A DIFFERENTIABLE STRUCTURE FOR A BUNDLE OF $C^*$-ALGEBRAS ASSOCIATED WITH A DYNAMICAL SYSTEM

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Let \((M, G)\) be a differentiable dynamical system, and \(\sigma\) be a transverse action for \((M, G)\). We have a differentiable bundle \((B, \pi, M, C)\) of \(C^*\)-algebras with respect to a flat family \(\mathcal{F}_\sigma\) of local coordinate systems and we have a flat connection \(\nabla\) in \(B\). If \(G\) is connected, the bundle \(B\) is a disjoint union of \(\rho_x(C^*_r(G))\) (\(x \in M\)), where \(G\) is the groupoid associated with \((M, G)\) and \(\rho_x\) is the regular representation of \(C^*_r(G)\). We show that, for \(f \in C^\infty_c(G)\), a cross section \(cs(f) : x \mapsto \rho_x(f)\) is differentiable with respect to the norm topology, and calculate a covariant derivative \(\nabla(cs(f))\). Though \(B\) is homeomorphic to the trivial bundle, the differentiable structure for \(B\) is not trivial in general. Let \(B^\sigma\) be a subbundle of \(B\) generated by elements \(f\) with the property \(\nabla(cs(f)) = 0\). We show the triviality of the differentiable structure for \(B^\sigma\) induced from that for \(B\) when \(C^*_r(G)\) is simple. We have a bundle \(RM(B)\) of right multiplier algebras and it contains \(B\) as a subbundle. Let \((M, G)\) be a Kronecker dynamical system and \(\sigma\) be a flow whose slope is rational. In this case, we have a subbundle \(D\) of \(RM(B)\) whose fibers are \(*\)-isomorphic to \(C(T)\). The flat connection \(\nabla^r\) in \(D\) is not trivial and the bundle \(B\) decomposes into the trivial bundle \(B^\sigma\) and the non-trivial bundle \(D\). Moreover, for a \(\sigma\)-invariant closed connected submanifold \(N\) of \(M\) with \(\dim N = 1\), we show that \(C^*_r(G|N)\) is \(*\)-isomorphic to \(C^*_r(D_x, \Phi_x)\), where \(\Phi_x\) is the holonomy group of \(\nabla^r\) with reference point \(x\). If \(G\) is not connected, we also have sufficiently many differentiable cross sections of \(B\) and calculate their covariant derivatives.

0. Introduction. In the theory of \(C^*\)-algebras, one sometimes study a stable \(C^*\)-algebra \(A \otimes K\) instead of studying a given \(C^*\)-algebra \(A\) itself, where \(K\) is the algebra of all compact operators on the infinite dimensional separable Hilbert space. There are many
other algebras $D$ such that $D \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Moreover, stable algebras do not have any identity elements. Therefore, given a stable $C^*$-algebra $C$, we want to find $C^*$-algebras $A$ with the property $A \otimes \mathcal{K} \cong C$, especially unital ones with the property. We do not know any general answer to the question, but there is a method to construct such algebras $A$ for foliation $C^*$-algebras. Let $(V, \mathcal{F})$ be a foliation and $C^*(V, \mathcal{F})$ be the foliation $C^*$-algebra introduced by A. Connes ([1], [3]). It follows from [10] that $C^*(V, \mathcal{F})$ is *-isomorphic to $C^*_r(\mathcal{G}|N) \otimes \mathcal{K}$, where $\mathcal{G}$ is the holonomy groupoid of $(V, \mathcal{F})$, where $N$ is a complete transverse submanifold and where the groupoid $\mathcal{G}|N$ is the reduction of $\mathcal{G}$ by $N$. Suppose that $V$ is compact. If we have $\dim N = \text{codim } \mathcal{F}$, then the $C^*$-algebra $C^*_r(\mathcal{G}|N)$ is unital. To give an example, if $(V, \mathcal{F})$ is a Kronecker foliation, then the $C^*$-algebra $C^*_r(\mathcal{G}|N)$ is the irrational rotation algebra $A_\theta$ for an appropriate $N$. This example plays an important role in the theory of non-commutative differential geometry by A. Connes. We refer the reader to the works of A. Connes [2], [3], that of A. Connes and M.A. Rieffel [4] and that of M.A. Rieffel [20]. M.A. Rieffel also studied the example in [17], [18] from the viewpoint of Morita equivalence. The author studied another example of $C^*_r(\mathcal{G}|N)$ in [12], [13].

From these considerations, we begin to study $C^*$-algebras of reductions of differentiable dynamical systems. Let $(M, G)$ be a differentiable dynamical system. We denote by $\mathcal{G}$ the topological groupoid $G \times M$ and denote by $C^*_r(\mathcal{G})$ the reduced $C^*$-algebra associated with $\mathcal{G}$. We have a regular representation $\rho_x$ of $C^*_r(\mathcal{G})$ on a Hilbert space $\mathcal{H}_x$ for every $x \in M$. For the moment we assume that $G$ is connected and that $C^*_r(\mathcal{G})$ is simple. We set $B_x = \rho_x(C^*_r(\mathcal{G}))$ and denote by $B$ the disjoint union of $C^*$-algebras $B_x$ ($x \in M$). We may consider elements $a$ of $C^*_r(\mathcal{G})$ to be cross sections $cs(a) : x \mapsto \rho_x(a)$ of the bundle $B$ on $M$. Continuous fields of $C^*$-algebras have been studied by many authors. We refer the reader to the book of J. Dixmier [5], those of J.M.G. Fell and R.S. Doran [8], [9], the work of B.D. Evans [6] and that of M.A. Rieffel [19]. Since we study $C^*$-algebras associated with differentiable dynamical systems, it is natural to consider differentiable structure for fields of $C^*$-algebras. In the previous paper [14], the author introduced the notion of differentiable bundles of $C^*$-algebras and
that of connections in them. A. Connes first introduced the notion of connections into the theory of $C^*$-algebras in \cite{2}. He defined the notion in the setting of projective modules. On the other hand, our definition of connections is in the setting of bundles of $C^*$-algebras and it is a literal translation of that in the setting of vector bundles, except that our connections are compatible with $*$-algebraic structures possessed by fibers.

In this paper, we introduce a notion of a transverse action $\sigma$ for $(M, G)$ and we construct a family $\mathcal{F}_\sigma$ of local coordinate systems for $B$ from local charts of $(M, G)$ compatible with $\sigma$. Then $\mathcal{F}_\sigma$ defines a differentiable structure for $B$. Next, we prove that the above cross section $cs(f)$ is differentiable with respect to the norm topology for every $f \in C^0_c(G)$. We define a flat connection $\nabla$ in $B$ with respect to $\mathcal{F}_\sigma$. Though $B$ is homeomorphic to the trivial bundle $M \times C^*_c(G)$, the differentiable structure for $B$ is not trivial, that is, $\nabla$ is not trivial. Let $B^\sigma$ be the subbundle of $B$ generated by elements $f$ with the property $\nabla(cs(f)) = 0$. Then $B^\sigma$ is trivial, that is, the restriction of $\nabla$ to $B^\sigma$ is trivial. We denote by $RM(B_x)$ the right multiplier algebra of $B_x$ and denote by $RM(B)$ the disjoint union of Banach algebras $RM(B_x)$ ($x \in M$). There exists a differentiable structure for $RM(B)$ such that $B$ is a subbundle of $RM(B)$ and such that $\nabla$ extends to a flat connection $\nabla^r$ in $RM(B)$. In the case where $(M, G)$ is a Kronecker dynamical system, we give a decomposition of $B$ into a trivial part and a non-trivial part. There exists a subbundle $D$ of $RM(B)$ such that every fiber $D_x$ is *-isomorphic to the commutative $C^*$-algebra $C(T)$ and such that $B^\sigma_xD_x$ generates $B_x$. Let $\nabla^r$ be the restriction of $\nabla^r$ to $D$ and let $\Phi_x$ be the holonomy group of $\nabla^r$ with reference point $x$. Note that $\Phi_x$ is a subgroup of the group Aut$(D_x)$ of all *-automorphisms of $D_x$. Let $N$ be a $\sigma$-invariant closed connected submanifold of $M$ with dim $N = 1$. Then we show that the $C^*$-algebra $C^*_r(G\mid N)$ is *-isomorphic to the reduced crossed product $C^*_r(D_x, \Phi_x)$ of $D_x$ by $\Phi_x$. This result means that $B$ decomposes into the trivial bundle $B^\sigma$ and the non-trivial bundle $D$ and that $D$ corresponds to the reduction of $(M, G)$ by $N$. This situation was studied by M.A. Rieffel in \cite{17}, \cite{18} from the viewpoint of projective modules. Our result describes the same situation from the viewpoint of vector bundles.

When $G$ is not connected, we also define a differentiable bundle
B associated with a transverse action for (M, G) and define a flat connection \( \nabla \) in \( B \). But, in this case, \( B_x \) is larger than \( \rho_x(C^*_r(\mathcal{G})) \) and cross sections \( cs(f) \) may not be differentiable. We define a cross section \( cs_\mu(f) \) of \( B \) for \( f \in C^\infty_r(\mathcal{G}) \) and every connected component \( m \) of \( G \), and we show that the cross sections \( cs_\mu(f) \) are differentiable. The *-algebra \( \mathcal{D}_x \) generated by elements of the form \( cs_\mu(f)_x \) is dense in \( B_x \) with respect to the strong operator topology. The above results are valid even if \( G \) is discrete.

To find a transverse action for a given dynamical system \( (M, G) \), it may be useful to consider the universal covering space \( \tilde{M} \) of \( M \). Suppose that the action of \( G \) on \( M \) lifts to an action of \( G \) on \( \tilde{M} \). (If \( G \) is simply connected, this assumption is satisfied.) If there exists a transverse action for \( (\tilde{M}, G) \) and if it is compatible with the covering map, then we have a transverse action for \( (M, G) \). But we do not know any interesting examples of transverse actions for dynamical systems \( (M, G) \) such that the connected components \( G_e \) of \( G \) are not abelian, and it is difficult to find such examples. This is the problem for further investigation.

1. Preliminaries. (a) Commutative dynamical systems. Let \( (M, G) \) be a topological transformation group. We assume that a topological space \( M \) and a topological group \( G \) are second countable, Hausdorff and locally compact. We denote by \( \mathcal{G} \) a topological groupoid \( G \times M \) with the following operations; \( s(g, x) = (e, x), r(g, x) = (e, gx), (g', gx)(g, x) = (g'g, x), (g, x)^{-1} = (g^{-1}, gx) \) for \( x \in M \) and \( g, g' \in G \), where \( e \) is the unit of \( G \). We set \( \mathcal{G}_x = \{(g, x) \in \mathcal{G}; g \in G\} \) for \( x \in M \). Let \( \mu \) be a right Haar measure on \( G \) and \( \Delta \) be the modular function of \( G \). We define a right Haar system \( \{\nu_x; x \in M\} \) on \( \mathcal{G} \) by \( \nu_x = \mu \times \delta_x \). Let \( C_c(\mathcal{G}) \) be the *-algebra of continuous functions with compact supports, where the product and the involution are defined as follows:

\[
(f_1 * f_2)(g, x) = \int_G f_1(g'^{-1}, g'gx)f_2(g'g, x) \, d\mu(g'),
\]

\[
f^*(g, x) = \overline{f(g^{-1}, gx)}
\]

for \( f, f_1, f_2, \in C_c(\mathcal{G}) \) and \( (g, x) \in \mathcal{G} \). We denote by \( \mathcal{H}_x \) the Hilbert space \( L^2(\mathcal{G}_x, \nu_x) \) for \( x \in M \). We define the regular representation
\( \rho_x \) of \( C_c(\mathcal{G}) \) on \( \mathcal{H}_x \) by

\[
(\rho_x(f)\xi)(g, x) = \int_G f(g^{-1}, g'x)\xi(g', x) \, d\mu(g')
\]

for \( f \in C_c(\mathcal{G}), \xi \in \mathcal{H}_x \) and \((g, x) \in \mathcal{G}_x\). We define the reduced norm \( ||f|| \) by \( ||f|| = \sup_{x \in M} ||\rho_x(f)|| \). We denote by \( C^*_r(\mathcal{G}) \) the completion of \( C_c(\mathcal{G}) \) by the reduced norm. The representation \( \rho_x \) extends to a representation of \( C^*_r(\mathcal{G}) \), which we denote again by \( \rho_x \). For details of groupoids and their \( C^* \)-algebras, we refer the reader to [1], [3] and [16].

**Lemma 1.1.** Let \( f \) be an element of \( C_c(\mathcal{G}) \) and \( D \) be a compact set in \( G \) such that \( \text{supp } f \subseteq D \times M \). Then the following inequality holds: \( ||\rho_x(f)|| \leq I_D||f||_\infty \), where \( ||f||_\infty \) is the supremum norm of \( f \) and \( I_D = \int_D \Delta^{1/2}(g) \, d\mu(g) \).

**Proof.** Let \( \chi_D \) be the characteristic function of \( D \). For \( \xi, \eta \in \mathcal{H}_x \), we have

\[
\int_G |f(g^{-1}, g'x)\xi(g', x)\eta(g, x)| \, d\mu(g) \\
\leq \left( \int_G |f(g^{-1}, g'x)||\xi(g', x)|^2 \, d\mu(g) \right)^{1/2} \\
\cdot \left( \int_G |f(g^{-1}, g'x)||\eta(g, x)|^2 \, d\mu(g) \right)^{1/2} \\
\leq ||f||_\infty \chi_D(g^{-1})||\xi||\Delta^{1/2}(g')||\eta||\xi||.
\]

Then we have \( ||(\rho_x(f)\xi)\eta|| \leq I_D||f||_\infty||\xi||||\eta||\xi||. \)

We introduce a \(*\)-algebra of functions on \( G \times G \). Let \( \tilde{C} \) be the set of bounded continuous functions \( K \) on \( G \times G \) with the following property; there exists a compact set \( D \) in \( G \) such that \( \text{supp } K \subseteq G \times D \). The set \( D \) may vary when \( K \) varies. Then \( \tilde{C} \) is a \(*\)-algebra with the following product and involution;

\[
(K_1 * K_2)(g, g') = \int_G K_1(g, g^{-1})K_2(g''g, g''g') \, d\mu(g''),
\]

\[
K^*(g, g') = \overline{K(g^{-1}g, g^{-1})}
\]

for \( K, K_1, K_2 \in \tilde{C} \) and \((g, g') \in G \times G \). We denote by \( \mathcal{H} \) the Hilbert space \( L^2(G, \mu) \). We define a \(*\)-representation \( \rho \) of \( \tilde{C} \) on \( \mathcal{H} \) by

\[
(\rho(K)\xi)(g) = \int_G K(g, g^{-1})\xi(g'g) \, d\mu(g')
\]
for $K \in \mathcal{C}$, $\xi \in \mathcal{H}$ and $g \in G$. We can prove the following lemma by a similar computation to that in the proof of Lemma 1.1.

**Lemma 1.2.** Let $K$ be an element of $\mathcal{C}$ and $D$ be a compact set in $G$ such that $\text{supp } K \subset G \times D$. Then the following inequality holds: $||\rho(K)|| \leq I_D||K||_\infty$.

(b) Differentiable bundles of $C^*$-algebras. With a few modifications on the definitions in [14, §1], we summarize the necessary facts. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and $x_1, \ldots, x_n$ be the canonical coordinate functions of $\mathbb{R}^n$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $f$ be a map of $\Omega$ into a Banach space $C$. If there exists $\lim_{h \to 0} h^{-1}(f(x + he_i) - f(x))$ with respect to the norm in $C$, then we denote the limit by $(\partial f/\partial x_i)(x)$. We say that $f$ is differentiable of class $(C^\infty)'$ on $\Omega$ if the partial derivatives $\partial^\alpha f/\partial x^\alpha$ exist and are continuous on $\Omega$ for all multi-indices $\alpha$.

**Definition 1.3.** (c.f. [14, Definition 1.1]). Let $M$ be a finite dimensional real manifold of class $C^\infty$ and $\mathcal{A}$ be the complete atlas defining the structure of $M$. A map $f$ of $M$ into a Banach space $C$ is said to be of class $C^\infty$ if $f \circ \varphi^{-1}$ is of class $(C^\infty)'$ on $\varphi(U)$ for every $(U, \varphi) \in \mathcal{A}$.

We assume that a real manifold $M$ is second countable, Hausdorff and of class $C^\infty$. Let $B$ be a topological space, $C$ be a $C^*$-algebra and $\pi$ be a continuous map of $B$ onto $M$. We set $B_x = \pi^{-1}(x)$ for $x \in M$ and suppose that $B_x$ is a $C^*$-algebra. (It is easy to rewrite the rest of this section for Banach algebras $C$ and $B_x$. We leave it to the reader.) Let $\{U_i\}$ be an open covering of $M$ indexed by a set $I$ and $\psi_i$ be a homeomorphism of $\pi^{-1}(U_i)$ onto $U_i \times C$ such that $p_1 \circ \psi_i(b) = \pi(b)$ for $b \in \pi^{-1}(U_i)$, where $p_1 : U_i \times C \to U_i$ is the projection. For $x \in U_i$, we define a map $\psi_{i,x}$ of $B_x$ into $C$ by $\psi_{i,x}(b) = p_2 \circ \psi_i(b)$ for $b \in B_x$, where $p_2 : U_i \times C \to C$ is the projection. We denote by $\mathcal{F}$ the set of pairs $(U_i, \psi_i)$ ($i \in I$).

**Definition 1.4.** (c.f. [14, Definition 1.2]). A quartet $(B, \pi, M, C)$ is called a differentiable bundle of $C^*$-algebras with respect to $\mathcal{F}$ if $\mathcal{F}$ satisfies the following conditions:

(i) For every $i \in I$ and $x \in U_i$, $\psi_{i,x}$ is a $*$-isomorphism between $C^*$-algebras.

(ii) For $i, j \in J$ with $U_i \cap U_j \neq \emptyset$ and for a map $f$ of $U_i \cap U_j$ into
C, define the map $f_{i,j}$ of $U_i \cap U_j$ into $C$ by $f_{i,j}(x) = \psi_{i,x} \circ \psi_{j,x}^{-1} \circ f(x)$. If $f$ is of class $C^\infty$, then $f_{i,j}$ is of class $C^\infty$.

Let $\mathcal{F}$ be a family satisfying the above condition (i). We say that $\mathcal{F}$ is a flat family of $C^*$-coordinate systems if it satisfies the following conditions:

(iii) For every $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ and for every connected component $U$ of $U_i \cap U_j$, there exists a $*$-automorphism $\alpha$ of the $C^*$-algebra $C$ such that $\alpha = \psi_{i,x} \circ \psi_{j,x}^{-1}$ for all $x \in U$.

Let $\xi$ be a map of an open set $U$ of $M$ into $\pi^{-1}(U)$ such that $\pi(\xi_x) = x$ for $x \in U$. For $i \in I$ with $U_i \cap U \neq \emptyset$, define the map $\xi_i$ of $U_i \cap U$ into $C$ by $\xi_i(x) = \psi_{i,x}(\xi_x)$. We say that $\xi$ is a differentiable cross section on $U$ if $\xi_i$ is of class $C^\infty$ for every $i \in I$ with $U_i \cap U \neq \emptyset$. We denote by $\Gamma(B)$ the $*$-algebra of all differentiable cross sections on $M$. Let $TM$ be the tangent bundle on $M$, $\Gamma(TM)$ be the space of $C^\infty$ vector fields on $M$ and $T^*M$ be the cotangent bundle on $M$. We denote by $T^*M \otimes B$ the tensor product of $T^*M$ and $B$ as real vector bundles. Let $\xi$ be a cross section of $T^*M \otimes B$. If $x_1, \ldots, x_n$ is a local coordinate system in $M$, then we have $\xi_x = \sum(dx_k)_x \otimes b^k_x$ with $b^k_x \in B_x$. We say that $\xi$ is differentiable if the cross sections $x \mapsto b^k_x$ are differentiable. Let $\Gamma(T^*M \otimes B)$ be the two-sided $\Gamma(B)$-module of differentiable cross sections of $T^*M \otimes B$. We define the involution on $\Gamma(T^*M \otimes B)$ by $\xi^* = \sum(dx_k)_x \otimes (b^k_x)^*$. We denote by $C^\infty(M, \mathbb{R})$ the space of real-valued $C^\infty$ functions on $M$.

**Definition 1.5.** (c.f. [14, Definition 1.3]). Let $(B, \pi, M, C)$ be a differentiable bundle of $C^*$-algebras and $\mathcal{D}$ be a $*$-subalgebra of $\Gamma(B)$ such that $f\xi \in \mathcal{D}$ for $f \in C^\infty(M; \mathbb{R})$ and $\xi \in \mathcal{D}$. A linear map $\nabla$ of $\mathcal{D}$ into $\Gamma(T^*M \otimes B)$ is called a connection in $B$ with domain $\mathcal{D}$ if it satisfies the following conditions: (i) $\nabla(f\xi) = df \otimes \xi + f\nabla\xi$, (ii) $\nabla(\xi\eta) = (\nabla\xi)\eta + \xi(\nabla\eta)$, (iii) $(\nabla\xi)(X) \in \mathcal{D}$, (iv) $\nabla(\xi^*) = (\nabla\xi)^*$ for $\xi, \eta \in \mathcal{D}$, $f \in C^\infty(M; \mathbb{R})$ and $X \in \Gamma(TM)$.

Suppose that the family $\mathcal{F}$ is flat. Let $\nabla$ be a connection in $B$ with domain $\Gamma(B)$ and $(V, x_1, \ldots, x_n)$ be a local coordinate system in $M$. For $\xi \in \Gamma(B)$ and $i \in I$, we set $\xi_i(x) = \psi_{i,x}(\xi_x)$. We say that $\nabla$ is a flat connection if we have

$$\psi_{i,x}((\nabla\xi)(X)_x) = \sum_{k=1}^n a_k(x) \frac{\partial \xi_i}{\partial x_k}(x) \quad (x \in V \cap U_i),$$

for $X \in \Gamma(TM)$ with $X_x = \sum a_k(x)(\partial/\partial x_k)_x$ (c.f. [14, Definition 1.6],

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LEMMA 1.6. If \((B, \pi, M, C)\) is a differentiable bundle of \(C^*\)-algebras with respect to a flat family \(\mathcal{F}\), then there exists a unique flat connection in \(B\).

2. Transverse actions and bundles of \(C^*\)-algebras. Let \(M\) be an \(n\)-dimensional real manifold of class \(C^\infty\) and \(G\) be a \(p\)-dimensional real Lie group of class \(C^\infty\). In the following sections, we assume that \(M\) and \(G\) are second countable and Hausdorff and that \(0 < n \leq \infty\) and \(0 \leq p < \infty\). If \(p = 0\), then \(G\) is a countable discrete group. Moreover we assume that \(M\) is connected. Suppose that \((M, G)\) is a differentiable dynamical system, that is, \((M, G)\) is a transformation group and the map \((g, x) \mapsto gx\) of \(G \times M\) into \(M\) is of class \(C^\infty\). Let \(G_e\) be the connected component of the unit \(e\) in \(G\). We denote by \(\mathcal{N}\) the countable discrete group \(G/G_e\) and denote by \(G_m\) the connected component of \(G\) corresponding to \(m \in \mathcal{N}\).

We take notations from §1, and also use the following notations;
\[ g_m = G_m \chi_M, \quad g_m^* = g_m^* \chi_M, \quad n_m = L^2(G_m, \mu|G_m), \quad n_m^* = L^2(G_m, \chi|G_m), \quad f_m \in \Lambda^*(G) \]
and we set \[ ||\zeta|| = \sup_m ||f_m||_\infty \]

LEMMA 2.1. For \(\zeta = (f_m)_{m \in \mathcal{N}} \in \Lambda(\mathcal{G})\), the sum \(\tilde{\rho}_x(\zeta) = \sum_{m \in \mathcal{N}} \rho_x(f_m)P^m_x\) converges with respect to the strong operator topology in \(\mathcal{B}(\mathcal{H}_x)\), and the following inequality holds:
\[ ||\tilde{\rho}_x(\zeta)|| \leq J_D ||\zeta|| \]
where \(D\) is any compact set in \(G\) such that \(\text{supp} f_m \subset D \times M\) for all \(m \in \mathcal{N}\). We set \(||\zeta|| = \sup_m ||f_m||_\infty\).

Proof. We set \(D_m = D \cap G_m\). There exist elements \(m(1), \ldots, m(k)\) of \(\mathcal{N}\) such that \(D\) is the disjoint union of non-empty sets \(D_{m(1)}, \ldots, D_{m(k)}\). Then we have
\[ \rho_x(f_m)P^m_x = \sum_{l \in A(m)} P^l_x \rho_x(f_m)P^m_x \]
where \(A(m) = \{m(i) : i = 1, \ldots, k\}\). If we have \(P^l_x \rho_x(f_m)P^m_x \zeta(g, x) \neq 0\), then there exists \(g' \in G_m\) such that \(gg'^{-1} \in D_{m^{-1}}\). This implies that we have \(lm^{-1} = m(i)\) for some \(i\) with \(1 \leq i \leq k\). We set
$B(l) = \{m(i)^{-1}l \in \mathcal{N}; i = 1, \ldots, n\}$. We have

$$\left\| \sum_{m \in \mathcal{N}} \rho_x(f_m) P_x^m \xi \right\|^2 \leq k \sum_{l \in \mathcal{N}} \sum_{m \in B(l)} \left\| P_x^l \rho_x(f_m) P_x^m \xi \right\|^2.$$

Note that $m \in B(l)$ if and only if $l \in A(m)$. Thus we have

$$\sum_{l \in \mathcal{N}} \sum_{m \in B(l)} \left\| P_x^l \rho_x(f_m) P_x^m \xi \right\|^2 \leq I_D^2 \left\| \xi \right\|^2 \sum_{m \in \mathcal{N}} \left\| P_x^m \xi \right\|^2.$$

Let $B_x$ be the $C^*$-subalgebra of $B(\mathcal{H}_x)$ generated by $\{\tilde{\rho}_x(\zeta); \zeta \in \mathcal{N}(G)\}$. Since we have $\tilde{\rho}_x(\zeta) = \rho_x(f)$ for $\zeta = (f_m)$ with $f_m = f$ for all $m \in \mathcal{N}$, $B_x$ contains $\rho_x(C^*_r(G))$. If $G$ is connected, then we have $B_x = \rho_x(C^*_r(G))$. If $G$ is not connected, then $B_x$ may not be separable. For $x \in M$ and $f \in C_c(G)$, we define $K^f_x = \tilde{\mathcal{C}}$ by $K^f_x(g, g') = f(g', g^{-1}gx)$. For $m \in \mathcal{N}$, we define $\chi_m \in C^\infty(G \times G)$ as follows; $\chi_m(g, g') = 1$ if $g'g \in G_m$ and $\chi_m(g, g') = 0$ otherwise. For $\zeta = (f_m) \in \mathcal{N}(G)$, we define $K^\zeta_x = \tilde{\mathcal{C}}$ by $K^\zeta_x = \sum_{m \in \mathcal{N}} K^f_{x_m} \chi_m$. We denote by $C_x$ the $C^*$-subalgebra of $B(\mathcal{H})$ generated by $\{\rho(K^\zeta_x); \zeta \in \mathcal{N}(G)\}$. We define an isometry $T$ of $\mathcal{H}_x$ onto $\mathcal{H}$ by $(T\eta)(g) = \eta(g, x)$ for $\eta \in \mathcal{H}_x$. We set $\psi_x(a) = TaT^*$ for $a \in B_x$.

For $g \in G_e$ and $a \in C_x$, we set $\psi(x, g)(a) = R_gaR_g^*$, where $R$ is the right regular representation of $G$ on $\mathcal{H}$. Then we have:

**Lemma 2.2.** For $x \in M$, there exists a unique spatial isomorphism $\psi_x$ of $B_x$ onto $C_x$ such that $\psi_x(\tilde{\rho}_x(\zeta)) = \rho(K^\zeta_x)$ for $\zeta \in \mathcal{N}(G)$.

**Lemma 2.3.** For $x \in M$ and $g \in G_e$, there exists a unique spatial isomorphism $\psi(x, g)$ of $C_x$ onto $C_{g_x}$ such that $\psi(x, g)(\rho(K^\zeta_{g_x})) = \rho(K^\zeta_{g_x})$ for $\zeta \in \mathcal{N}(G)$.

We denote by $\text{Diff}_G(M)$ the group of diffeomorphisms of $M$ which commute with the action of the connected component $G_e$ on $M$. For $\alpha \in \text{Diff}_G(M)$ and $m \in \mathcal{N}$, there exists a diffeomorphism $\alpha_m$ such that $g \alpha_m(x) = \alpha_m(gx)$ for all $g \in G_m$ and $x \in M$. If $G$ is discrete, then we have $\text{Diff}_G(M) = \text{Diff}(M)$, the group of all diffeomorphisms on $G$. For $\alpha \in \text{Diff}_G(M)$ and $\zeta = (f_m) \in \mathcal{N}(G)$, we define $\tilde{\alpha}(\zeta) \in \mathcal{N}(G)$ by $\tilde{\alpha}(\zeta) = (\tilde{\alpha}_m(f_m))$, where $\tilde{\alpha}_m(f_m)(g, x) = f_m(g, \alpha_m^{-1}(x))$. For $\zeta \in \mathcal{N}(G)$, we have $K^{\zeta^{-1}_\alpha} = K^{\tilde{\alpha}_\zeta}$. Thus we have:
Lemma 2.4. For $\alpha \in \text{Diff}_G(M)$ and $x \in M$, $C_x = C_{\alpha(x)}$.

Remember that $\dim M = n$ and $\dim G = p$. We assume that $n \geq p$. Let $\sigma : \mathbb{R}^{n-p} \to \text{Diff}_G(M)$ be a differentiable action, that is, $\sigma$ is a homomorphism and the map $(x, t) \mapsto \sigma_t(x)$ is of class $C^\infty$.

Definition 2.5. Let $U$ be a connected open set in $M$. Suppose that there exists a $C^\infty$ diffeomorphism $\varphi$ of $U$ onto $S \times T$, where $S$ is an open set in $G_e$ with $e \in S$ and $T$ is an open set in $\mathbb{R}^{n-p}$ with $0 \in T$. Then the pair $(U, \varphi)$ is called a local chart of $(M, G)$ compatible with $\sigma$ if it satisfies the following conditions:

(i) $\varphi^{-1}(g, t) = g\varphi^{-1}(e, t)$,
(ii) $\varphi^{-1}(g, t) = \sigma_t(\varphi^{-1}(g, 0))$ for all $(g, t) \in S \times T$.

Let $(U, \varphi)$ be a local chart compatible with $\sigma$ as above. We set $x_0 = \varphi^{-1}(e, 0)$. For $x \in U$ with $\varphi(x) = (g, t)$, we have $g^{-1}x = \sigma_t(x_0)$. It follows from Lemmas 2.2, 2.3 and 2.4 that the map $\Psi(x, g^{-1}) \circ \tilde{\psi}_x$ is a spatial isomorphism of $B_x$ onto $C_{x_0}$ for $x \in U$ with $\varphi(x) = (g, t)$. We set $\psi_x = \Psi(x, g^{-1}) \circ \tilde{\psi}_x$. Then we have the following:

Proposition 2.6. Let $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ be local charts compatible with $\sigma$ and $U$ be a connected component of $U_1 \cap U_2$. If $\psi_{i,x}$ is the *-isomorphism of $B_x$ onto $C_{x_i}$ as above with respect to $(U_i, \varphi_i)$ with $x_i = \varphi_i^{-1}(e, 0)$ $(i = 1, 2)$, then there exists a *-isomorphism $\alpha$ of $C_{x_1}$ onto $C_{x_2}$ such that $\alpha = \psi_{2,x} \circ \psi_{1,x}^{-1}$ for all $x \in U$.

Proof. For $i = 1, 2$, we set $\varphi_i(U_i) = S_i \times T_i$ as in Definition 2.5. We fix $x \in U$ and suppose that $\varphi_i(x) = (g_i, t_i)$ $(i = 1, 2)$. Let $x'$ be an element of $U$ such that $\varphi_i(x') = (g'_i, t'_i)$ $(i = 1, 2)$. We set $g_0 = g'_1g_1^{-1}$ and $t_0 = t_1 - t'_1$. Let $U_0$ be a sufficiently small neighborhood of $x$ in $U$. For $x' \in U_0$, we have $g_0x = \sigma_{t_0}(x')$, $\varphi_2(g_0x) = (g_0g_2, t_2)$ and $\varphi_2(\sigma_{t_0}(x')) = (g'_2, t_0 + t'_2)$. Since we have $(g_0g_2, t_2) = (g'_2, t_0 + t'_2)$, we have $g_2^{-1}g_1 = g'_2^{-1}g'_1$. Since we have $\psi_{2,x} \circ \psi_{1,x}^{-1} = \Psi(x_1, g'_2^{-1}g'_1)$ and $\psi_{2,x'} \circ \psi_{1,x'}^{-1} = \Psi(x_1, g'_2^{-1}g'_1)$, we have $\psi_{2,x} \circ \psi_{1,x}^{-1} = \psi_{2,x'} \circ \psi_{1,x'}^{-1}$. Since $U_0$ is a neighborhood of $x$, this completes the proof of Proposition 2.6.

We denote by $B$ the disjoint union of $C^*$-algebras $\{B_x; x \in M\}$ and denote by $\pi$ the map of $B$ onto $M$ defined by $\pi(a) = x$ for $a \in B_x$. Let $\{(U_i, \varphi_i)\}$ be the set of all local charts of $(M, G)$ compatible
with \( \sigma \) indexed by a set \( I \) and let \( \psi_{i,x} \) be the \(*\)-isomorphism of \( B_x \) onto \( C_{x_i} \) constructed as above from \((U_i, \varphi_i)\) with \( x_i = \varphi_i^{-1}(e,0) \). We define a map \( \psi_i \) of \( \pi^{-1}(U_i) \) onto \( U_i \times C_{x_i} \) by \( \psi_i(a) = (x, \psi_{i,x}(a)) \) for \( a \in B_x \). Let \( \mathcal{F}_\sigma \) be the set of pairs \((U_i, \psi_i)\) \((i \in I)\) constructed as above.

**Definition 2.7.** A differentiable action \( \sigma \) is called a transverse action for \((M,G)\) if the family \( \{U_i; i \in I\} \) is an open covering of \( M \).

In the following we assume that \( \sigma \) is a transverse action for \((M,G)\). It follows from Proposition 2.6 that there exists a unique topology on \( B \) such that \( \pi \) is continuous and \( \psi_i \) is a homeomorphism for all \( i \in I \). Since \( M \) is connected, the \( \mathbb{C}^* \)-algebras \( C_x \) are mutually \(*\)-isomorphic. Therefore, for a fixed \( \tilde{x} \in M \), we set \( C = C_{\tilde{x}} \) and fix a \(*\)-isomorphism between \( C \) and \( C_{x_i} \) for every \( i \in I \), and then we identify \( C_{x_i} \) with \( C \) by this isomorphism. Thus we consider \( \psi_{i,x} \) to be a \(*\)-isomorphism of \( B_x \) onto \( C \) and \( \psi_i \) to be a homeomorphism of \( \pi^{-1}(U_i) \) onto \( U_i \times C \). By virtue of Proposition 2.6, we have the following theorem:

**Theorem 2.8.** Suppose that \( \sigma \) is a transverse action for \((M,G)\). Then the quartet \((B, \pi, M, C)\) constructed above is a differentiable bundle of \( \mathbb{C}^* \)-algebras with respect to the flat family \( \mathcal{F}_\sigma \) of \( \mathbb{C}^* \)-coordinate systems.

### 3. Differentiable cross sections.

For \( f \in \mathcal{C}_c^\infty(G) \) and \( m \in \mathcal{N} \), we define an element \([f]_m = (f_m)\) of \( \mathcal{N}(G) \) by \( f_m = f \) and \( f_k = 0 \) if \( k \neq m \), and define the cross section \( cs_m(f) \) of \( B \) by \( cs_m(f)_x = \tilde{\rho}_x([f]_m) \) \((x \in M)\), that is, \( cs_m(f)_x = \rho_x(f)P^m_x \). If \( G \) is connected, we set \( cs(f) = cs_e(f) \), where \( \mathcal{N} = \{e\} \), and we have \( cs(f)_x = \rho_x(f) \). Let \( \sigma^m : \mathbb{R}^{n-p} \rightarrow \text{Diff}(M) \) be a differentiable action such that \( \sigma^m = g \circ \sigma \circ g^{-1} \) for every \( g \in G_m \). We prepare a lemma for proving the differentiability of \( cs_m(f) \).

**Lemma 3.1.** For \( F \in \mathcal{C}_c^\infty(\mathbb{R}^{n-p} \times G) \) and \( t \in \mathbb{R}^{n-p} \), define an element \( F_t \) of \( \mathcal{C}_c^\infty(G) \) by \( F_t(g,x) = F(t,g,x) \). Let \( t_0 \) be an element of \( \mathbb{R}^{n-p} \). (i) The supremum norm \( ||F_t - F_{t_0}||_\infty \) converges to 0 as \( t \to t_0 \). (ii) Let \( J \) be an open interval in \( \mathbb{R} \) containing 0, and let \( t(\cdot) \) be a \( C^2 \) map of \( J \) into \( \mathbb{R}^{n-p} \) with \( t(0) = t_0 \). Define an element \( f \) of \( C_c^\infty(G) \) by \( f(g,x) = \sum_{i=1}^{n-p} \frac{\partial F}{\partial t_i}(t_0,g,x)(dt_i/dh)(0) \), where
Then \( \|h^{-1}(F(h) - F_0) - f\|_\infty \) converges to 0 as \( h \to 0 \).

The proof is elementary, and we omit it.

**Theorem 3.2.** The cross section \( cs_m(f) \) is differentiable, that is, \( cs_m(f) \in \Gamma(B) \), for every \( f \in C^\infty_c(G) \) and \( m \in \mathcal{N} \).

**Proof.** We fix \( i \in I \), that is, we fix \((U_i, \psi_i)\) in \( \mathcal{F}_\sigma \) and a local chart \((U_i, \varphi_i)\) compatible with \( \sigma \). Recall that \( \varphi_i \) is a diffeomorphism of \( U_i \) onto \( S \times T \), where \( S \) and \( T \) are open sets of \( G_e \) and \( \mathbb{R}^{n-p} \) respectively. Let \((U_0, \varphi_0)\) be a local coordinate system of \( M \) such that \( U_i \cap U_0 \neq \emptyset \). We set \( U = U_i \cap U_0 \) and \( V = \varphi_0(U) \). We define \( C^\infty \) map \( x(v) \) of \( V \) into \( U \) by \( x(v) = \varphi_0^{-1}(v) \) and define \( C^\infty \) maps \( g(v) \) of \( V \) into \( S \) and \( t(v) \) of \( V \) into \( T \) by \( \varphi_i(x(v)) = (g(v), t(v)) \). We set \( \xi = cs_m(f) \) and define maps \( \xi_i \) of \( U_i \) into \( C \) and \( \eta \) of \( V \) into \( C \) by \( \xi_i(x) = \psi_i(x) \) and \( \eta = \xi_i \circ \varphi_0^{-1} \) respectively. It follows from Lemmas 2.2 and 2.3 that we have \( \eta(v) = \rho(K_{\xi[i]}^{m-1}x(v)) \). We have \( g(v)^{-1}x(v) = \sigma_{t(v)}(x_i) \), where \( x_i = \varphi_i^{-1}(e, 0) \). We define an element \( F \) of \( C^\infty(\mathbb{R}^{n-p} \times G) \) by \( F(t, g, x) = f(g, \sigma_{t}^{m}(x)) \). We have

\[
\left\| K_{\sigma_{t(u)}(x_i)}^{f} \chi_{m} - K_{\sigma_{t(u)}(x_i)}^{f} \chi_{m} \right\|_\infty \leq \left\| F_{t(u)} - F_{t(u)} \right\|_\infty, \text{ for } u, v \in V.
\]

Let \( E \) be a compact set in \( G \) such that \( \text{supp } f \subset E \times M \). It follows from Lemma 1.2 that we have \( \|\eta(v) - \eta(u)\| \leq I_E \|F_{t(v)} - F_{t(u)}\|_\infty \). By virtue of Lemma 3.1 we know that \( \eta \) is continuous on \( V \).

Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \) and \( v_1, \ldots, v_n \) be coordinate functions of \( \mathbb{R}^n \) associated with \( e_1, \ldots, e_n \). We fix an element \( u \) of \( V \). For a fixed \( k = 1, \ldots, n \), let \( \delta > 0 \) be such that \( u + he_k \in V \) for \( |h| < \delta \). We denote by \( J \) the interval \( \{h; |h| < \delta\} \) in \( \mathbb{R} \). We define a \( C^\infty \) map \( \tau \) of \( J \) into \( T \) by \( \tau(h) = t(u + he_k) \). For \( j = 1, \ldots, n - p \), we define an element \( f_j^{m} \) of \( C^\infty_c(G) \) by \( f_j^{m}(g, x) = (\partial/\partial t_j)(f(g, \sigma_{t}^{m}(x)))_{t=0} \). We set \( t(v) = (t_1(v), \ldots, t_{n-p}(v)) \) and \( \tau(h) = (\tau_1(h), \ldots, \tau_{n-p}(h)) \). We define an element \( a \) of \( C^\infty_c(G) \) by \( a(g, x) = \sum_{j=1}^{n-p}(\partial F/\partial t_j)(\tau(0), g, x)(d\tau_j/dh)(0) \). It follows from Lemma 3.1 that \( h^{-1}(F_{\tau(h)} - F_{\tau(0)}) \) converges to \( a \) as \( h \to 0 \). Let \( \tilde{K}_h \)
be a function on $G \times G$ such that $\tilde{K}_h(g, g')$ is equal to

$$h^{-1}\left\{K_{\sigma_{r(h)}(x_i)}^f \chi_m - K_{\sigma_{r(0)}(x_i)}^f \chi_m\right\}(g, g')$$

$$- \sum_{j=1}^{n-p} \left(K_{\sigma_{r(0)}(x_i)}^m \chi_m\right)(g, g') \frac{\partial t_j}{\partial v_k}(u).$$

We have $||\tilde{K}_h||_\infty \leq ||h^{-1}(F_{\tau(h)} - F_{\tau(0)}) - a||_\infty$. We set $\xi^j = c_{s_m}(f_j^m)$ and define maps $\xi^j_1$ of $U_i$ onto $C$ and $\eta^j$ of $V$ into $C$ by $\xi^j_1(x) = \psi_{i,x}(\xi^j_1)$ and $\eta^j = \xi^j_1 \circ \varphi_0^{-1}$ respectively. It follows from Lemma 1.2 that we have

$$\left\|h^{-1}(\eta(u + he_k) - \eta(u)) - \sum_{j=1}^{n-p} \eta^j(u) \frac{\partial t_j}{\partial v_k}(u)\right\|$$

$$\leq I_E ||h^{-1}(F_{\tau(h)} - F_{\tau(0)}) - a||_\infty.$$ 

Therefore we have $(\partial \eta/\partial v_k)(u) = \sum_{j=1}^{n-p} \eta^j(u) (\partial t_j/\partial v_k)(u)$. As we have seen in the first half of this proof, $\eta^j$ is continuous on $V$. Therefore $\eta$ is of class $(C^1)'$ in the sense of §1. Similarly $\eta^j$ is of class $(C^1)'$ for $j = 1, \ldots, n - p$. Therefore we know that $\eta$ is of class $(C^\infty)'$ and that $\xi^j_1$ is of class $C^\infty$ in the sense of Definition 1.3. This completes the proof of Theorem 3.2. \qed

Recall that $\Gamma(B)$ is not only a $*$-algebra but also a $C^\infty(M)$-module. We denote by $\mathcal{D}$ the $*$-subalgebra of $\Gamma(B)$ generated by elements of the form $\omega \cdot c_{s_m}(f)$ with $f \in C^\infty_c(\mathcal{G})$, $m \in \mathcal{N}$ and $\omega \in C^\infty(M)$. Then $\mathcal{D}$ is also a $C^\infty(M)$-submodule of $\Gamma(B)$. For $x \in M$, we set $\mathcal{D}_x = \{ \xi \in B_x; \xi \in \mathcal{D}\}$. Note that $\mathcal{D}_x$ is the $*$-subalgebra of $B_x$ generated by elements of the form $\rho_x(f) P_x^m$ with $f \in C^\infty_c(\mathcal{G})$ and $m \in \mathcal{N}$. If $\mathcal{N}$ is finite, then $\mathcal{D}_x$ is dense in the norm topology of $B_x$ for every $x \in M$. If $\mathcal{N}$ is infinite, then $\mathcal{D}_x$ may not be dense in the norm topology, but it is dense in the strong operator topology of $B_x$ by Lemma 2.1.

4. Flat connections. It follows from Lemma 1.6 that there exists a unique flat connection $\nabla$ in $B$. In this section we calculate $\nabla(c_{s_m}(f))$ explicitly.

**Lemma 4.1.** For $j = 1, \ldots, n - p$, there exists an element $w^j$ of $\Gamma(T^*M)$ such that $w^j_x(X_x) = X_x(t_j \circ p_2 \circ \varphi)$ ($X \in \Gamma(TM)$, $x \in U$)
for every local chart \((U, \varphi)\) of \((M, G)\) compatible with \(\sigma\), where \(p_2\) is the projection of \(G \times \mathbb{R}^{n-p}\) onto \(\mathbb{R}^{n-p}\) and \(t_j\) is the \(j\)-th coordinate function of \(\mathbb{R}^{n-p}\).

**Proof.** Let \(\{\omega_k; k = 1, 2, \ldots\}\) be a partition of unity on \(M\) subordinate to the cover \(\{U_i; i \in I\}\). Let \(i(k)\) be an element of \(I\) such that \(\text{supp} \omega_k \subset U_{i(k)}\). We define \(w^j\) by \(w^j = \sum_{k=1}^{\infty} \omega_k d(t_j \circ p_2 \circ \varphi_{i(k)})\). □

**Theorem 4.2.** The flat connection \(\nabla\) in \(B\) satisfies the following equation;

\[
\nabla(cs_m(f)) = \sum_{j=1}^{n-p} w^j \otimes cs_m(f^m_j) \quad (f \in C^\infty_c(G) \ m \in \mathcal{N}),
\]

where \(f^m_j(g, x) = (\partial/\partial t_j)(f(g, \sigma_q^m(x)))\big|_{t=0}\). In particular, a cross section \((\nabla \xi)(X)\) is an element of \(D\) for every \(\xi \in D\) and \(X \in \Gamma(TM)\).

**Proof.** Let \(\{\omega_k\}\) be the partition of unity as in the proof of Lemma 4.1 and \(i(k)\) be an element of \(I\) such that \(\text{supp} \omega_k \subset U_{i(k)}\). Let \((V, \psi)\) be a local coordinate system of \(M\) and \(x_1, \ldots, x_n\) be coordinate functions associated with \((V, \psi)\). We set \(\xi = cs_m(f)\) and \(\xi^j = cs_m(f^m_j)\), we set \(\tilde{\xi}_{i(k)}(x) = \psi_{i(k), x}(\xi_x)\) and \(\tilde{\xi}^j_{i(k)} = \psi_{i(k), x}(\xi^j_x)\), and then we set \(\eta = \tilde{\xi}_{i(k)} \circ \psi^{-1}\) and \(\eta^j = \tilde{\xi}^j_{i(k)} \circ \psi^{-1}\). We set \(\tilde{t}^j_{i(k)} = t_j \circ p_2 \circ \varphi_{i(k)}\). It follows from the proof of Theorem 3.2 that we have \((\partial \eta/\partial v_l) = \sum_{j=1}^{n-p} \eta^j (\partial \tilde{t}^j_{i(k)} \circ \psi^{-1}/\partial v_l)\). Since we have \(\sum_{k=1}^{\infty} (\partial \omega_k/\partial x_l) = 0\,\), we have

\[
\sum_{k=1}^{\infty} \omega_{i(k), x}^{-1} \left(\frac{\partial (\omega_k \tilde{\xi}_{i(k)})}{\partial x_l}(x)\right) = \sum_{k=1}^{\infty} \sum_{j=1}^{n-p} \omega_k(x) \xi^j_x \left(\frac{\partial \tilde{t}^j_{i(k)}}{\partial x_l}(x)\right).
\]

Let \(X\) be an element of \(\Gamma(TM)\). It follows from Lemma 4.1 that we have \((\nabla \xi)(X)_x = \sum_{j=1}^{n-p} w^j_x(X_x) \xi^j_x\). This completes the proof of Theorem 4.2. □

In the rest of this section, we assume that \(G\) is connected and that \(C^*_c(G)\) is simple. The following proposition shows that the bundle \(B\) is topologically trivial, but the differentiable structure for \(B\) is not trivial as we shall see in the next section.
PROPOSITION 4.3. Suppose that $G$ is connected and that $C^\star_r(G)$ is simple. Then the bundle $B$ is isomorphic to the product bundle $M \times C^\star_r(G)$ as topological vector bundles.

Proof. We set $A = C^\star_r(G)$. Since $G$ is connected, we have $\hat{\rho}_x = \rho_x$. Since $A$ is simple, $\rho_x$ is a $\ast$-isomorphism of $A$ onto $B_x$. For $i \in I$, we define a $\ast$-isomorphism $\Theta_{i,x}$ of $A$ onto $C$ by $\Theta_{i,x} = \psi_{i,x} \circ \rho_x$, where $(U_i, \psi_i) \in F_\sigma$. For $a \in A$, we define a map $\eta_a$ of $U_i$ into $C$ by $\eta_a(x) = \Theta_{i,x}(a)$. For $f \in C^\infty_c(G)$, it follows from the proof of Theorem 3.2 that $\eta_f$ is continuous. Since $\Theta_{i,x}$ is isometry, the map $(x, a) \mapsto \eta_a(x)$ is continuous on $U_i \times A$. For $c \in C$, we define a map $\tilde{\eta}_c$ of $U_i$ into $A$ by $\tilde{\eta}_c(x) = \Theta_{i,x}^{-1}(c)$. The map $(x, c) \mapsto \tilde{\eta}_c(x)$ is continuous on $U_i \times C$. We define a map $\Theta_i$ of $U_i \times A$ onto $U_i \times C$ by $\Theta_i(x, a) = (x, \eta_a(x))$. Then we have $\Theta_i^{-1}(x, c) = (x, \tilde{\eta}_c(x))$. Therefore $\Theta_i$ is a homeomorphism. We define a map $\Theta$ of $M \times A$ onto $B$ by $\Theta(x, a) = \rho_x(a)$. Then we have $\psi_i \circ \Theta = \Theta_i$ for every $i \in I$. Since the topology of $B$ is determined by $\{\psi_i\}$, $\Theta$ is a homeomorphism.

We denote by $C^\infty_c(G)^\sigma$ the $\ast$-algebra of all elements $f$ of $C^\infty_c(G)$ with the property that $\nabla(cs(f)) = 0$. It follows from Theorem 4.2 that $f$ is an element of $C^\infty_c(G)^\sigma$ if and only if we have $f(g, \sigma_t(x)) = f(g, x)$ for all $t \in \mathbb{R}^{n-p}$ and $(g, x) \in G$. Let $C^\ast_r(G)^\sigma$ be the $\ast$-subalgebra of $C^\ast_r(G)$ generated by $C^\infty_c(G)^\sigma$. We set $B^\sigma_x = \rho_x(C^\ast_r(G)^\sigma)$. We set $B^\sigma = \bigcup_{x \in M} B^\sigma_x$ and $\pi^\sigma = \pi|B^\sigma$, the restriction of $\pi$ to $B^\sigma$. For $(U_i, \psi_i) \in F_\sigma$, we set $\psi^\sigma_i = \psi_i | (\pi^\sigma)^{-1}(U_i)$ and $\psi^\sigma_{i,x} = \psi_{i,x} | B^\sigma_x$ $(x \in U_i)$. We denote by $F^\sigma_\sigma$ the set of $(U_i, \psi_i^\sigma)$ $(i \in I)$. Let $C^\sigma_x$ be the $\ast$-subalgebra of $C_x$ generated by elements $\rho(K^f_x) (f \in C^\infty_c(G)^\sigma)$. Then $\psi^\sigma_{i,x}$ is a $\ast$-isomorphism of $B^\sigma_x$ onto $C^\sigma_x$. Let $\tilde{x}$ be the element chosen in §2 so that we can identify $C^\sigma_{x_i}$ with $C = C^\sigma_{\tilde{x}}$. We set $C^\sigma = C^\sigma_{\tilde{x}}$. Then we may identify the subalgebra $C^\sigma_{x_i}$ of $C_{x_i}$ with the subalgebra $C^\sigma_{\tilde{x}}$ of $C$. Thus we consider $\psi^\sigma_{i,x}$ to be a $\ast$-isomorphism of $B^\sigma_x$ onto $C^\sigma$ and $\psi^\sigma_{i}$ to be a homeomorphism of $(\pi^\sigma)^{-1}(U_i)$ onto $U_i \times C^\sigma$. We denote by $\Theta^\sigma$ the restriction of $\Theta$ to $M \times C^\ast_r(G)^\sigma$, where $\Theta$ is the homeomorphism defined in the proof of Proposition 4.3. Then we have the following:

PROPOSITION 4.4. Suppose that $G$ is connected and $C^\ast_r(G)$ is simple. The quartet $(B^\sigma, \pi^\sigma, M, C^\sigma)$ is a differentiable bundle of $C^\ast$-algebras with respect to the family $F^\sigma_\sigma$. Moreover the differentiable structure for $B^\sigma$ is trivial in the following sense: There exists
a homeomorphism \( \Theta^\sigma \) of \( M \times C^*_\sigma(G) \) onto \( B^\sigma \) with the following property; for every \((U_i, \psi_i) \in \mathcal{F}_\sigma\), there exists a \(*\)-isomorphism \( \alpha_i \) of \( C^*_\sigma(G) \) onto \( C^\sigma \) such that \( \psi_i \circ \Theta_i^\sigma = \text{id}_i \times \alpha_i \), where \( \Theta_i^\sigma \) is the restriction of \( \Theta^\sigma \) to \( U_i \times C^*_\sigma(G) \) and \( \text{id}_i \) is the identity map of \( U_i \) onto itself.

We denote by \( RM(A) \) the Banach algebra of all right multipliers of a \( C^* \)-algebra \( A \) on a Hilbert space ([15, 3.12]). Let \( RM(B) \) be the disjoint union of Banach algebras \( RM(B_x) \) (\( x \in M \)) and \( \bar{\pi} \) be the map of \( RM(B) \) onto \( M \) defined by \( \bar{\pi}(a) = x \) for \( a \in RM(B_x) \). Let \((U_i, \psi_i)\) be an element of \( \mathcal{F}_\sigma \). It follows from Lemmas 2.2 and 2.3 that \( \psi_{i,x} \) is spatial for every \( x \in U_i \). Therefore we can extend \( \psi_{i,x} \) to an isomorphism \( \tilde{\psi}_{i,x} \) of \( RM(B_x) \) onto \( RM(C_{z_i}) \). We define a map \( \tilde{\psi}_i \) of \( \bar{\pi}^{-1}(U_i) \) onto \( U_i \times RM(C_{z_i}) \) by \( \tilde{\psi}_i(a) = (x, \tilde{\psi}_{i,x}(a)) \) for \( a \in RM(B_x) \). We denote by \( \mathcal{F}_\sigma \) the set of \((U_i, \tilde{\psi}_i)\) \( (i \in I) \). Moreover we may identify \( RM(C_{z_i}) \) with \( RM(C) \). Thus we consider \( \tilde{\psi}_{i,x} \) to be an isomorphism of \( RM(B_x) \) onto \( RM(C) \) and \( \tilde{\psi}_i \) to be a homeomorphism of \( \bar{\pi}^{-1}(U_i) \) onto \( U_i \times RM(C) \). Then the quartet \((RM(B), \bar{\pi}, M, RM(C))\) is a differentiable bundle of Banach algebras with respect to the flat family \( \mathcal{F}_\sigma \) of Banach coordinate systems. It follows from Lemma 1.6 that there exists a unique flat connection \( \bar{\nabla} \) in \( RM(B) \).

5. Examples. (a) Kronecker dynamical systems and irrational rotation algebras. Let \( M \) be the two-torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). For \( \mu \in \mathbb{R} \cup \{\infty\} \), we define an action \( F^\mu \) of \( \mathbb{R} \) on \( M \) by \( F^\mu_t(x_1, x_2) = (x_1 + t, x_2 + \mu t) \) if \( \mu \in \mathbb{R} \) and by \( F^{\infty}_t(x_1, x_2) = (x_1, x_2 + t) \) \((x_1, x_2) \in M, t \in \mathbb{R}\) Let \( G \) be the real line \( \mathbb{R} \) and \( \theta \) be an irrational number. We define an action of \( G \) on \( M \) by \( t \cdot x = F^\theta_t(x) \) for \( t \in G \) and \( x \in M \). For \( \mu \in \mathbb{Q} \cup \{\infty\} \), we define an action \( \sigma \) of \( \mathbb{R} \) on \( M \) by \( \sigma = F^\mu \). For \( x_0 = (x_0^0, x_0^0) \in M \) and \( \varepsilon > 0 \), we set \( S = T = \{ t \in \mathbb{R}; |t| < \varepsilon \} \). We define a map \( \varphi_0 \) of \( S \times T \) into \( M \) by \( \varphi_0(t_1, t_2) = t_1 \cdot \sigma_{t_2}(x_0) \). We set \( U = \varphi_0(S \times T) \). If \( \varepsilon \) is small enough, then \( \varphi_0 \) is a diffeomorphism onto \( U \). In this case, we set \( \varphi = \varphi_0^{-1} \) and \((U, \varphi)\) is a local chart of \((M, G)\) compatible with \( \sigma \). Therefore \( \sigma \) is a transverse action for \((M, G)\). It follows from Theorem 2.8 that there exists the differentiable bundle \((B, \pi, M, C)\) of \( C^* \)-algebras with respect to the flat family \( \mathcal{F}_\sigma \). Let \( \nabla \) be the flat connection in \( B \) (Lemma 1.6). For \( f \in C^\infty_c(G) \), it follows from Theorem 4.2 that we
have, $\nabla(cs(f)) = (adx_1 + bdx_2) \otimes cs(f_1)$, where $a = -\theta/(\mu - \theta)$, $b = 1/(\mu - \theta)$ and $f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2)$, if $\mu \in \mathbb{Q}$ and we have $\nabla(cs(f)) = (-\theta dx_1 + dx_2) \otimes cs(\partial f/\partial x_2)$ if $\mu = \infty$.

First, we suppose that $\mu = \infty$. For $u \in C(T)$, we define an operator $rm(u)_x$ on $H_x$ by $(rm(u)_x \zeta)(t, x) = i/(x^2 + \theta s)Cs(t)$ for $x = (x_1, x_2) \in M$, $\zeta \in H_x$ and $t \in G$. For $f \in C_c(G)$, we have $\rho_x(f)rm(u)_x = \rho_x(f \cdot u)$, where $(f \cdot u)(t, x_1, x_2) = f(t, x_1, x_2)u(x_2)$. Therefore $rm(u)_x$ is an element of $RM(B_x)$. We denote by $D_x$ the set of elements $rm(u)_x$ ($u \in C(T)$). Then $D_x$ is a $C^*$-subalgebra of $B(H_x)$ and $D_x$ is $*-$isomorphic to $C(T)$. Note that $f$ is an element of $C_c(\mathbb{R})$ if and only if there exists an element $\tilde{f}$ of $C_c(\mathbb{R})$ such that $f(t, x_1, x_2) = \tilde{f}(t, x_1)$. Therefore $B\times D$ generates $B_x$.

Let $[\alpha, \beta]$ be a closed interval in $\mathbb{R}$, and $\gamma : [\alpha, \beta] \to M$ be a smooth curve, that is, $\gamma$ extends to a $C^\infty$ map of $(a - \varepsilon, b + \varepsilon)$ into $M$ for some $\varepsilon > 0$, which we denote again by $\gamma$. We shall say that a
map ξ of [a, b] into D is a smooth curve in D if ξ extends to be a map of (a − ε, b + ε) into D, which we denote again by ξ, such that πr(ξ(t)) = γ(t) and the map t → ψr i,γ(t)(ξ(t)) is of class C∞ for every i ∈ I. Next suppose that γ is a piecewise smooth curve. By definition there exists a partition a = a₀ < a₁ < ⋯ < aₖ = b such that γ|[a_j, a_{j+1}] is smooth for every j ([21, Definition 1.41]).

We shall say that a map ξ of [a, b] into D is a piecewise smooth curve in D if ξ|[a_j, a_{j+1}] is smooth for every j. For a piecewise smooth curve ξ in D, we define ∇r(ξ)(γ(t)) ∈ Dγ(t) by ∇r(ξ)(γ(t)) = (ψr i,γ(t))⁻¹((d/dt)(ψr i,γ(t)(ξ(t)))) (c.f. [14, §1]). A horizontal curve ξ in D is a piecewise smooth curve in D such that ∇r(ξ)(γ(t)) = 0 for every t ∈ [a, b] (c.f. [11, Chapter II, §3]). Then we have the following:

**Lemma 5.1.** Let γ : [a, b] → M be a piecewise smooth curve with γ(a) = γ(b) = x. For every A ∈ Dₓ, there exists a unique horizontal curve ξₐ in D such that ξₐ(a) = A. For u ∈ C(T), define an element h(u) of C(T) by ξₐ(b) = rm(h(u))ₓ, where A = rm(u)ₓ. Then there exists an integer k such that h(u)(s) = u(s + kθ) (s ∈ T) for every u ∈ C(T).

**Proof.** We fix t₀ ∈ [a, b]. Let (U, ψr) and (U, ϕ) be as above with x₀ = γ(t₀). Let V be a connected neighborhood of t₀ such that γ(t) ∈ U for every t ∈ V. Then we have ξₐ(t) = (ψrᵢ,γ(t₀))⁻¹ ∘ ψrᵢ,γ(t₀)(ξₐ(t₀)) for t ∈ V. This implies the existence and the uniqueness of ξₐ. Let uᵢ be an element of C(T) such that ξₐ(t) = rm(uᵢ,γ(t)). We set γ(t) = (γ₁(t), γ₂(t)) and xᵢ(t₁, t₂) = γⱼ(t₁) − γⱼ(t₂) for j = 1, 2. Then we have uᵢ(−θx₁(t, t₀) + γ₂(t) + θs) = uᵢ₀(γ₂(t₀) + θs) (s ∈ T). Thus we have h(u)(s) = u(s + kθ) for an integer k. □

By virtue of Lemma 5.1, one can define a *-automorphism h_γ of Dₓ by h_γ(A) = ξₐ(b). This automorphism is called the parallel displacement along the curve γ. We denote by C(x) the set of piecewise smooth curves starting and ending at x. The holonomy group Φₓ of ∇r with reference point x is the group of all automorphisms h_γ (γ ∈ C(x)) (c.f. [11, Chapter II, §4]). We define an action α of Z on C(T) by αₖ(u)(t) = u(t + kθ) for u ∈ C(T), k ∈ Z and t ∈ T. It follows from Lemma 5.1 that (Dₓ, Φₓ) is isomorphic to (C(T), α). Therefore the reduced crossed product C_r*(Dₓ, Φₓ) is *-isomorphic to the irrational rotation algebra Aθ. On the other hand, let N be a
σ-invariant closed connected submanifold with dim \( N = 1 \). Then \( N \) is of the form \( \{x_1\} \times \mathbb{T} \) for some \( x_1 \in \mathbb{T} \), and \( C^*_r(G|N) \) is \(*\)-isomorphic to \( A_\theta \). Therefore \( C^*_r(G|N) \) is \(*\)-isomorphic to \( C^*_r(D_x, \Phi_x) \).

Next, we suppose that \( \mu \) is rational, say \( \mu = p/q \) for relatively prime integers \( p \) and \( q \). There exist integers \( a \) and \( b \) such that \( pb - qa = 1 \). We define a diffeomorphism \( S \) of \( M \) as follows; \( S(x_1, x_2) = (px_1 - qx_2, -ax_1 + bx_2) \) for \( (x_1, x_2) \in M \). We set \( \nu = (-a + b\theta)/(p - q\theta) \) and define actions \( \tilde{F} \) and \( \tilde{\sigma} \) by \( \tilde{F}_t = S \circ F_t^\theta \circ S^{-1} \) and \( \tilde{\sigma}_t = S \circ \sigma_t \circ S^{-1} \). Then we have \( \tilde{F}_t = F_{(p-q\theta)t}^\nu \) and \( \tilde{\sigma}_t = F_{t/q}^\infty \).

Since the system \((M, F^\theta, \sigma)\) is conjugate to \((M, \tilde{F}, \tilde{\sigma})\) by \( S \), we have a similar result to that obtained above. Note that \( C^*_r(G|N) \) is \(*\)-isomorphic to \( A_\nu \) for every \( \sigma \)-invariant closed connected submanifold \( N \) with \( \dim N = 1 \). We can summarize the conclusion just obtained as follows:

**Theorem 5.2.** Let \( \sigma \) be a transverse action for \((M, G)\) defined by \( \sigma = F^\mu \) for \( \mu \in \mathbb{Q} \cup \{\infty\} \) and let \((B, \pi, M, C)\) be a differentiable bundle of \( C^*\)-algebras with respect to \( F_\sigma \). Then there exists a subbundle \((D, \pi^r, M, C(T))\) of \((RM(B), \bar{\pi}, M, RM(C))\) with the following properties; (i) \( B_x^\sigma D_x \) generates \( B_x \) for every \( x \in M \), (ii) \( C^*_r(G|N) \) is \(*\)-isomorphic to \( C^*_r(D_x, \Phi_x) \) for every \( x \in M \), where \( N \) is a \( \sigma \)-invariant closed connected submanifold of \( M \) with \( \dim N = 1 \) and \( \Phi_x \) is the holonomy group of the flat connection \( \nabla^r \) in \( D \).

(b) An action of a semi-direct product group. Let \( S \) be an element of \( \text{SL}(2, \mathbb{Z}) \), \( \lambda \) be an eigenvalue of \( S \) and \((1, \theta)\) be an eigenvector of \( S \) with respect to \( \lambda \). We suppose that \( \theta \) is real and irrational. Let \( G \) be a semi-direct product group of \( \mathbb{Z} \) and \( \mathbb{R} \) defined by \((n, t)(m, s) = (n + M, \lambda^{-m}t + s)\) for \( (n, t), (m, s) \in \mathbb{Z} \times \mathbb{R} \). We may identify the group \( \mathcal{N} \) with \( \mathbb{Z} \) and a connected component \( G_m \) is the set of elements of the form \((m, t)\) \( (t \in \mathbb{R}) \) for \( m \in \mathbb{Z} \). Let \( M \) be the torus \( \mathbb{T}^2 \). Since we have \( SF_t^\theta = F_{\lambda t}^\theta S \), we can define an action of \( G \) on \( M \) by \((n, t) \cdot x = S^nF_t^\theta(x)\) for \( (n, t) \in G \) and \( x \in M \). Let \( \nu \) be the other eigenvalue of \( S \) and \((1, \mu)\) be an eigenvector of \( S \) with respect to \( \nu \). We set \( \sigma_t = F_t^\mu \) for \( t \in \mathbb{R} \). As in (a), \( \sigma \) is a transverse action for \((M, G)\). Let \((B, \pi, M, C)\) be a differentiable bundle of \( C^*\)-algebras with respect to the family \( F_\sigma \) and let \( \nabla \) be the flat connection in \( B \). It follows from Theorem 4.2 that we have
\( \nabla(cs_m(f)) = \nu^m(adx_1 + bdx_2) \otimes cs_m(f_1) \), where \( a = -\theta/(\mu - \theta) \), \( b = 1/(\mu - \theta) \) and \( f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2) \) for \( m \in \mathbb{Z} \) and \( f \in C^\infty_c(\mathcal{G}) \). We denote by \( N \) the submanifold \( \{0\} \times \mathbb{T} \) of \( M \) and denote by \( B(S, \lambda) \) the reduction \( C^\star \)-algebra \( C^\star_r(\mathcal{G}|N) \). This algebra was studied in [12, 13, 14]. It follows from [7] and [13] that it is a simple algebra. We do not have any results concerning relations between the bundle and the algebra \( B(S, \lambda) \). This is the problem for further investigation.

(c) Actions of discrete groups. Let \( (M, G) \) be a differentiable dynamical system with \( G \) discrete and let \( \sigma : \mathbb{R}^n \to \text{Diff} (M) \) be a differentiable action. Suppose that the differential of the map \( t \mapsto \sigma_t(x) \) at 0 is an isomorphism for every \( x \in M \). Then, for every \( x_0 \in M \), there exist a neighborhood \( U \) of \( x_0 \) and a neighborhood \( T \) of 0 in \( \mathbb{R}^n \) such that the map \( \varphi_0 : t \mapsto \sigma_t(x_0) \) is a diffeomorphism of \( T \) onto \( U \). We set \( \varphi = \varphi_0^{-1} \). Then \( (U, \varphi) \) is a local chart compatible with \( \sigma \) and \( \sigma \) is a transverse action for \( (M, G) \). Let \( (B, \pi, M, C) \) be a differentiable bundle of \( C^\star \)-algebras with respect to \( \mathcal{F}_\sigma \) and \( \nabla \) be the flat connection. It follows from Theorem 4.2 that we have, for \( g \in G \) and \( f \in C^\infty_c(\mathcal{G}) \), \( \nabla(cs_g(f)) = \sum_{k=1}^n d\varphi^k \otimes cs_g(f_k^g) \), where \( \varphi = (\varphi^1, \ldots, \varphi^n) \) and \( f_k^g(g', x) = (\partial/\partial t_k)f(g', g_\sigma(t^{-1}x))|_{t=0} \).
Differentiable Structure for a Bundle of C*-Algebras


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