A DIFFERENTIABLE STRUCTURE FOR A BUNDLE OF C*-ALGEBRAS ASSOCIATED WITH A DYNAMICAL SYSTEM

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Let $(M, G)$ be a differentiable dynamical system, and $\sigma$ be a transverse action for $(M, G)$. We have a differentiable bundle $(B, \pi, M, C)$ of $C^*$-algebras with respect to a flat family $F_\sigma$ of local coordinate systems and we have a flat connection $\nabla$ in $B$. If $G$ is connected, the bundle $B$ is a disjoint union of $\rho_x(C^*_r(G))$ ($x \in M$), where $G$ is the groupoid associated with $(M, G)$ and $\rho_x$ is the regular representation of $C^*_r(G)$. We show that, for $f \in C^\infty_c(G)$, a cross section $cs(f) : x \mapsto \rho_x(f)$ is differentiable with respect to the norm topology, and calculate a covariant derivative $\nabla(cs(f))$. Though $B$ is homeomorphic to the trivial bundle, the differentiable structure for $B$ is not trivial in general. Let $B^\sigma$ be a subbundle of $B$ generated by elements $f$ with the property $\nabla(cs(f)) = 0$. We show the triviality of the differentiable structure for $B^\sigma$ induced from that for $B$ when $C^*_r(G)$ is simple. We have a bundle $RM(B)$ of right multiplier algebras and it contains $B$ as a subbundle. Let $(M, G)$ be a Kronecker dynamical system and $\sigma$ be a flow whose slope is rational. In this case, we have a subbundle $D$ of $RM(B)$ whose fibers are $*$-isomorphic to $C(\mathbb{T})$. The flat connection $\nabla^r$ in $D$ is not trivial and the bundle $B$ decomposes into the trivial bundle $B^\sigma$ and the non-trivial bundle $D$. Moreover, for a $\sigma$-invariant closed connected submanifold $N$ of $M$ with $\dim N = 1$, we show that $C^*_r(G|N)$ is $*$-isomorphic to $C^*_r(D_x, \Phi_x)$, where $\Phi_x$ is the holonomy group of $\nabla^r$ with reference point $x$. If $G$ is not connected, we also have sufficiently many differentiable cross sections of $B$ and calculate their covariant derivatives.

0. Introduction. In the theory of $C^*$-algebras, one sometimes study a stable $C^*$-algebra $A \otimes K$ instead of studying a given $C^*$-algebra $A$ itself, where $K$ is the algebra of all compact operators on the infinite dimensional separable Hilbert space. There are many
other algebras \( D \) such that \( D \otimes \mathcal{K} \cong A \otimes \mathcal{K} \). Moreover, stable algebras do not have any identity elements. Therefore, given a stable \( C^* \)-algebra \( C \), we want to find \( C^* \)-algebras \( A \) with the property \( A \otimes \mathcal{K} \cong C \), especially unital ones with the property. We do not know any general answer to the question, but there is a method to construct such algebras \( A \) for foliation \( C^* \)-algebras. Let \((V, \mathcal{F})\) be a foliation and \( C^*(V, \mathcal{F}) \) be the foliation \( C^* \)-algebra introduced by A. Connes ([1], [3]). It follows from [10] that \( C^*(V, \mathcal{F}) \) is \(*\)-isomorphic to \( \mathcal{C}_r^* (\mathcal{G}|N) \otimes \mathcal{K} \), where \( \mathcal{G} \) is the holonomy groupoid of \((V, \mathcal{F})\), where \( N \) is a complete transverse submanifold and where the groupoid \( \mathcal{G}|N \) is the reduction of \( \mathcal{G} \) by \( N \). Suppose that \( V \) is compact. If we have \( \dim N = \text{codim} \mathcal{F} \), then the \( C^* \)-algebra \( \mathcal{C}_r^* (\mathcal{G}|N) \) is unital. To give an example, if \((V, \mathcal{F})\) is a Kronecker foliation, then the \( C^* \)-algebra \( \mathcal{C}_r^* (\mathcal{G}|N) \) is the irrational rotation algebra \( A_\theta \) for an appropriate \( N \). This example plays an important role in the theory of non-commutative differential geometry by A. Connes. We refer the reader to the works of A. Connes [2], [3], that of A. Connes and M.A. Rieffel [4] and that of M.A. Rieffel [20]. M.A. Rieffel also studied the example in [17], [18] from the viewpoint of Morita equivalence. The author studied another example of \( \mathcal{C}_r^* (\mathcal{G}|N) \) in [12], [13].

From these considerations, we begin to study \( C^* \)-algebras of reductions of differentiable dynamical systems. Let \((M, G)\) be a differentiable dynamical system. We denote by \( G \) the topological groupoid \( G \times M \) and denote by \( \mathcal{C}_r^* (G) \) the reduced \( C^* \)-algebra associated with \( G \). We have a regular representation \( \rho_x \) of \( \mathcal{C}_r^* (G) \) on a Hilbert space \( \mathcal{H}_x \) for every \( x \in M \). For the moment we assume that \( G \) is connected and that \( \mathcal{C}_r^* (G) \) is simple. We set \( B_x = \rho_x (\mathcal{C}_r^* (G)) \) and denote by \( B \) the disjoint union of \( C^* \)-algebras \( B_x \) (\( x \in M \)). We may consider elements \( a \) of \( \mathcal{C}_r^* (G) \) to be cross sections \( cs(a) : x \mapsto \rho_x (a) \) of the bundle \( B \) on \( M \). Continuous fields of \( C^* \)-algebras have been studied by many authors. We refer the reader to the book of J. Dixmier [5], those of J.M.G. Fell and R.S. Doran [8], [9], the work of B.D. Evans [6] and that of M.A. Rieffel [19]. Since we study \( C^* \)-algebras associated with differentiable dynamical systems, it is natural to consider differentiable structure for fields of \( C^* \)-algebras. In the previous paper [14], the author introduced the notion of differentiable bundles of \( C^* \)-algebras and
that of connections in them. A. Connes first introduced the notion of connections into the theory of $C^*$-algebras in [2]. He defined the notion in the setting of projective modules. On the other hand, our definition of connections is in the setting of bundles of $C^*$-algebras and it is a literal translation of that in the setting of vector bundles, except that our connections are compatible with $*$-algebraic structures possessed by fibers.

In this paper, we introduce a notion of a transverse action $\sigma$ for $(M, G)$ and we construct a family $\mathcal{F}_\sigma$ of local coordinate systems for $B$ from local charts of $(M, G)$ compatible with $\sigma$. Then $\mathcal{F}_\sigma$ defines a differentiable structure for $B$. Next, we prove that the above cross section $cs(f)$ is differentiable with respect to the norm topology for every $f \in C^\infty_c(G)$. We define a flat connection $\nabla$ in $B$ with respect to $\mathcal{F}_\sigma$. Though $B$ is homeomorphic to the trivial bundle $M \times C^*_r(G)$, the differentiable structure for $B$ is not trivial, that is, $\nabla$ is not trivial. Let $B^\sigma$ be the subbundle of $B$ generated by elements $f$ with the property $\nabla(cs(f)) = 0$. Then $B^\sigma$ is trivial, that is, the restriction of $\nabla$ to $B^\sigma$ is trivial. We denote by $RM(B_x)$ the right multiplier algebra of $B_x$ and denote by $RM(B)$ the disjoint union of Banach algebras $RM(B_x)$ ($x \in M$). There exists a differentiable structure for $RM(B)$ such that $B$ is a subbundle of $RM(B)$ and such that $\nabla$ extends to a flat connection $\overline{\nabla}^r$ in $RM(B)$. In the case where $(M, G)$ is a Kronecker dynamical system, we give a decomposition of $B$ into a trivial part and a non-trivial part. There exists a subbundle $D$ of $RM(B)$ such that every fiber $D_x$ is $*$-isomorphic to the commutative $C^*$-algebra $C(\mathbb{T})$ and such that $B^\sigma_x D_x$ generates $B_x$. Let $\nabla^r$ be the restriction of $\overline{\nabla}^r$ to $D$ and let $\Phi_x$ be the holonomy group of $\nabla^r$ with reference point $x$. Note that $\Phi_x$ is a subgroup of the group $\text{Aut}(D_x)$ of all $*$-automorphisms of $D_x$. Let $N$ be a $\sigma$-invariant closed connected submanifold of $M$ with $\dim N = 1$. Then we show that the $C^*$-algebra $C^*_r(G|N)$ is $*$-isomorphic to the reduced crossed product $C^*_r(D_x, \Phi_x)$ of $D_x$ by $\Phi_x$. This result means that $B$ decomposes into the trivial bundle $B^\sigma$ and the non-trivial bundle $D$ and that $D$ corresponds to the reduction of $(M, G)$ by $N$. This situation was studied by M.A. Rieffel in [17], [18] from the viewpoint of projective modules. Our result describes the same situation from the viewpoint of vector bundles.

When $G$ is not connected, we also define a differentiable bundle
B associated with a transverse action for \((M, G)\) and define a flat connection \(\nabla\) in \(B\). But, in this case, \(B_x\) is larger than \(\rho_x(C^*_\tau(G))\) and cross sections \(cs(f)\) may not be differentiable. We define a cross section \(cs_m(f)\) of \(B\) for \(f \in C^\infty_c(G)\) and every connected component \(m\) of \(G\), and we show that the cross sections \(cs_m(f)\) are differentiable. The \(*\)-algebra \(\mathcal{D}_x\) generated by elements of the form \(cs_m(f)_x\) is dense in \(B_x\) with respect to the strong operator topology. The above results are valid even if \(G\) is discrete.

To find a transverse action for a given dynamical system \((M, G)\), it may be useful to consider the universal covering space \(\tilde{M}\) of \(M\). Suppose that the action of \(G\) on \(M\) lifts to an action of \(G\) on \(\tilde{M}\). (If \(G\) is simply connected, this assumption is satisfied.) If there exists a transverse action for \((\tilde{M}, G)\) and if it is compatible with the covering map, then we have a transverse action for \((M, G)\). But we do not know any interesting examples of transverse actions for dynamical systems \((M, G)\) such that the connected components \(G_x\) of \(G\) are not abelian, and it is difficult to find such examples. This is the problem for further investigation.

1. Preliminaries. (a) **Commutative dynamical systems.** Let \((M, G)\) be a topological transformation group. We assume that a topological space \(M\) and a topological group \(G\) are second countable, Hausdorff and locally compact. We denote by \(\mathcal{G}\) a topological groupoid \(G \times M\) with the following operations; \(s(g, x) = (e, x)\), \(r(g, x) = (e, gx)\), \((g', gx)(g, x) = (g'g, x)\), \((g, x)^{-1} = (g^{-1}, gx)\) for \(x \in M\) and \(g, g' \in G\), where \(e\) is the unit of \(G\). We set \(\mathcal{G}_x = \{(g, x) \in \mathcal{G}; g \in G\}\) for \(x \in M\). Let \(\mu\) be a right Haar measure on \(G\) and \(\Delta\) be the modular function of \(G\). We define a right Haar system \(\{\nu_x; x \in M\}\) on \(\mathcal{G}\) by \(\nu_x = \mu \times \delta_x\). Let \(C_c(\mathcal{G})\) be the \(*\)-algebra of continuous functions with compact supports, where the product and the involution are defined as follows:

\[
(f_1 \ast f_2)(g, x) = \int_{G} f_1(g', g'g, x) f_2(g', g, x) \, d\mu(g'),
\]

\[
f^*(g', x) = f(g^{-1}, gx)
\]

for \(f, f_1, f_2, \in C_c(\mathcal{G})\) and \((g, x) \in \mathcal{G}\). We denote by \(\mathcal{H}_x\) the Hilbert space \(L^2(\mathcal{G}_x, \nu_x)\) for \(x \in M\). We define the regular representation
\( \rho_x \) of \( C_c(G) \) on \( \mathcal{H}_x \) by

\[
(\rho_x(f)\xi)(g, x) = \int_G f(gg'^{-1}, g'x)\xi(g', x) \, d\mu(g')
\]

for \( f \in C_c(G) \), \( \xi \in \mathcal{H}_x \) and \( (g, x) \in G_x \). We define the reduced norm \( \|f\| \) by \( \|f\| = \sup_{x \in M} \|\rho_x(f)\| \). We denote by \( C^*_r(G) \) the completion of \( C_c(G) \) by the reduced norm. The representation \( \rho_x \) extends to a representation of \( C^*_r(G) \), which we denote again by \( \rho_x \). For details of groupoids and their \( C^* \)-algebras, we refer the reader to [1], [3] and [16].

**Lemma 1.1.** Let \( f \) be an element of \( C_c(G) \) and \( D \) be a compact set in \( G \) such that \( \text{supp} \ f \subset D \times M \). Then the following inequality holds: \( \|\rho_x(f)\| \leq I_D \|f\|_{\infty} \), where \( \|f\|_{\infty} \) is the supremum norm of \( f \) and \( I_D = \int_D \Delta^{1/2}(g) \, d\mu(g) \).

**Proof.** Let \( \chi_D \) be the characteristic function of \( D \). For \( \xi, \eta \in \mathcal{H}_x \), we have

\[
\int_G |f(g'^{-1}, g'gx)\xi(g'g, x)\eta(g, x)| \, d\mu(g) \\
\leq \left( \int_G |f(g'^{-1}, g'gx)|\|\xi(g'g, x)\|^2 \, d\mu(g) \right)^{1/2} \\
\cdot \left( \int_G |f(g'^{-1}, g'gx)|\|\eta(g, x)\|^2 \, d\mu(g) \right)^{1/2} \\
\leq \|f\|_{\infty} \chi_D(g'^{-1})\|\eta\|\Delta^{1/2}(g')\|\xi\|.
\]

Then we have \( \|\rho_x(f)\xi|\eta]\| \leq I_D \|f\|_{\infty} \|\eta\|\|\xi\|. \)

We introduce a \(*\)-algebra of functions on \( G \times G \). Let \( \tilde{C} \) be the set of bounded continuous functions \( K \) on \( G \times G \) with the following property; there exists a compact set \( D \) in \( G \) such that \( \text{supp} \ K \subset G \times D \). The set \( D \) may vary when \( K \) varies. Then \( \tilde{C} \) is a \(*\)-algebra with the following product and involution;

\[
(K_1 \ast K_2)(g, g') = \int_G K_1(g, g'^{-1})K_2(g'' g, g'' g') \, d\mu(g''), \\
K^*(g, g') = \overline{K(g'^{-1}g, g'^{-1})}
\]

for \( K, K_1, K_2 \in \tilde{C} \) and \( (g, g') \in G \times G \). We denote by \( \mathcal{H} \) the Hilbert space \( L^2(G, \mu) \). We define a \(*\)-representation \( \rho \) of \( \tilde{C} \) on \( \mathcal{H} \) by

\[
(\rho(K)\xi)(g) = \int_G K(g, g'^{-1})\xi(g'g) \, d\mu(g')
\]
for $K \in \mathcal{C}$, $\xi \in \mathcal{H}$ and $g \in G$. We can prove the following lemma by a similar computation to that in the proof of Lemma 1.1.

**Lemma 1.2.** Let $K$ be an element of $\mathcal{C}$ and $D$ be a compact set in $G$ such that $\text{supp } K \subset G \times D$. Then the following inequality holds: $||\rho(K)|| \leq I_D||K||_{\infty}$.

(b) *Differentiable bundles of $C^*$-algebras.* With a few modifications on the definitions in [14, §1], we summarize the necessary facts. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and $x_1, \ldots, x_n$ be the canonical coordinate functions of $\mathbb{R}^n$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $f$ be a map of $\Omega$ into a Banach space $C$. If there exists $\lim_{h \to 0} h^{-1}(f(x + he_i) - f(x))$ with respect to the norm in $C$, then we denote the limit by $(\partial f/\partial x_i)(x)$. We say that $f$ is differentiable of class $(C^\infty)'$ on $\Omega$ if the partial derivatives $\partial^\alpha f/\partial x^\alpha$ exist and are continuous on $\Omega$ for all multi-indices $\alpha$.

**Definition 1.3.** (c.f. [14, Definition 1.1]). Let $M$ be a finite dimensional real manifold of class $C^\infty$ and $A$ be the complete atlas defining the structure of $M$. A map $f$ of $M$ into a Banach space $C$ is said to be of class $C^\infty$ if $f \circ \phi^{-1}$ is of class $(C^\infty)'$ on $\phi(U)$ for every $(U, \phi) \in A$.

We assume that a real manifold $M$ is second countable, Hausdorff and of class $C^\infty$. Let $B$ be a topological space, $C$ be a $C^*$-algebra and $\pi$ be a continuous map of $B$ onto $M$. We set $B_x = \pi^{-1}(x)$ for $x \in M$ and suppose that $B_x$ is a $C^*$-algebra. (It is easy to rewrite the rest of this section for Banach algebras $C$ and $B_x$. We leave it to the reader.) Let $\{U_i\}$ be an open covering of $M$ indexed by a set $I$ and $\psi_i$ be a homeomorphism of $\pi^{-1}(U_i)$ onto $U_i \times C$ such that $p_1 \circ \psi_i(b) = \pi(b)$ for $b \in \pi^{-1}(U_i)$, where $p_1 : U_i \times C \to U_i$ is the projection. For $x \in U_i$, we define a map $\psi_{i,x}$ of $B_x$ into $C$ by $\psi_{i,x}(b) = p_2 \circ \psi_i(b)$ for $b \in B_x$, where $p_2 : U_i \times C \to C$ is the projection. We denote by $\mathcal{F}$ the set of pairs $(U_i, \psi_i)$ ($i \in I$).

**Definition 1.4.** (c.f. [14, Definition 1.2]). A quartet $(B, \pi, M, C)$ is called a differentiable bundle of $C^*$-algebras with respect to $\mathcal{F}$ if $\mathcal{F}$ satisfies the following conditions:

(i) For every $i \in I$ and $x \in U_i$, $\psi_{i,x}$ is a $*$-isomorphism between $C^*$-algebras.

(ii) For $i, j \in J$ with $U_i \cap U_j \neq \emptyset$ and for a map $f$ of $U_i \cap U_j$ into
C, define the map $f_{i,j}$ of $U_i \cap U_j$ into $C$ by $f_{i,j}(x) = \psi_{i,x} \circ \psi_{j,x}^{-1} \circ f(x)$. If $f$ is of class $C^\infty$, then $f_{i,j}$ is of class $C^\infty$.

Let $\mathcal{F}$ be a family satisfying the above condition (i). We say that $\mathcal{F}$ is a flat family of $C^*$-coordinate systems if it satisfies the following conditions:

(iii) For every $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ and for every connected component $U$ of $U_i \cap U_j$, there exists a *-automorphism $\alpha$ of the $C^*$-algebra $C$ such that $\alpha = \psi_{i,x} \circ \psi_{j,x}^{-1}$ for all $x \in U$.

Let $\xi$ be a map of an open set $U$ of $M$ into $\pi^{-1}(U)$ such that $\pi(\xi_x) = x$ for $x \in U$. For $i \in I$ with $U_i \cap U \neq \emptyset$, define the map $\check{\xi}_i$ of $U_i \cap U$ into $C$ by $\check{\xi}_i(x) = \psi_{i,x}(\xi_x)$. We say that $\xi$ is a differentiable cross section on $U$ if $\check{\xi}_i$ is of class $C^\infty$ for every $i \in I$ with $U_i \cap U \neq \emptyset$. We denote by $\Gamma(B)$ the *-algebra of all differentiable cross sections on $M$. Let $TM$ be the tangent bundle on $M$, $\Gamma(TM)$ be the space of $C^\infty$ vector fields on $M$ and $T^*M$ be the cotangent bundle on $M$. We denote by $T^*M \otimes B$ the tensor product of $T^*M$ and $B$ as real vector bundles. Let $\xi$ be a cross section of $T^*M \otimes B$. If $x_1, \ldots, x_n$ is a local coordinate system in $M$, then we have $\xi_x = \sum (dx_k)_x \otimes b^k_x$ with $b^k_x \in B_x$. We say that $\xi$ is differentiable if the cross sections $x \mapsto b^k_x$ are differentiable. Let $\Gamma(T^*M \otimes B)$ be the two-sided $\Gamma(B)$-module of differentiable cross sections of $T^*M \otimes B$. We define the involution on $\Gamma(T^*M \otimes B)$ by $\xi^* = \sum (dx_k)_x \otimes (b^k_x)^*$. We denote by $C^\infty(M, \mathbb{R})$ the space of real-valued $C^\infty$ functions on $M$.

**Definition 1.5.** (c.f. [14, Definition 1.3]). Let $(B, \pi, M, C)$ be a differentiable bundle of $C^*$-algebras and $\mathcal{D}$ be a *-subalgebra of $\Gamma(B)$ such that $f\xi \in \mathcal{D}$ for $f \in C^\infty(M; \mathbb{R})$ and $\xi \in \mathcal{D}$. A linear map $\nabla$ of $\mathcal{D}$ into $\Gamma(T^*M \otimes B)$ is called a connection in $B$ with domain $\mathcal{D}$ if it satisfies the following conditions: (i) $\nabla(f\xi) = df \otimes \xi + f\nabla\xi$, (ii) $\nabla(\xi\eta) = (\nabla\xi)\eta + \xi(\nabla\eta)$, (iii) $(\nabla\xi)(X) \in \mathcal{D}$, (iv) $\nabla(\xi^*) = (\nabla\xi)^*$ for $\xi, \eta \in \mathcal{D}$, $f \in C^\infty(M; \mathbb{R})$ and $X \in \Gamma(TM)$.

Suppose that the family $\mathcal{F}$ is flat. Let $\nabla$ be a connection in $B$ with domain $\Gamma(B)$ and $(V, x_1, \ldots, x_n)$ be a local coordinate system in $M$. For $\xi \in \Gamma(B)$ and $i \in I$, we set $\check{\xi}_i(x) = \psi_{i,x}(\xi_x)$. We say that $\nabla$ is a flat connection if we have

$$\psi_{i,x}((\nabla\xi)(X)_x) = \sum_{k=1}^n a_k(x) \frac{\partial \check{\xi}_i}{\partial x_k}(x) \quad (x \in V \cap U_i),$$

for $X \in \Gamma(TM)$ with $X_x = \sum a_k(x)(\partial/\partial x_k)_x$ (c.f. [14, Definition 1.6],

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LEMMA 1.6. If \((B, \pi, M, C)\) is a differentiable bundle of \(C^*\)-algebras with respect to a flat family \(F\), then there exists a unique flat connection in \(B\).

2. Transverse actions and bundles of \(C^*\)-algebras. Let \(M\) be an \(n\)-dimensional real manifold of class \(C^\infty\) and \(G\) be a \(p\)-dimensional real Lie group of class \(C^\infty\). In the following sections, we assume that \(M\) and \(G\) are second countable and Hausdorff and that \(0 < n < \infty\) and \(0 \leq p < \infty\). If \(p = 0\), then \(G\) is a countable discrete group. Moreover we assume that \(M\) is connected. Suppose that \((M, G)\) is a differentiable dynamical system, that is, \((M, G)\) is a transformation group and the map \((g, x) \mapsto gx\) of \(G \times M\) into \(M\) is of class \(C^\infty\). Let \(G_e\) be the connected component of the unit \(e\) in \(G\). We denote by \(\mathcal{N}\) the countable discrete group \(G/G_e\) and denote by \(G_m\) the connected component of \(G\) corresponding to \(m \in \mathcal{N}\).

We take notations from §1, and also use the following notations; \(G_{m,x} = G_m \cap G_x\), \(\mathcal{H}_x = L^2(G_{m,x}, \nu_x|G_{m,x})\), for \(m \in \mathcal{N}\) and \(x \in M\). Let \(P_x^m \in \mathcal{B}(\mathcal{H}_x)\) be the projection on \(\mathcal{H}_x^m\) and \(P^m \in \mathcal{B}(\mathcal{H})\) be the projection on \(\mathcal{H}^m\). We denote by \(\mathcal{N}(G)\) the set of families \(\zeta = (f_m)_{m \in \mathcal{N}}\) with the following properties; (1) \(f_m \in C_c(G)\) \((m \in \mathcal{N})\), (2) \(\sup_{m \in \mathcal{N}} ||f_m||_{\infty} < +\infty\), (3) there exists a compact set \(D\) in \(G\) such that \(sup f_m \subset D \times M\) for all \(m \in \mathcal{N}\). We set \(||\zeta|| = \sup_{m} ||f_m||_{\infty}\).

LEMMA 2.1. For \(\zeta = (f_m)_{m \in \mathcal{N}} \in \mathcal{N}(G)\), the sum \(\tilde{\rho}_x(\zeta) = \sum_{m \in \mathcal{N}} \rho_x(f_m)P_x^m\) converges with respect to the strong operator topology in \(\mathcal{B}(\mathcal{H}_x)\), and the following inequality holds: \(||\tilde{\rho}_x(\zeta)|| \leq J_D||\zeta||\), where \(D\) is any compact set in \(G\) such that \(sup f_m \subset D \times M(m \in \mathcal{N})\), and \(J_D\) is a constant depending only on \(D\).

Proof. We set \(D_m = D \cap G_m\). There exist elements \(m(1), \ldots, m(k)\) of \(\mathcal{N}\) such that \(D = D_m\) is disjoint union of non-empty sets \(D_m(1), \ldots, D_m(k)\). Then we have \(\rho_x(f_m)P_x^m = \sum_{l \in A(m)} P_x^l \rho_x(f_m)P_x^m\), where \(A(m) = \{m(i); i = 1, \ldots, k\}\). If we have \((P_x^l \rho_x(f_m)P_x^m \xi)(g, x) \neq 0\), then there exists \(g' \in G_m\) such that \(gg'^{-1} \in D_{lm^{-1}}\). This implies that we have \(lm^{-1} = m(i)\) for some \(i\) with \(1 \leq i \leq k\). We set
$B(l) = \{m(i)^{-1}l \in \mathcal{N}; i = 1, \ldots, k\}$. We have

$$\left\| \sum_{m \in \mathcal{N}} \rho_x(f_m) P^m_x \xi \right\|^2 \leq k \sum_{l \in \mathcal{N}} \sum_{m \in B(l)} \left\| P^l_x \rho_x(f_m) P^m_x \xi \right\|^2.$$  

Note that $m \in B(l)$ if and only if $l \in A(m)$. Thus we have

$$\sum_{l \in \mathcal{N}} \sum_{m \in B(l)} \left\| P^l_x \rho_x(f_m) P^m_x \xi \right\|^2 \leq I_\mathcal{H} \left\| \xi \right\|^2 \sum_{m \in \mathcal{N}} \left\| P^m_x \xi \right\|^2.$$

Let $B_x$ be the $C^*$-subalgebra of $B(\mathcal{H}_x)$ generated by $\{\check{\rho}_x(\zeta); \zeta \in \mathcal{N}(\mathcal{G})\}$. Since we have $\check{\rho}_x(\zeta) = \rho_x(f)$ for $\zeta = (f_m)$ with $f_m = f$ for all $m \in \mathcal{N}$, $B_x$ contains $\rho_x(C^*_r(\mathcal{G}))$. If $G$ is connected, then we have $B_x = \rho_x(C^*_r(\mathcal{G}))$. If $G$ is not connected, then $B_x$ may not be separable. For $x \in M$ and $f \in C_c(\mathcal{G})$, we define $K^f_x \in \mathcal{C}$ by $K^f_x(g, g') = f(g', g'^{-1}gx)$. For $m \in \mathcal{N}$, we define $\chi_m \in C^\infty(G \times G)$ as follows; $\chi_m(g, g') = 1$ if $g'^{-1}g \in G_m$ and $\chi_m(g, g') = 0$ otherwise. For $\zeta = (f_m) \in \mathcal{N}(\mathcal{G})$, we define $K^\zeta_x \in \mathcal{C}$ by $K^\zeta_x = \sum_{m \in \mathcal{N}} K^f_m \chi_m$. We denote by $C_x$ the $C^*$-subalgebra of $B(\mathcal{H})$ generated by $\{\rho(K^\zeta_x); \zeta \in \mathcal{N}(\mathcal{G})\}$. We define an isometry $T$ of $\mathcal{H}_x$ onto $\mathcal{H}$ by $(T\eta)(g) = \eta(g, x)$ for $\eta \in \mathcal{H}_x$. We set $\check{\psi}_x(a) = TaT^*$ for $a \in B_x$. For $g \in G_x$ and $a \in C_x$, we set $\Psi(x, g)(a) = R_gaR_g^*$, where $R$ is the right regular representation of $G$ on $\mathcal{H}$. Then we have:

**Lemma 2.2.** For $x \in M$, there exists a unique spatial isomorphism $\psi_x$ of $B_x$ onto $C_x$ such that $\check{\psi}_x(\check{\rho}_x(\zeta)) = \rho(K^\zeta_x)$ for $\zeta \in \mathcal{N}(\mathcal{G})$.

**Lemma 2.3.** For $x \in M$ and $g \in G_x$, there exists a unique spatial isomorphism $\Psi(x, g)$ of $C_x$ onto $C_{gx}$ such that $\Psi(x, g)(\rho(K^\zeta_x)) = \rho(K^\zeta_{gx})$ for $\zeta \in \mathcal{N}(\mathcal{G})$.

We denote by $\text{Diff}_G(M)$ the group of diffeomorphisms of $M$ which commute with the action of the connected component $G_e$ on $M$. For $\alpha \in \text{Diff}_G(M)$ and $m \in \mathcal{N}$, there exists a diffeomorphism $\alpha_m$ such that $ga(x) = \alpha_m(gx)$ for all $g \in G_m$ and $x \in M$. If $G$ is discrete, then we have $\text{Diff}_G(M) = \text{Diff}(M)$, the group of all diffeomorphisms on $G$. For $\alpha \in \text{Diff}_G(M)$ and $\zeta = (f_m) \in \mathcal{N}(\mathcal{G})$, we define $\hat{\alpha}(\zeta) \in \mathcal{N}(\mathcal{G})$ by $\hat{\alpha}(\zeta) = (\hat{\alpha}_m(f_m))$, where $\hat{\alpha}_m(f_m)(g, x) = f_m(g, \alpha_m^{-1}(x))$. For $\zeta \in \mathcal{N}(\mathcal{G})$, we have $K^\zeta_{\alpha^{-1}}(\alpha) = K^\zeta_{\alpha(x)}$. Thus we have:
**Lemma 2.4.** For $\alpha \in \text{Diff}_G(M)$ and $x \in M$, $C_x = C_{\alpha(x)}$.

Remember that $\dim M = n$ and $\dim G = p$. We assume that $n \geq p$. Let $\sigma : \mathbb{R}^{n-p} \to \text{Diff}_G(M)$ be a differentiable action, that is, $\sigma$ is a homomorphism and the map $(x, t) \mapsto \sigma_t(x)$ is of class $C^\infty$.

**Definition 2.5.** Let $U$ be a connected open set in $M$. Suppose that there exists a $C^\infty$ diffeomorphism $\varphi$ of $U$ onto $S \times T$, where $S$ is an open set in $G_e$ with $e \in S$ and $T$ is an open set in $\mathbb{R}^{n-p}$ with $0 \in T$. Then the pair $(U, \varphi)$ is called a local chart of $(M, G)$ compatible with $\sigma$ if it satisfies the following conditions:

1. $\varphi^{-1}(g, t) = g \varphi^{-1}(e, t)$,
2. $\varphi^{-1}(g, t) = \sigma_t(\varphi^{-1}(g, 0))$ for all $(g, t) \in S \times T$.

Let $(U, \varphi)$ be a local chart compatible with $\sigma$ as above. We set $x_0 = \varphi^{-1}(e, 0)$. For $x \in U$ with $\varphi(x) = (g, t)$, we have $g^{-1}x = \sigma_t(x_0)$. It follows from Lemmas 2.2, 2.3 and 2.4 that the map $\Psi(x, g^{-1}) \circ \tilde{\psi}_x$ is a spatial isomorphism of $B_x$ onto $C_{x_0}$ for $x \in U$ with $\varphi(x) = (g, t)$. We set $\psi_x = \Psi(x, g^{-1}) \circ \psi_x$. Then we have the following:

**Proposition 2.6.** Let $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ be local charts compatible with $\sigma$ and $U$ be a connected component of $U_1 \cap U_2$. If $\psi_{i,x}$ is the $\ast$-isomorphism of $B_x$ onto $C_{x_i}$ as above with respect to $(U_i, \varphi_i)$ with $x_i = \varphi_i^{-1}(e, 0)$ ($i = 1, 2$), then there exists a $\ast$-isomorphism $\alpha$ of $C_{x_1}$ onto $C_{x_2}$ such that $\alpha = \psi_{2,x} \circ \psi_{1,x}^{-1}$ for all $x \in U$.

**Proof.** For $i = 1, 2$, we set $\varphi_i(U_i) = S_i \times T_i$ as in Definition 2.5. We fix $x \in U$ and suppose that $\varphi_i(x) = (g_i, t_i)$ ($i = 1, 2$). Let $x'$ be an element of $U$ such that $\varphi_i(x') = (g_i', t_i')$ ($i = 1, 2$). We set $g_0 = g_1^{-1}g'^{-1}_2$ and $t_0 = t_1 - t_1'$. Let $U_0$ be a sufficiently small neighborhood of $x$ in $U$. For $x' \in U_0$, we have $g_0x = \sigma_{t_0}(x')$, $\varphi_2(g_0x) = (g_0g_2, t_2)$ and $\varphi_2(\sigma_{t_0}(x')) = (g_2', t_0 + t_2')$. Since we have $(g_0g_2, t_2) = (g_2', t_0 + t_2')$, we have $g_2^{-1}g_1 = g_2'^{-1}g'. \psi_{i,x} \circ \psi_{i,x}^{-1} = \Psi(x_1, g_1^{-1}g_1')$ and $\psi_{2,x} \circ \psi_{1,x}^{-1} = \Psi(x_1, g_1'^{-1}g_1').$ Since $U_0$ is a neighborhood of $x$, this completes the proof of Proposition 2.6.

We denote by $B$ the disjoint union of $C^*$-algebras $\{B_x; x \in M\}$ and denote by $\pi$ the map of $B$ onto $M$ defined by $\pi(a) = x$ for $a \in B_x$. Let $\{(U_i, \varphi_i)\}$ be the set of all local charts of $(M, G)$ compatible
with $\sigma$ indexed by a set $I$ and let $\psi_{i,x}$ be the *-isomorphism of $B_x$ onto $C_{x_i}$ constructed as above from $(U_i, \varphi_i)$ with $x_i = \varphi_i^{-1}(e, 0)$. We define a map $\psi_i$ of $\pi^{-1}(U_i)$ onto $U_i \times C_{x_i}$ by $\psi_i(a) = (x, \psi_{i,x}(a))$ for $a \in B_x$. Let $\mathcal{F}_\sigma$ be the set of pairs $(U_i, \psi_i)$ ($i \in I$) constructed as above.

**Definition 2.7.** A differentiable action $\sigma$ is called a transverse action for $(M, G)$ if the family $\{U_i;i \in I\}$ is an open covering of $M$.

In the following we assume that $\sigma$ is a transverse action for $(M, G)$. It follows from Proposition 2.6 that there exists a unique topology on $B$ such that $\pi$ is continuous and $\psi_i$ is a homeomorphism for all $i \in I$. Since $M$ is connected, the $C^*$-algebras $C_x$ are mutually *-isomorphic. Therefore, for a fixed $\bar{x} \in M$, we set $C = C_{\bar{x}}$ and fix a *-isomorphism between $C$ and $C_{x_i}$ for every $i \in I$, and then we identify $C_{x_i}$ with $C$ by this isomorphism. Thus we consider $\psi_{i,x}$ to be a *-isomorphism of $B_x$ onto $C$ and $\psi_i$ to be a homeomorphism of $\pi^{-1}(U_i)$ onto $U_i \times C$. By virtue of Proposition 2.6, we have the following theorem:

**Theorem 2.8.** Suppose that $\sigma$ is a transverse action for $(M, G)$. Then the quartet $(B, \pi, M, C)$ constructed above is a differentiable bundle of $C^*$-algebras with respect to the flat family $\mathcal{F}_\sigma$ of $C^*$-coordinate systems.

**3. Differentiable cross sections.** For $f \in C_c^\infty(G)$ and $m \in \mathcal{N}$, we define an element $[f]_m = (f_m)$ of $\mathcal{N}(G)$ by $f_m = f$ and $f_k = 0$ if $k \neq m$, and define the cross section $cs_m(f)$ of $B$ by $cs_m(f)_x = \tilde{\rho}_x([f]_m) (x \in M)$, that is, $cs_m(f)_x = \rho_x(f)P^m_x$. If $G$ is connected, we set $cs(f) = cs_e(f)$, where $\mathcal{N} = \{e\}$, and we have $cs(f)_x = \rho_x(f)$. Let $\sigma^m : \mathbb{R}^{n-p} \to \text{Diff} (M)$ be a differentiable action such that $\sigma^m = g \circ \sigma \circ g^{-1}$ for every $g \in G_m$. We prepare a lemma for proving the differentiability of $cs_m(f)$.

**Lemma 3.1.** For $F \in C_c^\infty(\mathbb{R}^{n-p} \times G)$ and $t \in \mathbb{R}^{n-p}$, define an element $F_t$ of $C_c^\infty(G)$ by $F_t(g, x) = F(t, g, x)$. Let $t_0$ be an element of $\mathbb{R}^{n-p}$. (i) The supremum norm $\|F_t - F_{t_0}\|_\infty$ converges to 0 as $t \to t_0$. (ii) Let $J$ be an open interval in $\mathbb{R}$ containing 0, and let $t(\cdot)$ be a $C^2$ map of $J$ into $\mathbb{R}^{n-p}$ with $t(0) = t_0$. Define an element $f$ of $C_c^\infty(G)$ by $f(g, x) = \sum_{i=1}^{n-1} (\partial F/\partial t_i)(t_0, g, x)(dt_i/dh)(0)$, where
The proof is elementary, and we omit it.

**Theorem 3.2.** The cross section $c_{s_m}(f)$ is differentiable, that is, $c_{s_m}(f) \in \Gamma(B)$, for every $f \in C_c^\infty(G)$ and $m \in \mathcal{N}$.

**Proof.** We fix $i \in I$, that is, we fix $(U_i, \psi_i)$ in $\mathcal{F}_\sigma$ and a local chart $(U_i, \varphi_i)$ compatible with $\sigma$. Recall that $\varphi_i$ is a diffeomorphism of $U_i$ onto $S \times T$, where $S$ and $T$ are open sets of $G_e$ and $\mathbb{R}^{n-p}$ respectively. Let $(U_0, \varphi_0)$ be a local coordinate system of $M$ such that $U_i \cap U_0 \neq \emptyset$. We set $U = U_i \cap U_0$ and $V = \varphi_0(U)$. We define $C^\infty$ maps $x(v)$ of $V$ into $U$ by $x(v) = \varphi_0^{-1}(v)$ and define $C^\infty$ maps $g(v)$ of $V$ into $S$ and $t(v)$ of $V$ into $T$ by $\varphi_i(x(v)) = (g(v), t(v))$. We set $\xi = c_{s_m}(f)$ and define maps $\tilde{\xi}_i$ of $U_i$ into $C$ and $\eta$ of $V$ into $C$ by $\tilde{\xi}_i(x) = \psi_{i,x}(\xi_x)$ and $\eta = \tilde{\xi}_i \circ \varphi_0^{-1}$ respectively. It follows from Lemmas 2.2 and 2.3 that we have $\eta(v) = \rho(K_{g(v)}^{[f]m}x(v))$. We have $g(v)^{-1}x(v) = \sigma_t(v)(x_i)$, where $x_i = \varphi_i^{-1}(e, 0)$. We define an element $F$ of $C^\infty(\mathbb{R}^{n-p} \times G)$ by $F(t, g, x) = f(g, \sigma_t^m(x))$. We have

$$||K_{\sigma_t(v)(x_i)}^F \chi_m - K_{\sigma_t(u)(x_i)}^F \chi_m||_\infty \leq ||F_t(v) - F_t(u)||_\infty,$$ for $u, v \in V.\]
be a function on $G \times G$ such that $\tilde{K}_h(g, g')$ is equal to

$$h^{-1} \left\{ K_{\sigma_r(h)}(x) \chi_{m} - K_{\sigma_r(0)}(x) \chi_{m} \right\} (g, g') - \sum_{j=1}^{n-p} \left( K_{\sigma_r(0)}(x) \chi_{m} \right) (g, g') \frac{\partial t_j}{\partial v_k}(u).$$

We have $||\tilde{K}_h||_\infty \leq ||h^{-1}(F_{r(h)} - F_{r(0)}) - a||_\infty$. We set $\xi_i = c_s m(f_j^m)$ and define maps $\tilde{\xi}_i$ of $U_1$ onto $C$ and $\eta_i$ of $V$ into $C$ by $\tilde{\xi}_i(x) = \psi_{i,x}(\xi_i)$ and $\eta_i = \xi_i \circ \varphi_0^{-1}$ respectively. It follows from Lemma 1.2 that we have

$$\left\| h^{-1}(\eta(u + he_k) - \eta(u)) - \sum_{j=1}^{n-p} \eta_j(u) \frac{\partial t_j}{\partial v_k}(u) \right\| \leq I \| h^{-1}(F_{r(h)} - F_{r(0)}) - a \|_\infty.$$

Therefore we have $(\partial \eta/\partial v_k)(u) = \sum_{j=1}^{n-p} \eta_j(u) (\partial t_j/\partial v_k)(u)$. As we have seen in the first half of this proof, $\eta^j$ is continuous on $V$. Therefore $\eta$ is of class $(C^1)'$ in the sense of §1. Similarly $\eta^j$ is of class $(C^1)'$ for $j = 1, \ldots, n - p$. Therefore we know that $\eta$ is of class $(C^\infty)'$ and that $\tilde{\xi}_i$ is of class $C^\infty$ in the sense of Definition 1.3. This completes the proof of Theorem 3.2.

Recall that $\Gamma(B)$ is not only a $*$-algebra but also a $C^\infty(M)$-module. We denote by $\mathcal{D}$ the $*$-subalgebra of $\Gamma(B)$ generated by elements of the form $\omega \cdot c_s m(f)$ with $f \in C^\infty_c(G)$, $m \in \mathcal{N}$ and $\omega \in C^\infty(M)$. Then $\mathcal{D}$ is also a $C^\infty(M)$-submodule of $\Gamma(B)$. For $x \in M$, we set $\mathcal{D}_x = \{ \xi_x \in B_x; \xi \in \mathcal{D} \}$. Note that $\mathcal{D}_x$ is the $*$-subalgebra of $B_x$ generated by elements of the form $\rho_x(f)P_x^m$ with $f \in C^\infty_c(G)$ and $m \in \mathcal{N}$. If $\mathcal{N}$ is finite, then $\mathcal{D}_x$ is dense in the norm topology of $B_x$ for every $x \in M$. If $\mathcal{N}$ is infinite, then $\mathcal{D}_x$ may not be dense in the norm topology, but it is dense in the strong operator topology of $B_x$ by Lemma 2.1.

4. Flat connections. It follows from Lemma 1.6 that there exists a unique flat connection $\nabla$ in $B$. In this section we calculate $\nabla(c_s m(f))$ explicitly.

**Lemma 4.1.** For $j = 1, \ldots, n - p$, there exists an element $w^j_x$ of $\Gamma(T^*M)$ such that $w^j_x(X_x) = X_x(t_j \circ p_2 \circ \varphi) (X \in \Gamma(TM), x \in U)$
for every local chart \((U, \varphi)\) of \((M, G)\) compatible with \(\sigma\), where \(p_2\) is the projection of \(G_e \times \mathbb{R}^{n-p}\) onto \(\mathbb{R}^{n-p}\) and \(t_j\) is the \(j\)-th coordinate function of \(\mathbb{R}^{n-p}\).

**Proof.** Let \(\{\omega_k; k = 1, 2, \ldots\}\) be a partition of unity on \(M\) subordinate to the cover \(\{U_i; i \in I\}\). Let \(i(k)\) be an element of \(I\) such that \(\text{supp} \ \omega_k \subset U_{i(k)}\). We define \(w^j\) by \(w^j = \sum_{k=1}^{\infty} \omega_k d(t_j \circ p_2 \circ \varphi_{i(k)})\).

**Theorem 4.2.** The flat connection \(\nabla\) in \(B\) satisfies the following equation:

\[
\nabla(c_{sm}(f)) = \sum_{j=1}^{n-p} w^j \otimes c_{sm}(f^m_j) \quad (f \in C^\infty_c(G) \ m \in \mathcal{N}),
\]

where \(f^m_j(g, x) = (\partial/\partial t_j)(f(g, \sigma^m_t(x)))|_{t=0}\). In particular, a cross section \((\nabla \xi)(X)\) is an element of \(\mathcal{D}\) for every \(\xi \in \mathcal{D}\) and \(X \in \Gamma(TM)\).

**Proof.** Let \(\{\omega_k\}\) be the partition of unity as in the proof of Lemma 4.1 and \(i(k)\) be an element of \(I\) such that \(\text{supp} \ \omega_k \subset U_{i(k)}\). Let \((V, \psi)\) be a local coordinate system of \(M\) and \(x_1, \ldots, x_n\) be coordinate functions associated with \((V, \psi)\). We set \(\xi = c_{sm}(f)\) and \(\xi^j = c_{sm}(f^m_j)\), we set \(\tilde{\xi}_{i(k)}(x) = \psi_{i(k), x}(\xi_x)\) and \(\tilde{\xi}^j_{i(k)} = \psi_{i(k), x}(\xi^j_x)\), and then we set \(\eta = \tilde{\xi}_{i(k)} \circ \psi^{-1}\) and \(\eta^j = \tilde{\xi}^j_{i(k)} \circ \psi^{-1}\). We set \(\tilde{t}^j_{i(k)} = t_j \circ p_2 \circ \varphi_{i(k)}\). It follows from the proof of Theorem 3.2 that we have \((\partial \eta/\partial v_i) = \sum_{j=1}^{n-p} \eta^j(\partial \tilde{t}^j_{i(k)} \circ \psi^{-1}/\partial v_i)\). Since we have \(\sum_{k=1}^{\infty} (\partial \omega_k/\partial x_i) = 0\), we have

\[
\sum_{k=1}^{\infty} \psi_{i(k), x}^{-1} \left(\frac{\partial(w_k \tilde{\xi}_{i(k)}(x))}{\partial x_i}\right) = \sum_{k=1}^{\infty} \sum_{j=1}^{n-p} \omega_k(x) \xi_x^j \frac{\partial \tilde{t}^j_{i(k)}}{\partial x_i}(x).
\]

Let \(X\) be an element of \(\Gamma(TM)\). It follows from Lemma 4.1 that we have \((\nabla \xi)(X)_x = \sum_{j=1}^{n-p} w_x^j(X_x) \xi_x^j\). This completes the proof of Theorem 4.2.

In the rest of this section, we assume that \(G\) is connected and that \(C^*_r(G)\) is simple. The following proposition shows that the bundle \(B\) is topologically trivial, but the differentiable structure for \(B\) is not trivial as we shall see in the next section.
**Proposition 4.3.** Suppose that $G$ is connected and that $C^*_r(G)$ is simple. Then the bundle $B$ is isomorphic to the product bundle $M \times C^*_r(G)$ as topological vector bundles.

**Proof.** We set $A = C^*_r(G)$. Since $G$ is connected, we have $\tilde{\rho}_x = \rho_x$. Since $A$ is simple, $\rho_x$ is a $*$-isomorphism of $A$ onto $B_x$. For $i \in I$, we define a $*$-isomorphism $\Theta_{i,x}$ of $A$ onto $B_x$ by $\Theta_{i,x} = \psi_{i,x} \circ \rho_x$, where $(U_i, \psi_i) \in \mathcal{F}_\sigma$. For $a \in A$, we define a map $\eta_a$ of $U_i$ into $C$ by $\eta_a(x) = \Theta_{i,x}(a)$. For $f \in C^*_c(G)$, it follows from the proof of Theorem 3.2 that $\eta_f$ is continuous. Since $Q^x$ is isometry, the map $(x, a) \mapsto \eta_a(x)$ is continuous on $U_i \times A$. For $c \in C$, we define a map $\tilde{\eta}_c$ of $U_i$ into $A$ by $\tilde{\eta}_c(x) = \Theta_{i,x}^{-1}(c)$. The map $(x, c) \mapsto \tilde{\eta}_c(x)$ is continuous on $U_i \times C$. We define a map $\Theta_i$ of $U_i \times A$ onto $U_i \times C$ by $\Theta_i(x, a) = (x, \eta_a(x))$. Then we have $\Theta_i^{-1}(x, c) = (x, \tilde{\eta}_c(x))$. Therefore $\Theta_i$ is a homeomorphism. We define a map $\Theta$ of $M \times A$ onto $B$ by $\Theta(x, a) = \rho_x(a)$. Then we have $\psi_i \circ \Theta = \Theta_i$ for every $i \in I$. Since the topology of $B$ is determined by $\{\psi_i\}$, $\Theta$ is a homeomorphism. 

We denote by $C^*_c(G)^\sigma$ the $*$-algebra of all elements $f$ of $C^*_c(G)$ with the property that $\nabla(cs(f)) = 0$. It follows from Theorem 4.2 that $f$ is an element of $C^*_c(G)^\sigma$ if and only if we have $f(g, \sigma_t(x)) = f(g, x)$ for all $t \in \mathbb{R}^{n-p}$ and $(g, x) \in \mathcal{G}$. Let $C^*_r(G)^\sigma$ be the $*$-subalgebra of $C^*_r(G)$ generated by $C^*_c(G)^\sigma$. We set $B_x^\sigma = \rho_x(C^*_r(G)^\sigma)$. We set $B^\sigma = \bigcup_{x \in M} B_x^\sigma$ and $\pi^\sigma = \pi|B^\sigma$, the restriction of $\pi$ to $B^\sigma$. For $(U_i, \psi_i) \in \mathcal{F}_\sigma$, we set $\psi_i^\sigma = \psi_i|\pi^\sigma|^{-1}(U_i)$ and $\psi_{i,x}^\sigma = \psi_{i,x}|B_x^\sigma$ $(x \in U_i)$. We denote by $\mathcal{F}_\sigma$ the set of $(U_i, \psi_i^\sigma) (i \in I)$. Let $C_x^\sigma$ be the $C^*$-subalgebra of $C_x$ generated by elements $\rho(K_f^x)$ $(f \in C^*_c(G)^\sigma)$. Then $\psi_{i,x}^\sigma$ is a $*$-isomorphism of $B_x^\sigma$ onto $C_x^\sigma$. Let $\hat{x}$ be the element chosen in §2 so that we can identify $C_{x_i}$ with $C = C_{\hat{x}}$. We set $C^\sigma = C_{\hat{x}}^\sigma$. Then we may identify the subalgebra $C_{x_i}^\sigma$ of $C_{x_i}$ with the subalgebra $C^\sigma$ of $C$. Thus we consider $\psi_{i,x}^\sigma$ to be a $*$-isomorphism of $B_x^\sigma$ onto $C^\sigma$ and $\psi_i^\sigma$ to be a homeomorphism of $(\pi^\sigma)^{-1}(U_i)$ onto $U_i \times C^\sigma$. We denote by $\Theta^\sigma$ the restriction of $\Theta$ to $M \times C^*_r(G)^\sigma$, where $\Theta$ is the homeomorphism defined in the proof of Proposition 4.3. Then we have the following:

**Proposition 4.4.** Suppose that $G$ is connected and $C^*_r(G)$ is simple. The quartet $(B^\sigma, \pi^\sigma, M, C^\sigma)$ is a differentiable bundle of $C^*$-algebras with respect to the family $\mathcal{F}_\sigma$. Moreover the differentiable structure for $B^\sigma$ is trivial in the following sense: There exists
a homeomorphism \( \Theta^\sigma \) of \( M \times C^*_r(\mathcal{G})^\sigma \) onto \( B^\sigma \) with the following property; for every \((U_i, \psi^\sigma_i) \in \mathcal{F}_\sigma^\prime\), there exists a *-isomorphism \( \alpha_i \) of \( C^*_r(\mathcal{G})^\sigma \) onto \( C^\sigma \) such that \( \psi^\sigma_i \circ \Theta^\sigma_i = \text{id}_i \times \alpha_i \), where \( \Theta^\sigma_i \) is the restriction of \( \Theta^\sigma \) to \( U_i \times C^*_r(\mathcal{G})^\sigma \) and \( \text{id}_i \) is the identity map of \( U_i \) onto itself.

We denote by \( RM(A) \) the Banach algebra of all right multipliers of a \( C^* \)-algebra \( A \) on a Hilbert space ([15, 3.12]). Let \( RM(B) \) be the disjoint union of Banach algebras \( RM(B_x) \) (\( x \in M \)) and \( \tilde{\pi} \) be the map of \( RM(B) \) onto \( M \) defined by \( \tilde{\pi}(a) = x \) for \( a \in RM(B_x) \). Let \((U_i, \psi_i)\) be an element of \( \mathcal{F}_\sigma \). It follows from Lemmas 2.2 and 2.3 that \( \tilde{\psi}_i \) is spatial for every \( x \in U_i \). Therefore we can extend \( \tilde{\psi}_i \) to an isomorphism \( \tilde{\psi}_i^\prime \) of \( RM(B_x) \) onto \( RM(C_{x_i}) \).

We define a map \( \tilde{\psi}_i \) of \( \tilde{\pi}^{-1}(U_i) \) onto \( U_i \times RM(C_{x_i}) \) by \( \tilde{\psi}_i(a) = (x, \tilde{\psi}_i^\prime(a)) \) for \( a \in RM(B_x) \). We denote by \( \mathcal{F}_\sigma \) the set of \((U_i, \tilde{\psi}_i)\) (\( i \in I \)). Moreover we may identify \( RM(C_{x_i}) \) with \( RM(C) \). Thus we consider \( \tilde{\psi}_i \) to be an isomorphism of \( RM(B_x) \) onto \( RM(C) \) and \( \tilde{\psi}_i \) to be a homeomorphism of \( \tilde{\pi}^{-1}(U_i) \) onto \( U_i \times RM(C) \). Then the quartet \((RM(B), \tilde{\pi}, M, RM(C))\) is a differentiable bundle of Banach algebras with respect to the flat family \( \mathcal{F}_\sigma \) of Banach coordinate systems. It follows from Lemma 1.6 that there exists a unique flat connection \( \tilde{\nabla} \) in \( RM(B) \).

5. Examples. (a) Kronecker dynamical systems and irrational rotation algebras. Let \( M \) be the two-torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). For \( \mu \in \mathbb{R} \cup \{\infty\} \), we define an action \( F^\mu \) of \( \mathbb{R} \) on \( M \) by \( F^\mu_t(x_1, x_2) = (x_1 + t, x_2 + \mu t) \) if \( \mu \in \mathbb{R} \) and and \( F^\infty_t(x_1, x_2) = (x_1, x_2 + t) \) (\( (x_1, x_2) \in M, t \in \mathbb{R} \)). Let \( G \) be the real line \( \mathbb{R} \) and \( \theta \) be an irrational number.

We define an action of \( G \) on \( M \) by \( t \cdot x = F^\theta_t(x) \) for \( t \in G \) and \( x \in M \). For \( \mu \in \mathbb{Q} \cup \{\infty\} \), we define an action \( \sigma \) of \( \mathbb{R} \) on \( M \) by \( \sigma = F^\mu \). For \( x_0 = (x_0^1, x_0^2) \in M \) and \( \varepsilon > 0 \), we set \( S = T = \{t \in \mathbb{R}; |t| < \varepsilon\} \). We define a map \( \varphi_0 \) of \( S \times T \) into \( M \) by \( \varphi_0(t_1, t_2) = t_1 \cdot \sigma_{t_2}(x_0) \). We set \( U = \varphi_0(S \times T) \). If \( \varepsilon \) is small enough, then \( \varphi_0 \) is a diffeomorphism onto \( U \). In this case, we set \( \varphi = \varphi_0^{-1} \) and \((U, \varphi)\) is a local chart of \((M, G)\) compatible with \( \sigma \). Therefore \( \sigma \) is a transverse action for \((M, G)\). It follows from Theorem 2.8 that there exists the differentiable bundle \((B, \pi, M, C)\) of \( C^* \)-algebras with respect to the flat family \( \mathcal{F}_\sigma \). Let \( \nabla \) be the flat connection in \( B \) (Lemma 1.6). For \( f \in C_\infty^c(\mathcal{G}) \), it follows from Theorem 4.2 that we
have, $\nabla(cs(f)) = (adx_1 + bdx_2) \otimes cs(f_1)$, where $a = -\theta/(\mu - \theta)$, $b = 1/(\mu - \theta)$ and $f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2)$, if $\mu \in \mathbb{Q}$ and we have $\nabla(cs(f)) = (-\theta dx_1 + dx_2) \otimes cs(\partial f/\partial x_2)$ if $\mu = \infty$.

First, we suppose that $\mu = \infty$. For $u \in C(T)$, we define an operator $rm(u)_x$ on $\mathcal{H}_x$ by $(rm(u)_x) (t, x) = u(x_2 + \theta t) \zeta(t, x)$ for $x = (x_1, x_2) \in M$, $\zeta \in \mathcal{H}_x$ and $t \in G$. For $f \in C_c(G)$, we have $\rho_x(f) rm(u)_x = \rho_x(f \cdot u)$, where $(f \cdot u)(t, x_1, x_2) = f(t, x_1, x_2)u(x_2)$. Therefore $rm(u)_x$ is an element of $RM(B_x)$. We denote by $D_x$ the set of elements $rm(u)_x (u \in C(T))$. Then $D_x$ is a $C^*$-subalgebra of $B(\mathcal{H}_x)$ and $D_x$ is $*$- isomorphic to $C(T)$. Note that $f$ is an element of $C_c^\infty(G)$ if and only if there exists an element $\tilde{f}$ of $C_c^\infty(\mathbb{R} \times T)$ such that $f(t, x_1, x_2) = \tilde{f}(t, x_1)$. Therefore $B_x^\sigma D_x$ generates $B_x$. Let $D$ be the disjoint union of $D_x (x \in M)$, $\pi^r$ be the restriction of $\tilde{\pi}$ to $D$ and $\psi^r_i$ be the restriction of $\tilde{\psi}_i$ to $(\pi^r)^{-1}(U_i)$ for $(U_i, \tilde{\psi}_i) \in \mathcal{F}_\sigma$. We denote by $\mathcal{F}_\sigma^r$ the set of $(U_i, \psi^r_i) (i \in I)$. Then the quartet $(D, \pi^r, M, C(T))$ is a differentiable bundle of $C^*$-algebras with respect to the flat family $\mathcal{F}_\sigma^r$ of $C^*$-coordinate systems. We denote by $\nabla^r$ the unique flat connection in $D$ (Lemma 1.6). Let $(U, \psi)$ be an element of $\mathcal{F}_\sigma$ constructed from the above local chart $(U, \varphi)$. We denote by $\psi^r$ the restriction of $\tilde{\psi}$ to $(\pi^r)^{-1}(U)$ and denote by $\psi^r_x$ the restriction of $\psi_x$ to $D_x$ for $x \in U$. For $x = (x_1, x_2) \in U$, we have $\psi^r_x(rm(u)_x) (s) = u(-\theta(x_1 - x_0^1) + x_2 + \theta s) \zeta(s)$ for $u \in C(T)$, $\zeta \in \mathcal{H}$ and $s \in \mathbb{R}$. Let $(x_1, x_2, x_3)$ be a natural coordinate system of $U \times T$ as a subset of $T^3 = \mathbb{R}^3/Z^3$. We denote by $C_0^\infty(U \times T)$ the set of all $C^\infty$ functions $f$ on $U \times T$ with the property that partial derivatives $\partial^\alpha f/\partial \tilde{x}^\alpha$ are bounded for every multi-index $\alpha$ and every natural coordinate system $\tilde{x}$. For $v \in C(U \times T)$, we define a map $rm(v)$ of $U$ into $D$ by $rm(v)_x = rm(v_x)_x$ for $x \in U$, where $v_x$ is an element of $C(T)$ defined by $v_x(x_3) = v(x, x_3)$. As in the proof of Theorem 3.2, we can show that $rm(v)$ is a differentiable cross section of $D$ on $U$ for $v \in C_0^\infty(U \times T)$, and we have $\nabla^r(rm(v)) = dx_1 \otimes rm(v_1) + dx_2 \otimes rm(v_2)$, where $v_1 = \partial v/\partial x_1 - \theta(\partial v/\partial x_3)$ and $v_2 = \partial v/\partial x_2 + \partial v/\partial x_3$. Moreover we have $\nabla^r(rm(v)) = 0$ if and only if there exists an element $u$ of $C^\infty(T)$ such that $v(x_1, x_2, x_3) = u(\theta(x_1 - x_0^1) - (x_2 - x_0^2) + x_3)$.

Let $[a, b]$ be a closed interval in $\mathbb{R}$, and $\gamma : [a, b] \to M$ be a smooth curve, that is, $\gamma$ extends to be a $C^\infty$ map of $(a - \varepsilon, b + \varepsilon)$ into $M$ for some $\varepsilon > 0$, which we denote again by $\gamma$. We shall say that a
map $\xi$ of $[a, b]$ into $D$ is a smooth curve in $D$ if $\xi$ extends to be a map of $(a - \varepsilon, b + \varepsilon)$ into $D$, which we denote again by $\xi$, such that $\pi^r(\xi(t)) = \gamma(t)$ and the map $t \mapsto \psi_t^r, \gamma(t)(\xi(t))$ is of class $C^\infty$ for every $i \in I$. Next suppose that $\gamma$ is a piecewise smooth curve. By definition there exists a partition $a = a_0 < a_1 < \cdots < a_k = b$ such that $\gamma|[a_j, a_{j+1}]$ is smooth for every $j$ ([21, Definition 1.41]).

We shall say that a map $\xi$ of $[a, b]$ into $D$ is a piecewise smooth curve in $D$ if $\xi|[a_j, a_{j+1}]$ is smooth for every $j$. For a piecewise smooth curve $\xi$ in $D$, we define $\nabla^r(\xi)(\gamma(t)) \in D_\gamma(t)$ by $\nabla^r(\xi)(\gamma(t)) = (\psi_t^r, \gamma(t))^{-1}((d/dt)(\psi_t^r, \gamma(t)(\xi(t))))$ (c.f. [14, §1]). A horizontal curve $\xi$ in $D$ is a piecewise smooth curve in $D$ such that $\nabla^r(\xi)(\gamma(t)) = 0$ for every $t \in [a, b]$ (c.f. [11, Chapter II, §3]). Then we have the following:

**Lemmas 5.1.** Let $\gamma : [a, b] \to M$ be a piecewise smooth curve with $\gamma(a) = \gamma(b) = x$. For every $A \in D_x$, there exists a unique horizontal curve $\xi_A$ in $D$ such that $\xi_A(a) = A$. For $u \in C(\mathbb{T})$, define an element $h(u)$ of $C(\mathbb{T})$ by $\xi_A(b) = rm(h(u))_x$, where $A = rm(u)_x$. Then there exists an integer $k$ such that $h(u)(s) = u(s + k\theta)$ ($s \in \mathbb{T}$) for every $u \in C(\mathbb{T})$.

**Proof.** We fix $t_0 \in [a, b]$. Let $(U, \psi^r)$ and $(U, \varphi)$ be as above with $x_0 = \gamma(t_0)$. Let $V$ be a connected neighborhood of $t_0$ such that $\gamma(t) \in U$ for every $t \in V$. Then we have $\xi_A(t) = (\psi_t^r, \gamma(t_0))^{-1} \circ \psi_t^r(\xi_A(t_0))$ for $t \in V$. This implies the existence and the uniqueness of $\xi_A$. Let $u_t$ be an element of $C(\mathbb{T})$ such that $\xi_A(t) = rm(u_t)_\gamma(t)$. We set $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $x_j(t_1, t_2) = \gamma_j(t_1) - \gamma_j(t_2)$ for $j = 1, 2$. Then we have $u_t(-\theta x_1(t, t_0) + \gamma_2(t) + \theta s) = u_{t_0}(\gamma_2(t_0) + \theta s)$ ($s \in \mathbb{T}$). Thus we have $h(u)(s) = u(s + k\theta)$ for an integer $k$.

By virtue of Lemma 5.1, one can define a *-automorphism $\hat{h}_\gamma$ of $D_x$ by $\hat{h}_\gamma(A) = \xi_A(b)$. This automorphism is called the parallel displacement along the curve $\gamma$. We denote by $C(x)$ the set of piecewise smooth curves starting and ending at $x$. The holonomy group $\Phi_x$ of $\nabla^r$ with reference point $x$ is the group of all automorphisms $\hat{h}_\gamma$ ($\gamma \in C(x)$) (c.f. [11, Chapter II, §4]). We define an action $\alpha$ of $\mathbb{Z}$ on $C(\mathbb{T})$ by $\alpha_k(u)(t) = u(t + k\theta)$ for $u \in C(\mathbb{T})$, $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. It follows from Lemma 5.1 that $(D_x, \Phi_x)$ is isomorphic to $(C(\mathbb{T}), \alpha)$. Therefore the reduced crossed product $C_r^*(D_x, \Phi_x)$ is *-isomorphic to the irrational rotation algebra $A_\theta$. On the other hand, let $N$ be a
σ-invariant closed connected submanifold with dim $N = 1$. Then $N$ is of the form $\{x_1\} \times \mathbb{T}$ for some $x_1 \in \mathbb{T}$, and $C^*_r(\mathcal{G}|N)$ is $*$-isomorphic to $A_\theta$. Therefore $C^*_r(\mathcal{G}|N)$ is $*$-isomorphic to $C^*_r(D_x, \Phi_x)$.

Next, we suppose that $\mu$ is rational, say $\mu = p/q$ for relatively prime integers $p$ and $q$. There exist integers $a$ and $b$ such that $pb - qa = 1$. We define a diffeomorphism $S$ of $M$ as follows; $S(x_1, x_2) = (px_1 - qx_2, -ax_1 + bx_2)$ for $(x_1, x_2) \in M$. We set $\nu = (-a + b\theta)/(p - q\theta)$ and define actions $\tilde{F}$ and $\tilde{\sigma}$ by $\tilde{F}_t = S \circ F^\theta_t \circ S^{-1}$ and $\tilde{\sigma}_t = S \circ \sigma_t \circ S^{-1}$. Then we have $\tilde{F}_t = F^\nu_{(p-q\theta)t}$ and $\tilde{\sigma}_t = F^\infty_{t/q}$. Since the system $(M, F^\theta, \sigma)$ is conjugate to $(M, \tilde{F}, \tilde{\sigma})$ by $S$, we have a similar result to that obtained above. Note that $C^*_r(\mathcal{G}|N)$ is $*$-isomorphic to $A_\nu$ for every σ-invariant closed connected submanifold $N$ with dim $N = 1$. We can summarize the conclusion just obtained as follows:

**Theorem 5.2.** Let $\sigma$ be a transverse action for $(M, G)$ defined by $\sigma = F^\mu_\nu$ for $\mu \in \mathbb{Q} \cup \{\infty\}$ and let $(B, \pi, M, C)$ be a differentiable bundle of $C^*$-algebras with respect to $F_\sigma$. Then there exists a subbundle $(D, \pi^r, M, C(\mathbb{T}))$ of $(RM(B), \bar{\pi}, M, RM(C))$ with the following properties; (i) $B_x^r D_x$ generates $B_x$ for every $x \in M$, (ii) $C^*_r(\mathcal{G}|N)$ is $*$-isomorphic to $C^*_r(D_x, \Phi_x)$ for every $x \in M$, where $N$ is a σ-invariant closed connected submanifold of $M$ with dim $N = 1$ and $\Phi_x$ is the holonomy group of the flat connection $\nabla^r$ in $D$.

(b) An action of a semi-direct product group. Let $S$ be an element of $\text{SL}(2, \mathbb{Z})$, $\lambda$ be an eigenvalue of $S$ and $(1, \theta)$ be an eigenvector of $S$ with respect to $\lambda$. We suppose that $\theta$ is real and irrational. Let $G$ be a semi-direct product group of $\mathbb{Z}$ and $\mathbb{R}$ defined by $(n, t)(m, s) = (n + M, \lambda^{-m}t + s)$ for $(n, t), (m, s) \in \mathbb{Z} \times \mathbb{R}$. We may identify the group $\mathcal{N}$ with $\mathbb{Z}$ and a connected component $G_m$ is the set of elements of the form $(m, t)$ ($t \in \mathbb{R}$) for $m \in \mathbb{Z}$. Let $M$ be the torus $\mathbb{T}^2$. Since we have $SF^\theta_t = F^\theta_{\lambda^{-t}}S$, we can define an action of $G$ on $M$ by $(n, t) \cdot x = S^n F^\theta_t(x)$ for $(n, t) \in G$ and $x \in M$. Let $\nu$ be the other eigenvalue of $S$ and $(1, \mu)$ be an eigenvector of $S$ with respect to $\nu$. We set $\sigma_t = F^\mu_t$ for $t \in \mathbb{R}$. As in (a), $\sigma$ is a transverse action for $(M, G)$. Let $(B, \pi, M, C)$ be a differentiable bundle of $C^*$-algebras with respect to the family $F_\sigma$ and let $\nabla$ be the flat connection in $B$. It follows from Theorem 4.2 that we have
\[ \nabla (cs_m(f)) = \nu^m (adx_1 + bdx_2) \otimes cs_m(f_1), \text{ where } a = -\theta/(\mu - \theta), \\
b = 1/(\mu - \theta) \text{ and } f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2) \text{ for } m \in \mathbb{Z} \text{ and } f \in C^\infty_c(\mathcal{G}). \text{ We denote by } N \text{ the submanifold } \{0\} \times \mathbb{T} \text{ of } M \text{ and denote } \\
B(S, \lambda) \text{ the reduction } C^*-\text{algebra } C^*_r(\mathcal{G}|N). \text{ This algebra was} \\
\text{studied in } [12, 13, 14]. \text{ It follows from } [7] \text{ and } [13] \text{ that it is a} \\
simple \text{algebra. We do not have any results concerning relations} \\
between \text{the bundle and the algebra } B(S, \lambda). \text{ This is the problem} \\
for \text{further investigation.} \\

(c) Actions of discrete groups. Let \((M, G)\) be a differentiable dynamical system with \(G\) discrete and let \(\sigma : \mathbb{R}^n \rightarrow \text{Diff}(M)\) be a differentiable action. Suppose that the differential of the map \(t \mapsto \sigma_t(x)\) at 0 is an isomorphism for every \(x \in M\). Then, for every \(x_0 \in M\), there exist a neighborhood \(U\) of \(x_0\) and a neighborhood \(T\) of 0 in \(\mathbb{R}^n\) such that the map \(\varphi_0 : t \mapsto \sigma_t(x_0)\) is a diffeomorphism of \(T\) onto \(U\). We set \(\varphi = \varphi_0^{-1}\). Then \((U, \varphi)\) is a local chart compatible with \(\sigma\) and \(\sigma\) is a transverse action for \((M, G)\). Let \((B, \pi, M, C)\) be a differentiable bundle of \(C^*-\text{algebras}\) with respect to \(\mathcal{F}_\sigma\) and \(\nabla\) be the flat connection. It follows from Theorem 4.2 that we have, \\
\text{for } g \in G \text{ and } f \in C^\infty_c(\mathcal{G}), \nabla(cs_g(f)) = \sum_{k=1}^n d\varphi_k \otimes cs_g(f^g_k), \text{ where} \\
\varphi = (\varphi^1, \ldots, \varphi^n) \text{ and } f^g_k(g', x) = (\partial/\partial t_k)f(g', g \sigma_t(g^{-1}x))|_{t=0}. \\

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Received August 20, 1992 and in revised form July 22, 1993.

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