , and  $u_k - u_i \leq 2e$ , then  $\theta(O^+(L)) = \dot{F}$ .*

*Proof.* Put  $L_k \cong \pi^{r_k} A(\varepsilon_k \pi^{u_k - r_k}, -\varepsilon_k^{-1} \pi^{-u_k + r_k} \delta_k)$ . By Lemma 2.2 and Lemma 2.5, we can assume  $u_k - u_i$  is even and  $\text{ord } \delta_k < u_k - r_k + e$  and  $\text{ord } \delta_k + 2r_k - u_k - u_j > 2e + 1$ . So  $r_k - u_i > r_k - u_j > e + 1$ . It can be checked that any  $\tau_z \in O(\vartheta x_j \perp \vartheta x_k)$  is also in  $O(L_j \perp L_k)$ . So  $O(\vartheta x_j \perp \vartheta x_k) \subseteq O(L_j \perp L_k)$ . By the same reason,  $O(\vartheta x_i \perp \vartheta x_k) \subseteq O(L_i \perp L_k)$ . By the proof of Lemma 2.1, we obtain  $\theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_i \perp \vartheta x_k))\theta(O^+(\vartheta x_j \perp \vartheta x_k)) = \dot{F}$ . Thus  $\theta(O^+(L)) = \dot{F}$ . □

Before obtaining our main result, we first establish the following Witt- type result.

**PROPOSITION 2.1.** *Suppose  $L$  cannot be splitted by any hyperbolic plane and  $\theta(O^+(L)) \neq \dot{F}$ . If  $\sigma L_1 \subseteq L$  for some  $\sigma \in O(V)$ , then there is  $\tau$  a product of symmetries in  $O(L)$  such that  $\tau\sigma|_{L_1} = 1$ .*

*Proof.* When  $e = 1$ , it has been done in [OP1]. We assume  $e > 1$  and  $r_1 = 0$ . By Lemma 2.3 and [O, 93:18], we know all the Jordan components are one or two dimensions and none of them is hyperbolic plane. □

When  $\text{rank } L_1 = 2$ , write  $L_1 \cong A(\varepsilon_1 \pi^{u_1}, -\varepsilon_1^{-1} \pi^{-u_1} \delta_1)$  adapted to a basis  $\{x_1, y_1\}$  with  $D(1 + \delta_1) = \delta_1 \vartheta$ . Put  $\sigma x_1 = ax_1 + by_1 + z$  where  $a$  and  $b$  are in  $\vartheta$ ,  $z \in L_2 \perp \cdots \perp L_t$ .

(1)  $\text{ord } Q(y_1) \geq e$ .

When  $u_k + u_1 \equiv 1 \pmod 2$  for some  $2 \leq k \leq t$ , then

$$u_k - u_1 \geq 2e + 3 - 2u_1 \geq 3, \quad r_k \geq u_k - e \geq e - u_1 + 3 \geq 3$$

by Lemma 2.2(2).

When  $u_k + u_1 \equiv 0 \pmod 2$  for some  $2 \leq k \leq t$ , then  $u_k \geq u_1 + 2$  and  $r_k > (u_k - u_1)/2 \geq 1$  by Lemma 1.1.

So  $\text{ord } Q(z) - \text{ord } Q(x_1) \geq 2$ . Note  $Q(x_1) = a^2 Q(x_1) + 2ab + b^2 Q(y_1) + Q(z)$ ,  $Q(\sigma x_1 - x_1) = 2((1 - a)Q(x_1) - b)$ .

If  $\text{ord } b = 0$  and  $\text{ord } Q(x_1) = \text{ord } Q(y_1) = e$ , then  $\tau_{\sigma x_1 - x_1} \in O(L)$ . Otherwise,  $a \equiv 1 \pmod p$  and we assume  $\text{ord } b \leq 1$  because we can consider  $\tau_{\pi^{[(e-u_1)/2]x_1+y_1}}\sigma(x_1)$  instead of  $\sigma x_1$  if necessary and  $\tau_{\pi^{[(e-u_1)/2]x_1+y_1}} \in O(L)$ .

- (i)  $u_1 \geq 1$  or  $\text{ord } b = 0$ . Then  $\tau_{\sigma x_1 - x_1} \in O(L)$ .
- (ii)  $u_1 = 0$  and  $\text{ord } b = 1$  and  $u_2 \geq 3$ . Since  $u_k \geq u_2 \geq 3$  for all  $k \geq 3$  by Lemma 1.1 and Lemma 2.2(2),  $\text{ord } Q(z) \geq 3$  and  $\text{ord } (1 - a^2) \geq 3$ . Therefore  $\text{ord } (1 - a) \geq 2$  and  $\tau_{\sigma x_1 - x_1} \in O(L)$ .
- (iii)  $u_1 = 0$  and  $\text{ord } b = 1$  and  $u_2 = 2$ . By the above arguments we only need to consider  $e$  is odd and  $\text{ord } (1 - a) = 1$ . Note  $D(-\varepsilon_1\varepsilon_2) = p^t$  with  $t > e - (u_1 + u_2)/2$  by Lemma 2.2(3). Write  $a = 1 + l\pi$  with  $l \in U$  and  $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$  with  $\xi, \lambda \in U$ .

Let

$$\eta = \xi + \pi^{(e-1)/2} \in U, \delta = \varepsilon_2(h^2 - \lambda\pi^{t-e+1} + 2\pi^{-e}\xi h\pi^{(e-1)/2+1}) \in U.$$

We have  $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{e-1}$  and  $\tau_{\varepsilon_2\pi x_1 + \eta x_2}$  is in  $O(L)$ . Write  $\tau_{\varepsilon_2\pi x_1 + \eta x_2}\sigma x_1 = a'x_1 + b'y_1 + z'$  with  $a' \equiv a(1 - 2\varepsilon_2^2\varepsilon_1\pi^{-e}\delta^{-1}\pi) \pmod{p^2}$  and  $z' \in L_2 \perp \dots \perp L_t$ . Note  $\text{ord } (1 - a') \geq 2$  by a direct computation. Therefore  $\tau_{\sigma' x_1 - x_1} \in O(L)$  with  $\sigma' = \tau_{\varepsilon_2\pi x_1 + \eta x_2}\sigma$  by the same argument as above.

(2)  $\text{ord } Q(y_1) < e$ .

When  $u_1 + u_k \equiv 1 \pmod 2$  for some  $2 \leq k \leq t$ , then  $u_1 + u_k \geq 2e + 3$  by Lemma 2.2(2) and  $r_k \geq u_k - e \geq e + 3 - u_1$ .

When  $u_1 + u_k \equiv 0 \pmod 2$  for some  $2 \leq k \leq t$ , then  $\text{ord } Q(y_1) + u_k \geq 2e + 3$  by Lemma 2.2(1) and  $r_k \geq u_k - e \geq e + 3 - \text{ord } Q(y_1)$ .

So  $\text{ord } Q(z) \geq 2e + 3 - \text{ord } Q(y_1)$  and  $a \equiv 1 \pmod p$ . We can assume  $\text{ord } b \leq e - \text{ord } Q(y_1)$  because we consider  $\tau_{2x_1+y_1}\sigma(x_1)$  instead of  $\sigma x_1$  if necessary and  $\tau_{2x_1+y_1} \in O(L)$ . We claim  $\text{ord } b = e - \text{ord } Q(y_1)$ . If  $\text{ord } b < e - \text{ord } Q(y_1)$ , then  $\text{ord } (b^2Q(y_1)) < \text{ord } 2ab < \text{ord } Q(z)$  and  $\text{ord } ((1 - a^2)Q(x_1)) = \text{ord } (b^2Q(y_1)) < 2e$ . Therefore  $\text{ord } (1 - a) = \text{ord } (1 + a) < e$  and  $\text{ord } Q(x_1) \equiv \text{ord } Q(y_1) \pmod 2$ . It is a contradiction since  $\text{ord } Q(y_1) < e$ . So we have  $\text{ord } ((1 - a^2)Q(x_1)) = \text{ord } (2ab + b^2Q(y_1) + Q(z)) \geq 2e - \text{ord } Q(y_1)$ .

If  $\text{ord } (1 - a) < e$ , then  $\text{ord } (1 + a) = \text{ord } (1 - a)$  and

$$\text{ord}((1 - a)Q(x_1)) \geq e - (\text{ord } Q(y_1) - \text{ord } Q(x_1))/2 > e - \text{ord } Q(y_1)$$

and

$$\text{ord}(1 - a) \geq e - (\text{ord } Q(y_1) + \text{ord } Q(x_1))/2 > e - \text{ord } Q(y_1).$$

Therefore  $\tau_{\sigma x_1 - x_1} \in O(L)$ .

If  $\text{ord}(1 - a) \geq e$ , then  $\tau_{\sigma x_1 - x_1} \in O(L)$ .

Now we can assume  $\sigma x_1 = x_1$ ,  $\sigma y_1 = \alpha x_1 + \beta y_1 + w$  by the above arguments. Here  $\alpha, \beta \in \mathfrak{v}$ ,  $w \in L_2 \perp \cdots \perp L_t$ . So we have  $1 = \alpha Q(x_1) + \beta$  and  $Q(\sigma y_1 - y_1) = 2\alpha(Q(x_1)Q(y_1) - 1)$ .

If  $\text{ord } \alpha \leq r_2$ , then  $\tau_{\sigma y_1 - y_1} \in O(L)$ .

If  $\text{ord } \alpha > r_2$  and  $u_1 + u_2 \geq 2e$ , then  $r_2 \geq u_2 - e \geq e - u_1$ . Put  $u = x_1 - Q(x_1)y_1$ , so  $\tau_u(x_1) = x_1$  and  $\tau_u\sigma(y_1) = \alpha'x_1 + \beta'y_1 + w'$  with  $\text{ord } \alpha' = e - u_1 < r_2$  and  $\tau_u \in O(L)$ . Therefore  $\tau_{\tau_u\sigma y_1 - y_1} \in O(L)$ .

If  $\text{ord } \alpha > r_2$  and  $u_1 + u_2 < 2e$ , then  $u_1 \equiv u_2 \pmod 2$  and  $D(-\varepsilon_1\varepsilon_2) = p^t$  with  $t > e - (u_2 + u_1)2$  by Lemma 2.2(3). Write  $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$  with  $\xi, \lambda \in U$  and

$$\begin{aligned} \eta &= \xi + \pi^{\lceil e/2 - (u_1 + u_2)/4 \rceil} \in \mathfrak{v}, \\ \delta &= \varepsilon_2(1 + 2\xi\pi^{-\lceil e/2 - (u_1 + u_2)/4 \rceil} - \lambda\pi^{t - 2\lceil e/2 - (u_1 + u_2)/4 \rceil}) \in U. \end{aligned}$$

So  $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{2\lceil e/2 - (u_1 + u_2)/4 \rceil}$ . Put

$$u = \pi^{(u_2 - u_1)/2}\varepsilon_2x_1 - \pi^{(u_2 - u_1)/2}\varepsilon_2Q(x_1)y_1 + \eta x_2 \in L.$$

Then  $\tau_u \in O(L)$  whenever  $\text{ord } Q(y_1) \geq e$ ; or  $\text{ord } Q(y_1) < e$  but  $u_2 + \text{ord } Q(y_1) > 2e + 1$  by Lemma 2.2(1). Note  $\tau_u x_1 = x_1$ ,  $\tau_u\sigma y_1 = \alpha'x_1 + \beta'y_1 + w'$  with

$$\begin{aligned} \text{ord } \alpha' &= e + u_2 - u_1 - (u_2 + 2\lceil e/2 - (u_1 + u_2)/4 \rceil) \\ &\leq (u_2 - u_1)/2 + 1 \leq r_2 \end{aligned}$$

by Lemma 1.1 and  $w' \in L_2 \perp \cdots \perp L_t$ . Therefore  $\tau_{\tau_u\sigma y_1 - y_1} \in O(L)$ . We have  $\tau_{\sigma y_1 - y_1}\sigma|_{L_1} = 1$  or  $\tau_{\tau_u\sigma y_1 - y_1}\tau_u\sigma|_{L_1} = 1$ .

When  $\text{rank } L_1 = 1$ , write  $\sigma x_1 = ax_1 + z$  with  $a \in \mathfrak{v}$  and  $z \in L_2 \perp \cdots \perp L_t$ . So  $Q(\sigma x_1 \pm x_1) = 2(1 \pm a)Q(x_1)$ . Since  $(1+a) + (1-a) = 2$ ,  $\text{ord}(1 - a) \leq e$  or  $\text{ord}(1 + a) \leq e$ . Note  $\tau_{\sigma x_1 \pm x_1} \in O(L)$  whenever  $\text{ord}(1 \mp a) \leq r_2$ . We only need to consider the following cases by Lemma 2.2.

- (1)  $u_2 \equiv 1 \pmod 2$  and  $u_2 < e$  and  $\text{rank } L_2 = 1$ .

By Lemma 2.4 and [X, Theorem 3.1], we have  $\text{rank } L_3 = 1$  and  $r_3 = u_3 > 2e$ . Write  $\sigma x_1 = ax_1 + bx_2 + w$  with  $b \in \vartheta$  and  $w \in L_3 \perp \cdots \perp L_t$ . We can assume  $\text{ord } b + u_2 < e$ . So  $\text{ord } (1 - a^2) = 2 \text{ord } b + u_2 < 2e$ . Therefore  $\text{ord } (1 - a) = \text{ord } (1 + a) = \text{ord } b + u_2/2 < \text{ord } b + u_2 < r_3$  and  $\tau_{\sigma x_1 - x_1} \in O(L)$ .

(2)  $u_2 \equiv 0 \pmod 2$  and  $D(-\varepsilon_1\varepsilon_2) = p^t$  with  $t > e - u_2/2$  and  $\text{ord } (1 - a) > r_2$ . Write  $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$  with  $\xi, \lambda \in U$  and  $\eta = \xi + \pi^{[e/2 - u_2/4]} \in \vartheta$  and  $\delta = \varepsilon_2(1 + 2\xi\pi^{-[e/2 - u_2/4]} - \lambda\pi^{t - 2[e/2 - u_2/4]}) \in U$ .

So  $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{2[e/2 - u_2/4]}$ . Put  $u = \pi^{u_2/2}\varepsilon_2x_1 + \eta x_2 \in L$ ; then  $\tau_u \in O(L)$ . Consider  $\tau_u\sigma x_1 = a'x_1 + z'$  with

$$a' = a - 2\varepsilon_1\varepsilon_2^2a\delta^{-1}\pi^{-2[e/2 - u_2/4]} - 2\eta\varepsilon_2\delta^{-1}B(x_2, z)\pi^{-u_2/2 - 2[e/2 - u_2/4]}$$

and  $z' \in L_2 \perp \cdots \perp L_t$ . Note  $\text{ord } (1 - a') = e - 2[e/2 - u_2/4] \leq u_2/2 + 1 \leq r_2$  by Lemma 1.1. So  $\tau_{\tau_u\sigma x_1 - x_1} \in O(L)$ .

(3)  $u_2 \equiv 0 \pmod 2$  and  $D(-\varepsilon_1\varepsilon_2) = p^t$  with  $t \leq e - u_2/2$ .

By Lemma 2.2, we have  $\text{rank } L_2 = 1$ . Write  $\sigma x_1 = ax_1 + bx_2 + w$  with  $w \in L_3 \perp \cdots \perp L_t$ , we only need to consider  $\text{ord } b + u_2 < e$ .

If  $u_k + u_2 \equiv \text{mod } 2$  for some  $3 \leq k \leq t$ , then  $u_k - u_2 > 2e + 1$  by [X, Theorem 3.1] and Lemma 2.2(2) and  $r_k \geq u_k - e > e + 1 + u_2$ . If  $u_k + u_2 \equiv 0 \pmod 2$  for some  $3 \leq k \leq t$ , then  $u_k > 2e$  by Lemma 2.1 and Lemma 2.6 and  $r_k \geq u_k - e > e$ .

Since  $\text{ord } (1 - a^2) = 2 \text{ord } b + u_2 < 2e$ ,

$$\text{ord}(1 - a) = \text{ord}(1 + a) = \text{ord } b + u_2/2 < \text{ord } b + u_2 < e < r_k.$$

for  $k = 3, \dots, t$ . Therefore  $\tau_{\sigma x_1 - x_1} \in O(L)$ .

**COROLLARY 2.1.** *If  $\sigma x_1 \in L$  for some  $\sigma \in O(V)$ , then there is  $\tau$  a product of symmetries in  $O(L)$  such that  $\tau\sigma x_1 = x_1$ .*

*Proof.* It follows from the proof of Proposition 2.1. □

**REMARK 2.1.** The assumptions in Proposition 2.1 cannot be removed.

**THEOREM 2.1.** *If  $\theta(O^+(L)) \neq \dot{F}$ , then  $O(L) = X(L)$ .*

*Proof.* It follows from Proposition 2.1 and [OP2, 2.5] and induction on rank  $L$ . □

REMARK 2.2. In fact we have proved a slightly stronger result. If  $L$  does not satisfy the hypotheses of the above lemmas (Lemma 2.1, 2.2, 2.3, 2.4, 2.5, 2.6) and [X, Theorem 3.1], then Proposition 2.1 and Corollary 2.1 and Theorem 2.1 are still true.

REMARK 2.3. [EH1, Prop. 2.1] can follow from Remark 2.2.

**Acknowledgment.** I would like to thank Prof. L. J. Gerstein for helpful conversations and I also wish to thank Prof. J. S. Hsia who gave me a lot of help to improve the exposition of this paper.

#### REFERENCES

- [EH] A.G. Earnest and J.S. Hsia, *Spinor norms of local integral rotations, II*, Pacific J. Math., **61** (1975), 71-86, (see also *ibid.* **115** (1984), 493-494).
- [EH1] ———, *Spinor genera under field extensions, II : 2 unramified in bottom field*, Amer. J. Math., **100** (1978), 523-537.
- [H] J.S. Hsia, *Spinor norms of local integral rotations, I*, Pacific J. Math., **57** (1975), 199-206.
- [HSX] J.S. Hsia, Y.Y. Shao and F. Xu, *Representations by spinor genera, II*, in preparation.
- [K] M.Kneser, *Klassenzahlen indefiniter quadratischer formen in drei oder mehr veränderlichen*, Arch. Math., **7** (1956), 323-332.
- [L] T.Y. Lam, *Algebraic theory of quadratic forms*, Benjamin (1973).
- [O] O.T. O'Meara, *Introduction to quadratic forms*, Springer, New-York, 1963.
- [O1] ———, *Integral equivalence of quadratic forms in ramified local fields*, Amer. J. Math., **79** (1957), 157-186.
- [OP1] O.T. O'Meara and B. Pollack, *Generation of local integral orthogonal groups, I*, Math. Z., **87** (1965), 385-400.
- [OP2] ———, *Generation of local integral orthogonal groups, II*, Math. Z., **93** (1966), 171-188.
- [R] C. Riehm, *On the integral representations of quadratic forms over local fields*, Amer. J. Math., **86** (1964), 25-62.
- [X] F. Xu, *Integral spinor norms in dyadic local fields, I*, Pacific J. Math., **157** (1993), 179-200.

Received July 10, 1992.

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 167 No. 2 February 1995

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