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JERRY MICHAEL SHURMAN

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JERRY SHURMAN

This paper defines a nonholomorphic Eisenstein series for a totally real algebraic number field F and the special orthogonal group with respect to a bilinear form $S = \begin{pmatrix} T & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}$, where $T \in M_n(F)$ and its embedded images $T^v \in M_n(\mathbb{R})$ under archimedean places v of F have signature $(1, n-1)$. This group has an associated product of tube domains $\mathcal{H}^{\mathfrak{a}} = \prod_{v \in \mathfrak{a}} \mathcal{H}_v$, the product taken over archimedean places of F and each $\mathcal{H}_v \subset \mathbb{C}^n$. The series is denoted $E(z, s; k, \psi, \mathfrak{b})$ or simply $E(z, s)$, with $z \in \mathcal{H}^{\mathfrak{a}}$, $s \in \mathbb{C}$ a complex parameter, $k \in \mathbb{Z}$ the weight, ψ a Hecke character on the ideles of F , and the level \mathfrak{b} an integral ideal in F . E has the Fourier expansion

$$E(z, s) = (-1)^{dk} 2^{d(k+2s)} \sum_{h \in L'} a(h, y, s) e \left(\sum_{v \in \mathfrak{a}} T^v(x_v, h_v) \right),$$

where $d = [F : \mathbb{Q}]$, L' is the lattice dual to \mathfrak{o}_F^n under T , $e(x) = e^{2\pi i x}$, and $z = (x_v + iy_v)_{v \in \mathfrak{a}} \in \mathcal{H}^{\mathfrak{a}}$. The Fourier coefficient $a(h, y, s)$ is the product $(N\mathfrak{d})^{-\frac{s}{2}} a_{\mathfrak{a}}(h, y, s) a_f(h, s)$ with $N\mathfrak{d}$ the norm of the different of F over \mathbb{Q} . The archimedean factor is $a_{\mathfrak{a}}(h, y, s) = \prod_{v \in \mathfrak{a}} \xi(y_v, h_v; k + s; T^v)$ with ξ a certain confluent hypergeometric function studied by Shimura. The nonarchimedean factor $a_f(h, s)$ is essentially a product and quotient of Hecke L -functions, depending on the parity of n and the nature of h . Specializing to $s = 0$ gives holomorphic and in special cases nearly holomorphic behavior.

1. Introduction and notation.

Introduction. This paper defines an Eisenstein series $E(z, s)$ of weight k for z in a tube domain and s a complex parameter, and computes its Fourier expansion explicitly. The series is of interest as a special case of the nearly holomorphic functions studied by Shimura and Blüher.

Section 2 describes the action of a subgroup of the adelization of a certain orthogonal group on an associated complex domain. A tube domain \mathcal{H} is associated to a bilinear form S of signature $(2, n)$ on \mathbb{R}^{n+2} , and the identity component of $\mathrm{SO}(S, \mathbb{R})$, the special orthogonal group over \mathbb{R} with respect to S , acts on \mathcal{H} . Take a totally real algebraic number field F , a symmetric matrix S all of whose embedded images S^v in $M_{n+2}(\mathbb{R})$ under archimedean places v of F have signature $(2, n)$, and the algebraic group $G = \mathrm{SO}(S, F)$. Then $G_{\mathbf{A}^+}$, a suitable subgroup of the adelization of G , acts on $\mathcal{H}^{\mathbf{a}}$, a product of tube domains \mathcal{H}_v over the archimedean places v of F .

Section 3 defines an Eisenstein series $E(z, s)$ for $z \in \mathcal{H}^{\mathbf{a}}$ and $s \in \mathbb{C}$, and shows that it has a Fourier expansion. The series agrees with a series studied by Indik in the case $F = \mathbb{Q}$. $E(z, s)$ has an associated series $\tilde{E}(y, s)$ for y in a certain subset of $G_{\mathbf{A}^+}$. Harmonic analysis gives a Fourier expansion of $\tilde{E}(y, s)$ with coefficients $b(h, w_y, s)$, where h runs through a lattice in F^n and w_y depends on $y = \mathrm{Im}(z)$. This transforms back to a Fourier expansion of $E(z, s)$.

Section 4 expresses the global Fourier coefficient $a(h, y, s)$ of $E(z, s)$ as a simple factor multiplied by a product of local coefficients $a_v(h, y, s)$, the product being taken over all places of F . For archimedean v , $a_v(h, y, s)$ is equal to a certain confluent hypergeometric function ξ studied by Shimura.

Section 5 continues to study the local coefficients of $E(z, s)$. The coefficients at finite places v dividing \mathfrak{b} (where \mathfrak{b} , an integral ideal of F , is the level of $E(z, s)$) are equal to 1. The coefficients at finite places v not dividing \mathfrak{b} are power series $\alpha_v(h_v, X) = \sum_{\lambda} S_v(\lambda, h_v) X^{\lambda}$ evaluated at certain values of X , where the coefficients $S_v(\lambda, h_v)$ are sums of exponentials.

Section 6 expresses the power series $\alpha_v(h_v, X)$ as a simple rational expression of Euler factors of Hecke L -functions, which depend on the v -adic nature of the lattice vector h . In some cases $\alpha_v(h_v, X)$ is not expressed precisely, but then it is a polynomial of bounded degree. Taking the product of $\alpha_v(h_v, X)$ over finite places v not dividing \mathfrak{b} expresses the finite part of $a(h, y, s)$ as essentially a product and quotient of Hecke L -functions. Thus the Fourier coefficients of $E(z, s)$ are explicit expressions in well understood functions, up to some polynomial factors. The methods in this section are from Indik.

Section 7 specializes the Eisenstein series to $s = 0$ to obtain holomorphic and in special cases nearly holomorphic behavior. Also, for certain values of k and s , $E(z, s)$ is either finite or exhibits a simple pole with residue that is holomorphic up to a factor.

My warmest thanks to Goro Shimura for suggesting this problem as a Ph.D thesis and for all his generous help as my advisor.

Notation. \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} , and \mathbb{T} denote the integers, the rational, real and complex numbers, and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For an associative ring A with identity, A^* denotes the group of invertible elements of A . When A is commutative, $M_n(A)$ denotes the ring of n -by- n matrices with entries in A , $GL_n(R)$ means $M_n(R)^*$, and $SL_n(R)$ denotes the elements of $GL_n(R)$ with determinant 1. $\left(\begin{smallmatrix} - \\ \end{smallmatrix}\right)$ denotes the Jacobi symbol, and for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer n such that $n < x$.

2. Archimedean and adelic preliminaries.

The quadratic forms T and S and the complex domain \mathcal{H} . Let $n > 2$ be an integer, and let T , a symmetric element of $M_n(\mathbb{R})$, define a quadratic form of signature $(1, n - 1)$ on \mathbb{R}^n . Write $T(x, y) = {}^t x T y$ and $T[x] = T(x, x)$ for $x, y \in \mathbb{C}^n$. Set

$$(2.1) \quad S = \begin{pmatrix} T & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix},$$

defining a quadratic form of signature $(2, n)$ on \mathbb{R}^{n+2} , and write $S(x, y) = {}^t x S y$, $S[x] = S(x, x)$ for $x, y \in \mathbb{C}^{n+2}$.

Fix $\varepsilon \in \mathbb{R}^n$ such that $T[\varepsilon] = 1$. Define a set \mathcal{P} of “positive” elements in \mathbb{R}^n by

$$\mathcal{P} = \{ y \in \mathbb{R}^n : T[y] > 0 \text{ and } T(y, \varepsilon) > 0 \}$$

and a complex domain \mathcal{H} by

$$\mathcal{H} = \{ z = x + iy \in \mathbb{C}^n : y \in \mathcal{P} \}.$$

\mathcal{P} and \mathcal{H} are connected.

The action of $\text{SO}(S, \mathbb{R})^\circ$ on \mathcal{H} . Let $\mathcal{G} = \text{SO}(S, \mathbb{R})^\circ$, where “ \circ ” denotes the identity component and

$$\text{SO}(S, \mathbb{R}) = \left\{ \alpha \in \text{SL}_{n+2}(\mathbb{R}) : {}^t\alpha S \alpha = S \right\}.$$

Thus for $x, y \in \mathbb{C}^{n+2}$ and $\alpha \in \mathcal{G}$, $S(\alpha x, \alpha y) = S(x, y)$ and $S[\alpha x] = S[x]$.

For $z \in \mathbb{C}^n$, define $w(z) = \begin{pmatrix} z \\ \frac{1}{2}T[z] \\ 1 \end{pmatrix} \in \mathbb{C}^{n+2}$. Any S -isotropic

$w \in \mathbb{C}^{n+2}$ with bottom entry 1 is of this form. If $z \in \mathcal{H}$ and $\alpha \in \mathcal{G}$ then $\{ \text{Re}(\alpha w(z)), \text{Im}(\alpha w(z)) \}$ forms an orthogonal basis $\{ u, v \}$ (with $S[u] = S[v]$) of a subspace in \mathbb{R}^{n+2} where S is positive definite. Set $j(\alpha, z) = \alpha w(z)_{n+2}$, which is nonzero, and define $\alpha(z) \in \mathbb{C}^n$ by

$$(2.2) \quad w(\alpha(z)) = j(\alpha, z)^{-1} \alpha w(z).$$

Since $j(\alpha, z)^{-1} \alpha w(z)$ is S -isotropic and has bottom entry 1, such an $\alpha(z)$ indeed exists.

To show that $\alpha(z) \in \mathcal{H}$, first note that $0 < T[\text{Im}(\alpha(z))] = S[\text{Im}(w(\alpha(z)))]$ follows from (2.2) and the properties of $\{ u, v \}$. Also, $T(\text{Im}(\alpha(z)), \varepsilon) > 0$: because $T(\text{Im}(\alpha(z)), \varepsilon)$ can not vanish as T is negative definite on $\{ x \in \mathbb{R}^n : T(x, \varepsilon) = 0 \}$ but positive at $\text{Im}(\alpha(z))$, it suffices to show $T(\text{Im}(\alpha(z)), \varepsilon) > 0$ for one α from the connected group \mathcal{G} , and taking $\alpha = I_{n+2}$ completes the proof.

Not all of $\text{SO}(S, \mathbb{R})$ acts on \mathcal{H} because while \mathcal{G} fixes \mathcal{H} and $-\mathcal{H}$, the other component interchanges them. Taking $\alpha = \begin{pmatrix} I_n & \\ & -I_2 \end{pmatrix}$, so that $\alpha(z) = -z$, shows this. From (2.2), the action of \mathcal{G} on \mathcal{H} is associative and j is a factor of automorphy. The action is well known to be transitive.

The field F and the group G . Let F denote a totally real algebraic number field of degree d , \mathfrak{o}_F the ring of algebraic integers in F , and $\mathfrak{a} = \{ v_1, \dots, v_d \}$ the set of archimedean places of F . Each $v \in \mathfrak{a}$ is an embedding $v : F \hookrightarrow \mathbb{R}$. Take T a symmetric element of $M_n(\mathfrak{o}_F)$ such that T^v defines a form of signature $(1, n-1)$ on \mathbb{R}^n for each $v \in \mathfrak{a}$. Define S as in (2.1), so that the S^v for all $v \in \mathfrak{a}$ define forms of signature $(2, n)$. For each $v \in \mathfrak{a}$ take an $\varepsilon_v \in \mathbb{R}^n$ such that $T^v[\varepsilon_v] = 1$. Set

$$G = \text{SO}(S, F) = \left\{ \alpha \in \text{SL}_{n+2}(F) : {}^t\alpha S \alpha = S \right\}.$$

The action of $G_{\mathbf{A}+}$ on $\mathcal{H}^{\mathbf{a}}$. Let \mathbf{f} and \mathbf{a} denote the set of nonarchimedean and archimedean places of F , respectively. For $v \in \mathbf{f} \cup \mathbf{a}$ denote by F_v the v -completion of F and, if $v \in \mathbf{f}$, by \mathfrak{o}_v the v -closure of \mathfrak{o}_F in F_v ; if $v \in \mathbf{a}$, identify F_v with \mathbb{R} . Denote the adeles and ideles of F as $F_{\mathbf{A}}$ and $F_{\mathbf{A}}^*$ and identify F with its embedded images in $F_{\mathbf{A}}$ and F_v for any v . $F_{\mathbf{f}}$ denotes the adeles $(a_v)_{v \in \mathbf{f} \cup \mathbf{a}}$ such that $a_v = 0$ for $v \notin \mathbf{f}$, $F_{\mathbf{a}}$ is defined similarly, and $\mathfrak{o}_{\mathbf{f}}$ denotes the elements of $F_{\mathbf{f}}$ such that $a_v \in \mathfrak{o}_v$ for all $v \in \mathbf{f}$; $F_{\mathbf{f}}^*$, $F_{\mathbf{a}}^*$ and $\mathfrak{o}_{\mathbf{f}}^*$ are the similarly defined subgroups of $F_{\mathbf{A}}^*$. The image of \mathfrak{o}_F in \mathbb{R}^d under $x \mapsto (x^v)_{v \in \mathbf{a}}$ is a lattice Λ of volume $(N\mathfrak{d})^{\frac{1}{2}}$, where N denotes the norm from F to \mathbb{Q} and \mathfrak{d} denotes the different of F over \mathbb{Q} .

Define G_v to be the v -completion of G for $v \in \mathbf{f} \cup \mathbf{a}$. Thus if $v \in \mathbf{a}$, G_v can be identified with $\text{SO}(S^v, \mathbb{R})$. Take the adelization $G_{\mathbf{A}}$ of G ; put $G_{\mathbf{f}} = \prod_{v \in \mathbf{f}} G_v \cap G_{\mathbf{A}}$, $G_{\mathbf{a}} = \prod_{v \in \mathbf{a}} G_v$. Identify G with its embedded image in $G_{\mathbf{A}}$ and the same convention holds for other groups defined below. For $x \in G_{\mathbf{A}}$ define $x_{\mathbf{f}} \in G_{\mathbf{f}}$ and $x_{\mathbf{a}} \in G_{\mathbf{a}}$ by $x = x_{\mathbf{f}}x_{\mathbf{a}}$. Define

$$G_{\mathbf{A}+} = \{ x \in G_{\mathbf{A}} : x_v \in \text{SO}(S^v, \mathbb{R})^\circ \text{ for all } v \in \mathbf{a} \}$$

and $G_{\mathbf{a}+} = G_{\mathbf{a}} \cap G_{\mathbf{A}+}$, $G_+ = G \cap G_{\mathbf{A}+}$.

For each $v \in \mathbf{a}$, let \mathcal{H}_v be the complex domain of the previous section associated to T^v and ε_v . Denote $\prod_{v \in \mathbf{a}} \mathcal{H}_v$ as $\mathcal{H}^{\mathbf{a}}$ and define the action of $G_{\mathbf{a}+}$ on $\mathcal{H}^{\mathbf{a}}$ componentwise. The action extends to $G_{\mathbf{A}+}$ by defining $x \in G_{\mathbf{A}+}$ to act as $x_{\mathbf{a}}$.

3. The Eisenstein series $E(z, s; k, \psi, \mathfrak{b})$ and its Fourier expansion.

The series E on $\mathcal{H}^{\mathbf{a}}$. Fix an integer k . Take a Hecke character $\psi : F_{\mathbf{A}}^* \rightarrow \mathbb{T}$ ($\psi(F^*) = 1$) with $\psi(a) = \prod_{v \in \mathbf{a}} \text{sgn}(a_v)^k$ for $a \in F_{\mathbf{a}}^*$; let \mathfrak{c} denote the finite part of its conductor, ψ_v the v -component of ψ , and $\psi_{\mathfrak{l}} = \prod_{v|\mathfrak{l}} \psi_v$ for any integral ideal \mathfrak{l} . Let $\mathfrak{b} \subset F$ be an integral ideal divisible by \mathfrak{c} , by 2, and by $\det T$. Define $\mathcal{U} = \{ u \in F^{n+2} : S[u] = 0 \}$, and for $u \in \mathcal{U}$, $z \in \mathcal{H}^{\mathbf{a}}$, set

$$S(u, w(z)) = \prod_{v \in \mathbf{a}} S^v(u_v, w_v(z)), \text{ where } w_v(z) = \begin{pmatrix} z_v \\ \frac{1}{2}T^v[z_v] \\ 1 \end{pmatrix}. \text{ Our}$$

Eisenstein series is defined as follows:

$$E(z, s; k, \psi, \mathfrak{b}) = \sum_{(u,t) \in \mathcal{U} \times F_{\mathfrak{f}}^* / \sim} c(tu)\psi(t)^{-1}|t|^{k+2s}S(u, w(z))^{-k}|S(u, w(z))|^{-2s}$$

for $z \in \mathcal{H}^{\mathfrak{a}}$ and $s \in \mathbb{C}$, where $(u, t) \sim (u', t')$ means that for some $b \in F^*$, $u' = bu$ and $t' \mathfrak{o}_F = b^{-1}t\mathfrak{o}_F$ (so that $t' = eb_{\mathfrak{f}}^{-1}t$ with $e \in \mathfrak{o}_{\mathfrak{f}}^*$). Here $c : F_{\mathfrak{A}}^{n+2} \rightarrow \mathbb{C}$ is the locally constant function

$$c(x) = \begin{cases} \psi_{\mathfrak{b}}(x_{n+2}), & \text{if } x_{\mathfrak{f}} \in \mathfrak{o}_{\mathfrak{f}}^{n+2} \text{ and } x_{n+2} \text{ is prime to } \mathfrak{b} \\ 0, & \text{otherwise.} \end{cases}$$

This series is also denoted simply $E(z)$ or $E(z, s)$.

E is readily seen to be well-defined. The series converges for sufficiently large $\text{Re}(s)$ and has an analytic continuation, as shown in [Sh80]. In the special case $F = \mathbb{Q}$, E reduces to the series studied by Indik in [In].

Transformation of E . Define subgroups of $G_{\mathfrak{A}+}$ by

$$P_{\mathfrak{A}} = \left\{ \gamma \in G_{\mathfrak{A}+} : \gamma = \begin{pmatrix} * & * & * \\ & * & * \\ 0 & 0 & * \end{pmatrix} \right\};$$

$$C = \prod_v C_v, \text{ where } C_v = \begin{cases} \text{SO}(S, \mathfrak{o}_v) & \text{if } v \in \mathfrak{f}, \\ \text{stabilizer of } i\varepsilon_v & \text{if } v \in \mathfrak{a}; \end{cases}$$

$$D = \left\{ \gamma \in C : \gamma \equiv \begin{pmatrix} * & * & * \\ & * & * \\ 0 & 0 & d_{\gamma} \end{pmatrix} \pmod{\mathfrak{b}} \right\};$$

and $\Gamma_0(\mathfrak{b}) \subset G_+$ by

$$\Gamma_0(\mathfrak{b}) = G_+ \cap DG_{\mathfrak{a}} = \left\{ \gamma \in G_+ \cap \text{SO}(S, \mathfrak{o}_F) : \gamma \equiv \begin{pmatrix} * & * & * \\ & * & * \\ 0 & 0 & d_{\gamma} \end{pmatrix} \pmod{\mathfrak{b}} \right\}.$$

For $\gamma \in G_{\mathfrak{A}+}$ and $z \in \mathcal{H}^{\mathfrak{a}}$ define

$$J(\gamma, z) = j(\gamma, z)^k |j(\gamma, z)|^{2s} \quad \text{where } j(\gamma, z) = \prod_{v \in \mathfrak{a}} j(\gamma_v, z_v),$$

$$J_{\psi}(\gamma, z) = \psi_{\mathfrak{b}}(d_{\gamma})J(\gamma, z).$$

The relation $J(\alpha\beta, z) = J(\alpha, \beta z)J(\beta, z)$ holds for all $\alpha, \beta \in G_{\mathbf{A}^+}$, and the same relation holds for J_ψ when $\alpha, \beta \in DG_{\mathbf{a}^+}$.

For $\gamma \in \Gamma_0(\mathfrak{b})$ and $z \in \mathcal{H}^{\mathbf{a}}$ one easily verifies that

$$E(\gamma(z)) = J_\psi(\gamma, z)E(z).$$

If, in particular,

$$\gamma \in \Gamma_0(\mathfrak{b}) \cap N, \text{ where } N = \left\{ \begin{pmatrix} 1 & 0 & b \\ & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : b \in F^n \right\},$$

then $b \in \mathfrak{o}_F^n$, $\gamma(z) = z + b$, and $J_\psi(\gamma, z) = 1$. Thus, $E(z + b) = E(z)$ for $b \in \mathfrak{o}_F^n$.

The series \tilde{E} on $G_+DG_{\mathbf{a}^+}$. Define $\tilde{E}(y, s)$ for $y \in G_+DG_{\mathbf{a}^+}$ and $s \in \mathbb{C}$ by

$$\tilde{E}(y, s) = E(x(i\varepsilon), s)J_\psi(x, i\varepsilon)^{-1}$$

for $y = \alpha x$ with $\alpha \in G_+, x \in DG_{\mathbf{a}^+}$.

Here $i\varepsilon$ means $(i\varepsilon_v)_{v \in \mathbf{a}} \in \mathcal{H}^{\mathbf{a}}$. $\tilde{E}(y, s)$ is well defined. Denote this series also $\tilde{E}(y)$. Then

$$\tilde{E}(\alpha y w) = \tilde{E}(y)J_\psi(w, i\varepsilon)^{-1} \text{ for } \alpha \in G_+, y \in G_+DG_{\mathbf{a}^+}, w \in D.$$

To write \tilde{E} explicitly, first note that

$$S(u, w(x(i\varepsilon))) = j(x, i\varepsilon)^{-1}S(x^{-1}u, w(i\varepsilon)).$$

So for $\alpha \in G_+, x \in DG_{\mathbf{a}^+}$,

$$\begin{aligned} \tilde{E}(\alpha x) &= \sum_{(u,t)} c(tu)\psi(t)^{-1}|t|^{k+2s}J(x, i\varepsilon)S(x^{-1}u, w(i\varepsilon))^{-k} \\ &\quad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s}J_\psi(x, i\varepsilon)^{-1} \\ &= \sum \psi_{\mathfrak{b}}(d_{x^{-1}})c(tu)\psi(t)^{-1}|t|^{k+2s}S(x^{-1}u, w(i\varepsilon))^{-k} \\ &\quad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s} \\ &= \sum c(x^{-1}tu)\psi(t)^{-1}|t|^{k+2s}S(x^{-1}u, w(i\varepsilon))^{-k} \\ &\quad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s}. \end{aligned}$$

The Fourier expansions of \tilde{E} and E . Let $V = F^n$ and $V_{\mathbf{A}} = F_{\mathbf{A}}^n$. For $x, y \in V_{\mathbf{A}}$ define a complex number $\chi(T(x, y))$:

$$\begin{aligned} \chi(T(x, y)) &= \prod_{v \in f \cup \mathbf{a}} e_v(T(x_v, y_v)) \\ &= \prod_{v \in f} e_p(T\tau_{F_v/\mathbb{Q}_p}(T(x_v, y_v))) \prod_{v \in \mathbf{a}} e(T^v(x_v, y_v)), \end{aligned}$$

where $v \mid p$, $e_p(t) = e(\text{the fractional part of } -t)$ for $t \in \mathbb{Q}_p$, and $e(s) = e^{2\pi i s}$ for $s \in \mathbb{C}$. Define

$$\tau(v) = \begin{pmatrix} 1 & 0 & v \\ & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G_+DG_{\mathbf{a}+} \text{ for } v \in V_{\mathbf{A}} (\tau(v) \in G_+DG_{\mathbf{a}+}$$

since $v = v' + w$ with $v' \in V$, $w \in \prod_f \mathfrak{o}_f^n \times F_{\mathbf{a}}^n$), and fix a Haar measure μ on $V_{\mathbf{A}}$ so that $\mu(V_{\mathbf{A}}/V) = 1$.

Consider $\tilde{E}(\tau(v)w)$ with $v \in V_{\mathbf{A}}$ and $w \in G_{\mathbf{a}+}$ as a function on $V_{\mathbf{A}}$. Then for $u \in V$, $\tilde{E}(\tau(v+u)w) = \tilde{E}(\tau(u)\tau(v)w) = \tilde{E}(\tau(v)w)$, so \tilde{E} is a function on $V_{\mathbf{A}}/V$. This gives the expansion

$$\tilde{E}(\tau(v)w, s) = \sum_{h \in V} b(h, w, s) \chi(T(v, h)) \quad \text{for } v \in V_{\mathbf{A}}, w \in G_{\mathbf{a}+},$$

where

$$b(h, w, s) = \int_{v \in V_{\mathbf{A}}/V} \tilde{E}(\tau(v)w, s) \chi(-T(v, h)) d\mu(v) \quad \text{for } h \in V.$$

Define lattices $L = \mathfrak{o}_F^n \subset V$ and $L_v = \mathfrak{o}_v^n \subset V_v$ for $v \in f$. For $u \in L_v$, $\tilde{E}(\tau(v+u)w) = \tilde{E}(\tau(v)w\tau(u)) = \tilde{E}(\tau(v)w) J_{\psi}(\tau(u), i\varepsilon)^{-1} = \tilde{E}(\tau(v)w)$. Hence $b(h, w, s) = \int_{v \in V_{\mathbf{A}}/V} \tilde{E}(\tau(v+u)w, s) \chi(-T(v+u, h)) d\mu(v) = \chi(-T(u, h)) b(h, w, s)$; this shows that $b(h, w, s) \neq 0$ only when $\chi(-T(u, h)) = 1$, i.e., when $h \in L'$ with $L' =$ the dual lattice to L under T , defined by $L' = \{h \in V : T(h, L) \subset \mathfrak{d}^{-1}\}$, where \mathfrak{d} is the different of F over \mathbb{Q} . Thus,

$$\tilde{E}(\tau(v)w, s) = \sum_{h \in L'} b(h, w, s) \chi(T(v, h)) \text{ for } v \in V_{\mathbf{A}}, w \in G_{\mathbf{a}+}.$$

To express this on $\mathcal{H}^{\mathbf{a}}$ for $z = (z_v)_{v \in \mathbf{a}}$ with $z_v = x_v + iy_v$, put $w_y = (w_{y_v})_{v \in \mathbf{a}}$ with

$$w_{y_v} = \begin{pmatrix} A_v \\ \sqrt{T[y_v]} \\ \sqrt{T[y_v]}^{-1} \end{pmatrix},$$

where $A_v \varepsilon_v = y_v / \sqrt{T^v[y_v]}$ and $T^v(A_v x, A_v y) = T^v(x, y)$ for $x, y \in \mathbb{R}^n$, so that $w_{y_v}(i\varepsilon_v) = iy_v$ and hence $w_y(i\varepsilon) = iy$. Then

$$\begin{aligned} E(z, s) &= \tilde{E}(\tau(x)w_y, s) J_\psi(\tau(x)w_y, i\varepsilon) \\ &= \tilde{E}(\tau(x)w_y, s) J(w_y, i\varepsilon), \end{aligned}$$

so

$$E(z, s) = J(w_y, i\varepsilon) \sum_{h \in L'} b(h, w_y, s) e\left(\sum_{v \in \mathfrak{a}} T^v(x_v, h_v)\right).$$

4. Fourier coefficients of E : reduction to the local case.

The coefficient $b(h, w_y, s)$. For $h \in L'$ and $x + iy \in \mathcal{H}^{\mathfrak{a}}$ we have $b(h, w_y, s) = \int_{v \in V_{\mathfrak{A}}/V} \tilde{E}(\tau(v)w_y, s) \chi(-T(v, h)) d\mu(v)$. Choosing representatives v of $V_{\mathfrak{A}}/V$ such that $\tau(v) \in DG_{\mathfrak{a}}$ gives

$$\begin{aligned} & \cdot b(h, w_y, s) \\ &= \int_{v \in V_{\mathfrak{A}}/V} \left\{ \sum_{(u,t) \in \mathcal{U} \times F_{\mathfrak{f}}^* / \sim} c\left((\tau(v)w_y)^{-1}tu\right) \psi(t)^{-1} |t|^{k+2s} \right. \\ & \quad \cdot S\left((\tau(v)w_y)^{-1}u, w(i\varepsilon)\right)^{-k} \\ & \quad \left. \cdot \left| S\left((\tau(v)w_y)^{-1}u, w(i\varepsilon)\right) \right|^{-2s} \chi(-T(v, h)) \right\} d\mu(v). \end{aligned}$$

If $u_{n+2} = 0$ then $((\tau(x)w_y)^{-1}tu)_{n+2} = 0$ at \mathfrak{f} since $(\tau(x)w_y)_{\mathfrak{f}} \in P_{\mathfrak{A}}$. So normalize $u_{n+2} = 1$ and sum over $\{(w(v'), t) : v' \in V, t \in F_{\mathfrak{f}}^* / \mathfrak{o}_{\mathfrak{f}}^*\}$. This gives

$$\begin{aligned} & b(h, w_y, s) \\ &= \int_{v \in V_{\mathfrak{A}}/V} \left\{ \sum_{v' \in V} S\left((\tau(v)w_y)^{-1}w(v'), w(i\varepsilon)\right)^{-k} \right. \\ & \quad \cdot \left| S\left((\tau(v)w_y)^{-1}w(v'), w(i\varepsilon)\right) \right|^{-2s} \\ & \quad \cdot \sum_{t \in F_{\mathfrak{f}}^* / \mathfrak{o}_{\mathfrak{f}}^*} c\left((\tau(v)w_y)^{-1}tw(v')\right) \\ & \quad \left. \cdot \psi(t)^{-1} |t|^{k+2s} \chi(-T(v, h)) \right\} d\mu(v) \end{aligned}$$

$$\begin{aligned}
&= \int_{v \in V_{\mathbf{A}}/V} \left\{ \sum_{v' \in V} S(w_y^{-1}w(v' - v), w(i\varepsilon))^{-k} \right. \\
&\quad \cdot |S(w_y^{-1}w(v' - v), w(i\varepsilon))|^{-2s} \\
&\quad \cdot \sum_{t \in F_{\mathfrak{f}}^*/\mathfrak{o}_{\mathfrak{f}}^*} c(tw_y^{-1}w(v' - v))\psi(t)^{-1}|t|^{k+2s} \\
&\quad \left. \cdot \chi(-T(v - v', h)) \right\} d\mu(v) \\
&= \int_{v \in V_{\mathbf{A}}} \left\{ S(w_y^{-1}w(v), w(i\varepsilon))^{-k} |S(w_y^{-1}w(v), w(i\varepsilon))|^{-2s} \right. \\
&\quad \left. \cdot \sum_{t \in F_{\mathfrak{f}}^*/\mathfrak{o}_{\mathfrak{f}}^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v, h)) \right\} d\mu(v) \\
&= \int_{v \in V_{\mathbf{A}}} \left\{ S(w(v), j(w_y, i\varepsilon)w(iy))^{-k} |S(w(v), j(w_y, i\varepsilon)w(iy))|^{-2s} \right. \\
&\quad \left. \cdot \sum_{t \in F_{\mathfrak{f}}^*/\mathfrak{o}_{\mathfrak{f}}^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v, h)) \right\} d\mu(v) \\
&= J(w_y, i\varepsilon)^{-1} \int_{v \in V_{\mathbf{A}}} \left\{ S(w(v), w(iy))^{-k} |S(w(v), w(iy))|^{-2s} \right. \\
&\quad \left. \cdot \sum_{t \in F_{\mathfrak{f}}^*/\mathfrak{o}_{\mathfrak{f}}^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v, h)) \right\} d\mu(v).
\end{aligned}$$

LEMMA. $S(w(v), w(iy)) = \left(-\frac{1}{2}\right)^d T_{\mathbf{a}}[-v + iy]$, where $d = [F : \mathbb{Q}]$ and $T_{\mathbf{a}}[x] = \prod_{v \in \mathbf{a}} T^v[x_v]$ for $x \in V_{\mathbf{A}}$.

Proof. Immediate from

$$S(w(v), w(iy)) = \prod_{v \in \mathbf{a}} \left({}^t v_v \frac{1}{2} T^v[v_v] 1 \right) \begin{pmatrix} T^v & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} \begin{pmatrix} iy_v \\ \frac{1}{2} T^v[iy_v] \\ 1 \end{pmatrix}.$$

□

This gives

$$\begin{aligned}
 b(h, w_y, s) &= J(w_y, i\varepsilon)^{-1}(-1)^{dk}2^{d(k+2s)} \\
 &\cdot \int_{v \in V_{\mathbf{A}}} T_{\mathbf{a}}[-v + iy]^{-k} |T_{\mathbf{a}}[-v + iy]|^{-2s} \sigma(v, s) \chi(T(v, h)) d\mu(v) \\
 &= J(w_y, i\varepsilon)^{-1}(-1)^{dk}2^{d(k+2s)} \\
 &\cdot \int_{v \in V_{\mathbf{A}}} T_{\mathbf{a}}[v + iy]^{-k} |T_{\mathbf{a}}[v + iy]|^{-2s} \sigma(v, s) \chi(-T(v, h)) d\mu(v),
 \end{aligned}$$

where

$$\sigma(x, s) = \sum_{t \in F_{\mathbf{f}}^* / \mathfrak{o}_{\mathbf{f}}^*} c(tw(x)) \psi(t)^{-1} |t|^{k+2s} \text{ for } x \in V_{\mathbf{A}}, s \in \mathbb{C}.$$

The sum $\sigma(x, s)$. For $x \in V_{\mathbf{A}}$ and $v \in \mathbf{f}$ define a local ideal $\iota_v(x_v) \subset \mathfrak{o}_v$ by $\iota_v(x_v) = \mathfrak{p}_v^{i_v(x)}$, where \mathfrak{p}_v is the maximal ideal of \mathfrak{o}_v and $i_v(x) = -\min_{1 \leq i \leq n+2} \{ \nu_v(w(x)_i) \}$ with ν_v the normalized v -adic valuation on F_v . $\iota_v(x_v)$ is integral since $w(x_v)_{n+2} = 1$, and $\iota_v(x_v) = \mathfrak{o}_v$ for almost all v .

The product ideal $\iota(x) = \prod_{v \in \mathbf{f}} \iota_v(x_v) \subset \mathfrak{o}_{\mathbf{f}}$ is such that $tw(x) \in \mathfrak{o}_{\mathbf{f}}^{n+2}$ for $t \in F_{\mathbf{f}}^*$ if and only if $t \in \iota(x)$. Thus $c(tw(x)) \neq 0$ if and only if $t \in \iota(x)$ and $(tw(x))_{n+2} = t$ is prime to \mathfrak{b} , in which case $c(tw(x)) = \psi_{\mathfrak{b}}(t)$ and the summand of $\sigma(x, s)$ is $\prod_{\substack{v \in \mathbf{f} \\ v \nmid \mathfrak{b}}} \psi(t_v)^{-1} |t_v|_v^{k+2s}$.

Thus

$$\begin{aligned}
 \sigma(x, s) &= \sum_{\substack{t = \prod_{v \in \mathbf{f}} \mathfrak{p}_v^{j_v} : \iota(x) | t \\ v \nmid \mathfrak{b}}} \prod_{\substack{v \in \mathbf{f} \\ v \nmid \mathfrak{b}}} \psi(\mathfrak{p}_v^{j_v})^{-1} |\mathfrak{p}_v^{j_v}|_v^{k+2s} \\
 &= \sum_t \prod_v (\psi(\mathfrak{p}_v)^{-1} |\mathfrak{p}_v|_v^{k+2s})^{j_v}.
 \end{aligned}$$

(The sum is empty if $\iota(x)$ is nontrivial at \mathfrak{b} .) This has the Euler product expansion $\sigma(x, s) = \prod_{v \in \mathbf{f}} \sigma_v(x_v, s)$, where

$$\begin{aligned}
 \sigma_v(x_v, s) &= \begin{cases} \delta_v(x_v), & \text{if } v | \mathfrak{b} \\ (1 - \psi(\mathfrak{p}_v)^{-1} |\mathfrak{p}_v|_v^{k+2s})^{-1} (\psi(\mathfrak{p}_v)^{-1} |\mathfrak{p}_v|_v^{k+2s})^{i_v(x_v)}, & \text{if } v \nmid \mathfrak{b}. \end{cases}
 \end{aligned}$$

Here $\delta_v(x_v) = 1$ if $x \in L_v$ (so that $\iota_v(x_v) = \mathfrak{o}_v$), 0 if $x_v \notin L_v$ (so that $\iota_v(x_v) \neq \mathfrak{o}_v$).

The local coefficient $a_v(h, y, s)$. We now have for $z = (z_v) = (x_v + iy_v) \in \mathcal{H}^{\mathbf{a}}$,

$$E(z, s) = (-1)^{dk} 2^{d(k+2s)} \sum_{h \in L'} a(h, y, s) \mathbf{e} \left(\sum_{v \in \mathbf{a}} T^v(x_v, h_v) \right),$$

where

$$a(h, y, s) = \int_{x \in V_{\mathbf{A}}} T_{\mathbf{a}}[x + iy]^{-k} |T_{\mathbf{a}}[x + iy]|^{-2s} \sigma(x, s) \chi(-T(x, h)) d\mu(x),$$

with

$$\begin{aligned} T_{\mathbf{a}}[x + iy] &= \prod_{v \in \mathbf{a}} T^v[x_v + iy_v], & \sigma(x, s) &= \prod_{v \in \mathbf{f}} \sigma_v(x_v, s), \\ \chi(-T(x, h)) &= \prod_v \mathbf{e}_v(-T(x_v, h_v)), & d\mu(x) &= c_{\mu} \prod_v d\mu_v(x_v), \end{aligned}$$

where $\mu(V_{\mathbf{A}}/V) = 1$, $\mu = c_{\mu} \prod_v \mu_v$, $\mu_v(L_v) = 1$ for $v \in \mathbf{f}$, and μ_v is Euclidean measure on \mathbb{R}^n for $v \in \mathbf{a}$; these determine $c_{\mu} = N\mathfrak{d}^{-n/2}$.

So

$$a(h, y, s) = N\mathfrak{d}^{-n/2} \prod_v a_v(h, y, s),$$

where for $v \in \mathbf{a}$,

$$\begin{aligned} a_v(h, y, s) &= \int_{x \in V_v} T^v[x + iy_v]^{-k} |T^v[x + iy_v]|^{-2s} \mathbf{e}(-T^v(x, h_v)) d\mu_v(x) \\ &= \int_{x \in V_v} T^v[x + iy_v]^{-k-s} T^v[x - iy_v]^{-s} \mathbf{e}(-T^v(x, h_v)) d\mu_v(x) \\ &= \xi(y_v, h_v; k + s, s; T^v), \end{aligned}$$

with ξ the confluent hypergeometric function studied by Shimura in [Sh82]. For $v \in \mathbf{f}$, the local coefficient does not depend on y and so may be denoted $a_v(h, s)$. Setting $q_v = |\mathfrak{p}_v|_v^{-1}$ and $X_v(s) = \psi(\mathfrak{p}_v)^{-1} q_v^{-k-2s}$ gives

$$\begin{aligned} a_v(h, s) &= \int_{x \in V_v} \sigma_v(x, s) \mathbf{e}_v(-T(x, h_v)) d\mu_v(x) \\ &= \begin{cases} \int_{x \in V_v} \delta_v(x_v) \mathbf{e}_v(-T(x, h_v)) d\mu_v(x) & \text{if } v \mid \mathfrak{b} \\ (1 - X_v(s))^{-1} \int_{x \in V_v} X_v(s)^{i_v(x_v)} \mathbf{e}_v(-T(x, h_v)) d\mu_v(x) & \text{if } v \nmid \mathfrak{b}. \end{cases} \end{aligned}$$

5. Local Fourier coefficients of E .

The archimedean coefficient $\xi(y_v, h_v; k + s, s; T^v)$. In [Sh82], Shimura defines the functions

$$\xi(y, h; \alpha, \beta; T) = \int_{x \in \mathbb{R}^n} T[x + iy]^{-\alpha} T[x - iy]^{-\beta} e(-T(x, h)) dx,$$

where $y \in \mathcal{P}$, $h \in \mathbb{R}^n$, $(\alpha, \beta) \in \mathbb{C}^2$, T defines a form of signature $(1, n - 1)$ on \mathbb{R}^n ; and

$$\begin{aligned} \eta^*(y, h; \alpha, \beta; T) \\ = T[y]^{\alpha + \beta - \frac{n}{2}} \int_{x \in Q(h)} T[x + h]^{\alpha - \frac{n}{2}} T[x - h]^{\beta - \frac{n}{2}} e^{-T(y, x)} dx, \end{aligned}$$

where $Q(h) = \{x \in \mathbb{R}^n : x \pm h \in \mathcal{P}\}$. Both integrals converge when $Re(\alpha) > n/2 - 1$, $Re(\beta) > n/2 - 1$. He defines

$$\omega(y, h; \alpha, \beta; T) = \eta^*(y, h; \alpha, \beta; T)$$

$$\cdot \left\{ \begin{array}{ll} 2^{-2\alpha} \Gamma_n(\beta)^{-1} \delta(hy)^{\frac{n}{2} - \alpha}, & h \in \mathcal{P} \\ 2^{-2\beta} \Gamma_n(\alpha)^{-1} \delta(hy)^{\frac{n}{2} - \beta}, & -h \in \mathcal{P} \\ \left| \det T \right|^{\frac{1}{2}} 2^{-2\alpha - 2\beta} \Gamma(\alpha - \frac{n-2}{2})^{-1} \Gamma(\beta - \frac{n-2}{2})^{-1} \\ \quad \cdot \delta_+(hy)^{1 - \alpha + \frac{n-2}{4}} \delta_-(hy)^{1 - \beta + \frac{n-2}{4}}, & T[h] < 0 \\ \left| \det T \right|^{\frac{1}{2}} 2^{-2\alpha - 2\beta} \Gamma(\alpha + \beta - \frac{n}{2})^{-1} \Gamma(\beta - \frac{n-2}{2})^{-1} \\ \quad \cdot \delta(hy)^{\frac{n}{2} - \alpha}, & T[h] = 0, \\ & T(\varepsilon, h) > 0 \\ \left| \det T \right|^{\frac{1}{2}} 2^{-2\alpha - 2\beta} \Gamma(\alpha + \beta - \frac{n}{2})^{-1} \Gamma(\alpha - \frac{n-2}{2})^{-1} \\ \quad \cdot \delta(hy)^{\frac{n}{2} - \beta}, & T[h] = 0, \\ & T(\varepsilon, h) < 0 \\ \Gamma_n(\alpha + \beta - \frac{n}{2})^{-1}, & h = 0, \end{array} \right.$$

where ε is as in section 2 and

$$\Gamma_n(s) = \left| \det T \right|^{-\frac{1}{2}} 2^{2s-1} \pi^{\frac{n}{2}-1} \Gamma(s) \Gamma\left(s - \frac{n}{2} + 1\right),$$

$\delta_+(hy)$ = the product of all positive roots to

$$\lambda^2 - 2T(y, h)\lambda + T[y]T[h] = 0,$$

$\delta_-(hy) = \delta_+((-h)y)$, $\delta(hy) = \delta_+(hy)\delta_-(hy)$;

and proves the relation

(5.1)

$$\begin{aligned} &\xi(y, h; k + s, s; T) \\ &= |\det T|^{-\frac{1}{2}} (-1)^k 2^{n-2k-4s} T[y]^{\frac{n}{2}-k-2s} \omega(2\pi y, h; k + s, s; T) \\ &\quad \cdot \begin{cases} 2^{2k+2s+1} \pi^{2k+2s+1-\frac{n}{2}} \Gamma(k + s)^{-1} \Gamma(k + s + 1 - \frac{n}{2})^{-1} \\ \quad \cdot \delta_+(hy)^{k+s-\frac{n}{2}}, & h \in \mathcal{P} \\ 2^{2s+1} \pi^{2s+1-\frac{n}{2}} \Gamma(s)^{-1} \Gamma(s + 1 - \frac{n}{2})^{-1} \delta_-(hy)^{s-\frac{n}{2}}, & -h \in \mathcal{P} \\ 2^{k+2s+\frac{n}{2}+1} \pi^{k+2s+1-\frac{n}{2}} \Gamma(k + s)^{-1} \Gamma(s)^{-1} \\ \quad \cdot \delta_+(hy)^{k+s-1-\frac{n-2}{4}} \delta_-(hy)^{s-1-\frac{n-2}{4}}, & T[h] < 0 \\ 2^{k+s+2+\frac{n}{2}} \pi^{k+s+2-\frac{n}{2}} \Gamma(k + 2s - \frac{n}{2}) \Gamma(k + s)^{-1} & T(\varepsilon, h) > 0 \\ \quad \cdot \Gamma(s)^{-1} \Gamma(k + s + 1 - \frac{n}{2})^{-1} \delta_+(hy)^{k+s-\frac{n}{2}}, & T[h] = 0, \\ 2^{s+2+\frac{n}{2}} \pi^{s+2-\frac{n}{2}} \Gamma(k + 2s - \frac{n}{2}) \Gamma(k + s)^{-1} & T(\varepsilon, h) < 0 \\ \quad \cdot \Gamma(s)^{-1} \Gamma(s + 1 - \frac{n}{2})^{-1} \delta_-(hy)^{s-\frac{n}{2}}, & T[h] = 0, \\ 2\pi^{\frac{n}{2}+1} \Gamma(k + 2s - \frac{n}{2}) \Gamma(k + 2s + 1 - n) \Gamma(k + s)^{-1} \\ \quad \cdot \Gamma(s)^{-1} \Gamma(k + s + 1 - \frac{n}{2})^{-1} \Gamma(s + 1 - \frac{n}{2})^{-1}, & h = 0. \end{cases} \end{aligned}$$

The main result of [Sh82] is that ω can be continued as a holomorphic function in (α, β) to \mathbb{C}^2 . Thus, zeros and poles of ξ can be read off from the previous equation.

The next result will be used in Section 7.

- PROPOSITION 5.1. (a) $\omega(2\pi y, h; \alpha, 0; T) = 2^{-n} e(T(iy, h))$ if $h \in \mathcal{P}$;
 (b) $\omega(2\pi y, h; \alpha, 0; T) = \omega(2\pi y, h; n/2, \beta) = 2^{-1-n} \pi^{n/2-1} e(T(iy, h))$ if $T[h] = 0, T(h, \varepsilon) > 0$;
 (c) $\omega(2\pi y, 0; \alpha, \beta; T) = 1$.

Proof. (a) and part of (b) are shown in [Sh82, 4.35.IV]. The remainder of (b) follows from [Sh82, 4.12.IV, 4.29, 3.15], where m, n there are $n, n - 2$ here, respectively. (c) is [Sh82, 4.9]. \square

The finite coefficient $a_v(h, s)$ for $v \mid \mathfrak{b}$. For $v \mid \mathfrak{b}$,

$$\begin{aligned} a_v(h, s) &= \int_{x \in V_v} \delta_v(x) e_v(-T(x, h_v)) d\mu_v(x) \\ &= \int_{x \in L_v} e_v(-T(x, h_v)) d\mu_v(x) = \int_{x \in L_v} d\mu_v(x) = 1. \end{aligned}$$

Thus

$$a_v(h, s) = 1 \quad \text{if } v \mid \mathfrak{b}.$$

The finite coefficient $a_v(h, s)$ for $v \nmid \mathfrak{b}$. For $v \nmid \mathfrak{b}$,

$$a_v(h, s) = (1 - X_v(s))^{-1} \int_{x \in V_v} X_v(s)^{i_v(x)} \mathbf{e}_v(-T(x, h_v)) d\mu_v(x).$$

Since the integrand is invariant under $x \mapsto x + l$ for $l \in L_v$, this is

$$\begin{aligned} a_v(h, s) &= (1 - X_v(s))^{-1} \sum_{x \in V_v/L_v} X_v(s)^{i_v(x)} \mathbf{e}_v(-T(x, h_v)) \\ &= (1 - X_v(s))^{-1} \sum_{\lambda=0}^{\infty} \sum_{\substack{x \in V_v/L_v \\ i_v(x)=\lambda}} X_v(s)^\lambda \mathbf{e}_v(-T(x, h_v)) \\ &= (1 - X_v(s))^{-1} \sum_{\lambda=0}^{\infty} X_v(s)^\lambda \sum_{\substack{x \in V_v/L_v \\ i_v(x)=\lambda}} \mathbf{e}_v(-T(x, h_v)). \end{aligned}$$

Now sum by parts, $\sum_{\lambda=0}^{\nu} a_\lambda b_\lambda = \sum_{\lambda=0}^{\nu-1} A_\lambda (b_\lambda - b_{\lambda+1}) + A_\nu b_\nu$, where $A_\lambda = \sum_{j=0}^{\lambda} a_j$. Letting $a_\lambda = \sum_{\substack{x \in V_v/L_v \\ i_v(x)=\lambda}} \mathbf{e}_v(-T(x, h_v))$, $b_\lambda = X_v(s)^\lambda$ gives

$$\begin{aligned} A_\lambda &= \sum_{\substack{x \in V_v/L_v \\ i_v(x) \leq \lambda}} \mathbf{e}_v(-T(x, h_v)) \\ &= \sum_{\substack{x \in V_v/L_v \\ w(x) \in \mathfrak{p}_v^{-\lambda} \mathfrak{o}_v^{n+2}}} \mathbf{e}_v(-T(x, h_v)) \stackrel{\text{call}}{=} S_v(\lambda, h_v) \end{aligned}$$

and $b_\lambda - b_{\lambda+1} = (1 - X_v(s))X_v(s)^\lambda$. Hence

$$\begin{aligned} \sum_{\lambda=0}^{\nu} X_v(s)^\lambda \sum_{\substack{x \in V_v/L_v \\ i_v(x)=\lambda}} \mathbf{e}_v(-T(x, h_v)) \\ = (1 - X_v(s)) \left(\sum_{\lambda=0}^{\nu-1} X_v(s)^\lambda S_v(\lambda, h_v) \right) + X_v(s)^\nu S_v(\nu, h_v). \end{aligned}$$

The last term goes to 0 as $\nu \rightarrow \infty$ when $\text{Re}(k + 2s) > n$, giving

$$a_v(h, s) = \alpha_v(h_v, X_v(s)) \quad \text{if } v \nmid \mathfrak{b}$$

where $\alpha_v(h_v, X)$ is the power series

$$\alpha_v(h_v, X) = \sum_{\lambda=0}^{\infty} S_v(\lambda, h_v) X^\lambda.$$

The exponential sum $S_v(\lambda, h_v)$. Let π_v generate the maximal ideal \mathfrak{p}_v of \mathfrak{o}_v , and let $y = \pi_v^\lambda x$. Summing over y 's, the set of summation for $S_v(\lambda, h_v)$ becomes

$$\left\{ y \in V_v/\mathfrak{p}_v^\lambda L_v : \begin{pmatrix} \pi_v^{-\lambda} y \\ \frac{1}{2} \pi_v^{-2\lambda} T[y] \\ 1 \end{pmatrix} \in \mathfrak{p}_v^{-\lambda} \mathfrak{o}_v^{n+2} \right\} \\ = \left\{ y \in L_v/\mathfrak{p}_v^\lambda L_v : \frac{1}{2} T[y] \in \mathfrak{p}_v^\lambda \right\}.$$

Since $2 \mid \mathfrak{b}$ and $v \nmid \mathfrak{b}$ the $\frac{1}{2}$ is irrelevant, so the sum is

$$S_v(\lambda, h_v) = \sum_{\substack{y \in L_v/\mathfrak{p}_v^\lambda L_v \\ T[y] \in \mathfrak{p}_v^\lambda}} e_v \left(\frac{-T(y, h_v)}{\pi_v^\lambda} \right).$$

This is independent of the choice of π_v since the set being summed over is stable under multiplication by units.

6. The power series $\alpha_v(h_v, X)$.

Definitions. The methods in this section are from Indik [In].

From now on all work is local at a fixed place $v \nmid \mathfrak{b}$ (so that $v \nmid 2 \det T$), and v 's will be suppressed in the notation; for example, $F, V, L, \mathfrak{o}, \mathfrak{p}$ and \mathfrak{d} now denote the local objects $F_v, V_v, L_v, \mathfrak{o}_v, \mathfrak{p}_v$ and \mathfrak{d}_v (the local different of F_v over \mathbb{Q}_p). Locally T^{-1} is integral; so for $y \in V, \nu(Ty) = \nu(y)$ and hence $L' = \mathfrak{d}^{-1}L$. To study the sum $S(\lambda, h)$, begin with some definitions.

Extend the v -adic valuation ν on F to a function also called ν on V by

$$\nu(x) = \min_{1 \leq i \leq n} \{ \nu(x_i) \}, \quad \text{for } x \in V.$$

For $\lambda \geq 0$ and $a \in \mathfrak{o}$ define the sets

$$\sigma(\lambda, a) = \{ y \in L : T[y] \equiv a \pmod{\mathfrak{p}^\lambda} \}, \\ \sigma'(\lambda, a) = \{ y \in \sigma(\lambda, a) : \nu(y) = 0 \}, \\ \overline{\sigma(\lambda, a)} = \{ y \in L/\mathfrak{p}^\lambda L : T[y] \equiv a \pmod{\mathfrak{p}^\lambda} \}, \\ \overline{\sigma'(\lambda, a)} = \{ y \in \overline{\sigma(\lambda, a)} : \nu(y) = 0 \}.$$

When $a = 0$, write $\sigma(\lambda)$ for $\sigma(\lambda, a)$ and so on. We will sometimes use the sets $\sigma(\lambda, a), \dots$ defined as above but for forms R other than T , in which case they are denoted $\sigma_R(\lambda, a)$, etc.

Extend the definition of S to

$$S(\lambda, h) = \begin{cases} \sum_{y \in \sigma(\lambda)} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right) & \text{if } h \in L' \\ 0 & \text{if } h \notin L', \end{cases}$$

and define

$$S'(\lambda, h) = \sum_{y \in \sigma'(\lambda)} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right) \quad \text{for } h \in L',$$

i.e., just sum over primitive vectors.

Recall that $q = |\mathfrak{p}|^{-1} = \#(\mathfrak{o}/\mathfrak{p})$.

PROPOSITION 6.1. *For symmetric $R \in M_n(\mathfrak{o}/\mathfrak{p})$ defining a non-degenerate bilinear form on $(\mathfrak{o}/\mathfrak{p})^n$,*

$$\# \overline{\sigma_R(1)} - q^{n-1} = \begin{cases} q^{\frac{n}{2}-1}(q-1) \left(\frac{(-1)^{\frac{n}{2}} \det R}{\mathfrak{p}} \right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This is a standard textbook exercise. □

Recurrence formula for $\# \overline{\sigma'(\lambda, a)}$. Fix $\lambda \geq 1$ and $a \in \mathfrak{o}$, and recall that $v \nmid 2$.

LEMMA. *For $\tilde{y} \in \sigma'(\lambda, a)$, there exists $d \in L$ such that $T(\tilde{y}, d) = \frac{1}{2}$.*

Proof. $(Ty)_i \in \mathfrak{o}^*$ for some i , so take $d_i = \frac{1}{2}(Ty)_i^{-1}$ and $d_j = 0$ for $j \neq i$. □

LEMMA. *For $v \in \overline{\sigma'(\lambda + 1, a)}$, $\# \{l \in L/\mathfrak{p}L : T(v, l) \in \mathfrak{p}\} = q^{n-1}$.*

Proof. $(Ty)_i \in \mathfrak{o}^*$ for some i ; consequently $T(v, l) \in \mathfrak{p}$ if and only if $l_i = (Tv)_i^{-1} \left(-\sum_{j \neq i} (Tv)_j l_j \right) + k$ with $k \in \mathfrak{p}$. This determines the value of $l_i \pmod{\mathfrak{p}}$ once the l_j for $j \neq i$ have been chosen. □

PROPOSITION 6.2. $\# \overline{\sigma'(\lambda + 1, a)} = q^{n-1} \# \overline{\sigma'(\lambda, a)}$. *Consequently, $\# \sigma'(\lambda, a) = q^{(n-1)(\lambda-1)} \# \sigma'(1, a)$ for $\lambda \geq 1$, and this value depends only on $a \pmod{\mathfrak{p}}$.*

Proof. Let $\pi_\lambda^{\lambda+1} : L/\mathfrak{p}^{\lambda+1}L \rightarrow L/\mathfrak{p}^\lambda L$ be the natural map. We will show that $\pi_\lambda^{\lambda+1} : \overline{\sigma'(\lambda + 1, a)} \rightarrow \overline{\sigma'(\lambda, a)}$ is surjective with multiplicity q^{n-1} .

Construct a function $\varphi : \overline{\sigma'(\lambda, a)} \rightarrow \overline{\sigma'(\lambda + 1, a)}$ as follows: Choose any lifting, denoted $\tilde{}$, from $L/\mathfrak{p}^\lambda L$ to L . Given $y \in \overline{\sigma'(\lambda, a)}$, there exists $d \in L$ such that $T(\tilde{y}, d) = \frac{1}{2}$, by the first lemma. Take $\varphi(y) = \tilde{y} + (a - T[y])d \pmod{\mathfrak{p}^{\lambda+1}L}$. Then $T[\varphi(y)] \equiv a \pmod{\mathfrak{p}^{\lambda+1}}$ is easy to check. Thus $\overline{\sigma'(\lambda, a)} \xrightarrow{\varphi} \overline{\sigma'(\lambda + 1, a)} \xrightarrow{\pi_\lambda^{\lambda+1}} \overline{\sigma'(\lambda, a)}$, and the composite is the identity since $\varphi(y) \equiv y \pmod{\mathfrak{p}^\lambda L}$. This shows that $\pi_\lambda^{\lambda+1} : \overline{\sigma'(\lambda + 1, a)} \rightarrow \overline{\sigma'(\lambda, a)}$ is surjective.

For $v \in \overline{\sigma'(\lambda + 1, a)}$ and $v' \in L/\mathfrak{p}^{\lambda+1}L$, $\pi_\lambda^{\lambda+1}(v') = \pi_\lambda^{\lambda+1}(v)$ if and only if $v' = v + \pi^\lambda l$ for some $l \in L/\mathfrak{p}L$, in which case $T[v'] \equiv a + 2\pi^\lambda T(v, l) \pmod{\mathfrak{p}^{\lambda+1}}$. This shows that $v' \in \overline{\sigma'(\lambda + 1, a)}$ if and only if $T(v, l) \in \mathfrak{p}$. The number of l satisfying this is q^{n-1} by the second lemma, so $\pi_\lambda^{\lambda+1} : \overline{\sigma'(\lambda + 1, a)} \rightarrow \overline{\sigma'(\lambda, a)}$ has multiplicity q^{n-1} , proving the proposition. \square

Recurrence formula for $S(\lambda, h)$.

LEMMA. $\sigma(\lambda) = \sigma'(\lambda) \cup \mathfrak{p}\sigma(\lambda - 2)$ for $\lambda \geq 2$, a disjoint union.

Proof. $\sigma(\lambda) \supset \sigma'(\lambda)$ and $\sigma(\lambda) \supset \mathfrak{p}\sigma(\lambda - 2)$ are clear, as is disjointness. Let $y \in \sigma(\lambda) - \sigma'(\lambda)$. Then $y = \pi x$ for some $x \in L$, and $\pi^2 T[x] = T[y] \in \mathfrak{p}^\lambda$ shows that $T[x] \in \mathfrak{p}^{\lambda-2}$, i.e., $x \in \sigma(\lambda - 2)$. \square

PROPOSITION 6.3. $S(\lambda, h) = S'(\lambda, h) + q^n S(\lambda - 2, h/\pi)$ for $\lambda \geq 2$ and $h \in L'$.

Proof.

$$S(\lambda, h) = S'(\lambda, h) + \sum_{\substack{y \in \mathfrak{p}\sigma(\lambda-2) \\ \pmod{\mathfrak{p}^\lambda L}}} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right)$$

by the lemma, so we need to evaluate this last sum, which is equal to

$$\sum_{\substack{y \in \sigma(\lambda-2) \\ \pmod{\mathfrak{p}^{\lambda-1}L}}} e_v \left(-\frac{T(y, h)}{\pi^{\lambda-1}} \right) \stackrel{\text{call}}{=} S.$$

The set $\sigma(\lambda - 2) \pmod{\mathfrak{p}^{\lambda-1}L}$ is stable under translation by any $\pi^{\lambda-2}l \in \mathfrak{p}^{\lambda-2}L$. So

$$S = \sum_{\substack{y \in \sigma(\lambda-2) \\ \pmod{\mathfrak{p}^{\lambda-1}L}}} e_v \left(-\frac{T(y + \pi^{\lambda-2}l, h)}{\pi^{\lambda-1}} \right) = e_v(-T(l, h/\pi)) S.$$

If $\frac{h}{\pi} \in L'$ then

$$S = \sum_{\substack{y \in \sigma(\lambda-2) \\ (\text{mod } \mathfrak{p}^{\lambda-1}L)}} e_v \left(-\frac{T(y, h/\pi)}{\pi^{\lambda-2}} \right) = q^n S(\lambda - 2, h/\pi).$$

If $\frac{h}{\pi} \notin L'$ then $T(L, h/\pi) \not\subset \mathfrak{d}^{-1}$, so for some $l \in L$ we have $\text{Tr}(T(l, h/\pi)) \notin \mathbb{Z}_p$, giving $e_v(-T(l, h/\pi)) \neq 1$, whence $S = 0$. Thus $S(\lambda, h) = S'(\lambda, h) + q^n S(\lambda - 2, h/\pi)$ in all cases. \square

COROLLARY 6.4. $S(\lambda, 0) - q^n S(\lambda - 2, 0) = q^{(n-1)(\lambda-1)} \overline{\#\sigma'(1)}$ for $\lambda \geq 2$. Equivalently, $\#\sigma(\lambda) - q^n \#\sigma(\lambda - 2) = q^{(n-1)(\lambda-1)} \#\sigma'(1)$.

Proof.

$$S(\lambda, 0) - q^n S(\lambda - 2, 0) = S'(\lambda, 0) = \overline{\#\sigma'(\lambda)} = q^{(n-1)(\lambda-1)} \overline{\#\sigma'(1)}$$

by the previous proposition. \square

The value of $\alpha(h, X)$ when $h = 0$.

PROPOSITION 6.5.

$$\alpha(0, X) = \frac{1 + (\overline{\#\sigma(1)} - q^{n-1})X - q^{n-1}X^2}{(1 - q^n X^2)(1 - q^{n-1}X)}.$$

Proof. Since $S(\lambda, 0) - q^n S(\lambda - 2, 0) = q^{(n-1)(\lambda-1)} \overline{\#\sigma'(1)}$ for $\lambda \geq 2$,

$$\begin{aligned} (1 - q^n X^2) \sum_{\lambda=0}^{\infty} S(\lambda, 0) X^\lambda &= 1 + S(1, 0) + \sum_{\lambda=2}^{\infty} (S(\lambda, 0) - q^n S(\lambda - 2, 0)) X^\lambda \\ &= 1 + \overline{\#\sigma(1)} X + \sum_{\lambda=2}^{\infty} q^{(n-1)(\lambda-1)} \overline{\#\sigma'(1)} X^\lambda, \end{aligned}$$

and since $\overline{\#\sigma(1)} = 1 + \overline{\#\sigma'(1)}$, this is

$$\begin{aligned} &= 1 + X + \sum_{\lambda=1}^{\infty} q^{(n-1)(\lambda-1)} \overline{\#\sigma'(1)} X^\lambda \\ &= 1 + X + \frac{\overline{\#\sigma'(1)} X}{1 - q^{n-1} X}. \end{aligned}$$

The result follows easily. □

DEFINITION. For n even, define a quadratic character θ by $\theta(\mathfrak{p}) = \left(\frac{(-1)^{\frac{n}{2}} \det T}{\mathfrak{p}} \right)$.

This gives for n even $\#\overline{\sigma}(1) - q^{n-1} = q^{\frac{n}{2}-1}(q-1)\theta(\mathfrak{p})$, so in the proposition the numerator becomes $(1 + q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1 - q^{\frac{n}{2}-1}\theta(\mathfrak{p})X)$, and the denominator, $(1 + q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1 - q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1 - q^{n-1}X)$. For n odd, $\#\overline{\sigma}(1) - q^{n-1} = 0$. Thus,

$$\alpha(h, X) = \begin{cases} \frac{1 - q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{(1 - q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1 - q^{n-1}X)} & \text{if } h = 0, n \text{ even} \\ \frac{1 - q^{n-1}X^2}{(1 - q^n X^2)(1 - q^{n-1}X)} & \text{if } h = 0, n \text{ odd.} \end{cases}$$

Formula for S .

DEFINITION. Let $\nu_{\mathfrak{d}} = \nu(\mathfrak{d})$, the valuation of the different.

PROPOSITION 6.6. For a set $\sigma \subset L/\mathfrak{p}^\lambda L$ such that $u\sigma = \sigma$ for all $u \in \mathfrak{o}^*$,

$$\sum_{y \in \sigma} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right) = \# \{ y \in \sigma : \nu(T(y, h)) \geq \lambda - \nu_{\mathfrak{d}} \} - \frac{1}{q-1} \# \{ y \in \sigma : \nu(T(y, h)) = \lambda - \nu_{\mathfrak{d}} - 1 \}.$$

Proof. We may assume $\lambda \geq 1$. Let $U_\lambda = \mathfrak{o}^*/\mathfrak{p}^\lambda = \mathfrak{o}/\mathfrak{p}^\lambda - \mathfrak{p}/\mathfrak{p}^\lambda$, with $\#U_\lambda = q^\lambda - q^{\lambda-1} = q^{\lambda-1}(q-1)$. Then

$$\begin{aligned} q^{\lambda-1}(q-1) \sum_{y \in \sigma} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right) &= \sum_{u \in U_\lambda} \sum_{y \in \sigma} e_v \left(-\frac{T(uy, h)}{\pi^\lambda} \right) \\ &= \sum_y \sum_u e_v \left(-\frac{T(uy, h)}{\pi^\lambda} \right) \\ &= \sum_y \left\{ \sum_{u \in \mathfrak{o}/\mathfrak{p}^\lambda} e_v \left(-\frac{T(uy, h)}{\pi^\lambda} \right) - \sum_{u \in \mathfrak{o}/\mathfrak{p}^{\lambda-1}} e_v \left(-\frac{T(uy, h)}{\pi^{\lambda-1}} \right) \right\}. \end{aligned}$$

Since the sums over $\mathfrak{o}/\mathfrak{p}^\lambda$ and $\mathfrak{o}/\mathfrak{p}^{\lambda-1}$ are character sums over finite groups, and since $\frac{T(uy, h)}{\pi^\lambda} \in \mathfrak{d}^{-1}$ for all u if and only if $\nu(T(y, h)) \geq$

$\lambda - \nu_{\mathfrak{d}}$, the inner sums yield

$$\begin{cases} 0 & \text{if } \nu(T(y, h)) < \lambda - \nu_{\mathfrak{d}} - 1 \\ -q^{\lambda-1} & \text{if } \nu(T(y, h)) = \lambda - \nu_{\mathfrak{d}} - 1 \\ q^{\lambda} - q^{\lambda-1} & \text{if } \nu(T(y, h)) \geq \lambda - \nu_{\mathfrak{d}}, \end{cases}$$

so

$$\begin{aligned} & q^{\lambda-1}(q-1) \sum_{y \in \sigma} e_v \left(-\frac{T(y, h)}{\pi^{\lambda}} \right) \\ &= (q^{\lambda} - q^{\lambda-1}) \# \{ y \in \sigma : \nu(T(y, h)) \geq \lambda - \nu_{\mathfrak{d}} \} \\ & \quad - q^{\lambda-1} \# \{ y \in \sigma : \nu(T(y, h)) = \lambda - \nu_{\mathfrak{d}} - 1 \}, \end{aligned}$$

giving the result. □

This shows that the coefficients of the power series $\alpha_v(h, X)$ are elements of \mathbb{Q} .

The value of $\alpha(h, X)$ when $T[h] = 0$. Now assume that $T[h] = 0$, $h \neq 0$.

DEFINITION. Given a nonzero $h \in L'$, define $\nu_h \in \mathbb{Z}$ and $h' \in L$ by $h = \pi^{\nu_h} h'$, where $\nu_h = \nu(h) \geq -\nu_{\mathfrak{d}}$ and $\nu(h') = 0$. Further define $\nu_{h\mathfrak{d}} = \nu_h + \nu_{\mathfrak{d}} \geq 0$.

There is an $x_0 \in L$ such that $T(x_0, h') = 1$; then setting $x = x_0 - \frac{1}{2}T[x_0]h'$ gives $T[x] = T[h'] = 0$, $T(x, h') = 1$, and $L = \mathfrak{o}h' + \mathfrak{o}x + W$, where $W = \{ w \in L : T(w, h') = T(w, x) = 0 \}$. Define $T' = T|_W$.

PROPOSITION 6.7. For a nonzero $h \in L'$ such that $T[h] = 0$,

$$\alpha(h, X) = \frac{1 + (\#\overline{\sigma_{T'}}(1) - q^{n-3})qX - q^{n-1}X^2}{1 - q^n X^2} G_{h,v}(X),$$

where

$$G_{h,v}(X) = \sum_{i=0}^{\nu_{h\mathfrak{d}}} (q^{n-1}X)^i = \frac{1 - (q^{n-1}X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1}X}.$$

Proof. For $y = ah' + bx + w \in L$, $T[y] = 2ab + T[w]$, so $y \in \sigma(\lambda)$ if and only if $T[w] \equiv -2ab \pmod{\mathfrak{p}^{\lambda}}$. Given $w \in W/\mathfrak{p}^{\lambda}W$ and $b \in \mathfrak{o}/\mathfrak{p}^{\lambda}$, there is an $a \in \mathfrak{o}/\mathfrak{p}^{\lambda}$ such that $T[w] \equiv -2ab \pmod{\mathfrak{p}^{\lambda}}$ if

and only if $\nu(T[w]) \geq \nu(b)$, in which case there are $q^{\min(\lambda, \nu(b))}$ such values a . Proposition 6.6 says,

$$(6.1) \quad S(\lambda, \pi^{\nu_h} h') = \# \left\{ y \in \overline{\sigma(\lambda)} : \nu(T(y, h')) \geq \lambda - \nu_{h\mathfrak{d}} \right\} - \frac{1}{q-1} \# \left\{ y \in \overline{\sigma(\lambda)} : \nu(T(y, h')) = \lambda - \nu_{h\mathfrak{d}} - 1 \right\}.$$

Setting $M = \max(0, \lambda - \nu_{h\mathfrak{d}})$ one finds that the first term of (6.1) is

$$\begin{aligned} & \sum_{m=M}^{\lambda} \# \left\{ b \in \mathfrak{o}/\mathfrak{p}^{\lambda} : \nu(b) = m \right\} \# \left\{ \sigma_{T'}(m) \pmod{\mathfrak{p}^{\lambda}L} \right\} q^m \\ &= \sum_{m=M}^{\lambda-1} q^{\lambda-m-1} (q-1) q^{(n-2)(\lambda-m)} \# \overline{\sigma_{T'}(m)} q^m + \# \overline{\sigma_{T'}(\lambda)} q^{\lambda} \\ &= \sum_{m=M}^{\lambda} q^{\lambda} q^{(n-2)(\lambda-m)} \# \overline{\sigma_{T'}(m)} - \sum_{m=M}^{\lambda-1} q^{\lambda-1} q^{(n-2)(\lambda-m)} \# \overline{\sigma_{T'}(m)} \\ &= q^{\lambda} \sum_{m=M}^{\lambda} q^{(n-2)(\lambda-m)} \# \overline{\sigma_{T'}(m)} \\ &\quad - q^{\lambda} \sum_{m=M+1}^{\lambda} q^{-1} q^{(n-2)(\lambda-m+1)} \# \overline{\sigma_{T'}(m-1)} \\ &= q^{\lambda} \sum_{m=M+1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^{\lambda} q^{(n-2)(\lambda-M)} \# \overline{\sigma_{T'}(M)} \\ &= \begin{cases} q^{\lambda} \sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^{\lambda} q^{(n-2)\lambda} & \text{if } \lambda \leq \nu_{h\mathfrak{d}} \\ q^{\lambda} \sum_{m=\lambda-\nu_{h\mathfrak{d}}+1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) \\ \quad + q^{\lambda} q^{(n-2)\nu_{h\mathfrak{d}}} \# \overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}})} & \text{if } \lambda > \nu_{h\mathfrak{d}}, \end{cases} \end{aligned}$$

where $\Delta(m) = \# \overline{\sigma_{T'}(m)} - q^{n-3} \# \overline{\sigma_{T'}(m-1)}$. The second term of (6.1) is 0 when $\lambda \leq \nu_{h\mathfrak{d}}$, and is

$$\begin{aligned} & - \frac{q^{\lambda-\nu_{h\mathfrak{d}}-1}}{q-1} \# \left\{ b \in \mathfrak{o}/\mathfrak{p}^{\lambda} : \nu(b) = \lambda - \nu_{h\mathfrak{d}} - 1 \right\} \\ & \quad \# \left\{ \sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1) \pmod{\mathfrak{p}^{\lambda}L} \right\} \\ &= - \frac{q^{\lambda-\nu_{h\mathfrak{d}}-1}}{q-1} q^{\nu_{h\mathfrak{d}}} (q-1) q^{(n-2)(\nu_{h\mathfrak{d}}+1)} \# \overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1)} \\ &= - q^{\lambda} q^{(n-2)\nu_{h\mathfrak{d}}} q^{n-3} \# \overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1)} \end{aligned}$$

when $\lambda > \nu_{h\delta}$. So

(6.2)

$$S(\lambda, h) = \begin{cases} q^\lambda \sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^\lambda q^{(n-2)\lambda} & \text{if } \lambda \leq \nu_{h\delta} \\ q^\lambda \sum_{m=\lambda-\nu_{h\delta}}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) & \text{if } \lambda > \nu_{h\delta}. \end{cases}$$

Now, $\Delta(m)$ satisfies

$$\begin{aligned} &\Delta(m+2) - q^{n-2}\Delta(m) \\ &= (\#\overline{\sigma_{T'}(m+2)} - q^{n-2}\#\overline{\sigma_{T'}(m)}) \\ &\quad - q^{n-3}(\#\overline{\sigma_{T'}(m+1)} - q^{n-2}\#\overline{\sigma_{T'}(m-1)}) \\ &= \#\overline{\sigma'_{T'}(m+2)} - q^{n-3}\#\overline{\sigma'_{T'}(m+1)} \\ &= 0, \end{aligned}$$

by Corollary 6.4 and Proposition 6.2 with T' in place of T . This shows that for $\lambda > \nu_{h\delta}$,

$$\begin{aligned} &S(\lambda+2, h) - q^n S(\lambda, h) \\ &= q^{\lambda+2} \sum_{m=\lambda-\nu_{h\delta}+2}^{\lambda+2} q^{(n-2)(\lambda+2-m)} \Delta(m) \\ &\quad - q^\lambda \sum_{m=\lambda-\nu_{h\delta}}^{\lambda} q^n q^{(n-2)(\lambda-m)} \Delta(m) \\ &= q^{\lambda+2} \sum_{m=\lambda-\nu_{h\delta}}^{\lambda} q^{(n-2)(\lambda-m)} (\Delta(m+2) - q^{n-2}\Delta(m)) \\ &= 0. \end{aligned}$$

So for $\nu_{h\delta} = 0$,

$$\begin{aligned} &(1 - q^n X^2) \sum_{\lambda=0}^{\infty} S(\lambda, h) X^\lambda \\ &= 1 + S(1, h)X + \sum_{\lambda=2}^{\infty} (S(\lambda, h) - q^n S(\lambda-2, h)) X^\lambda \\ &= 1 + (\#\overline{\sigma_{T'}(1)} - q^{n-3})qX + (q^2\#\overline{\sigma_{T'}(2)} - q^{n-1}\#\overline{\sigma_{T'}(1)} - q^n)X^2, \end{aligned}$$

giving the result in this case, as the relations $\overline{\#\sigma_{T'}(2)} = q^{n-2}\overline{\#\sigma_{T'}(0)} + q^{n-3}\overline{\#\sigma'_{T'}(1)}$ and $\overline{\#\sigma_{T'}(1)} = \overline{\#\sigma'_{T'}(1)} + 1$ show that the coefficient of X^2 is $-q^{n-1}$. Also when $\nu_{h\delta} = 0$, (6.2) shows that

$$\sum_{\lambda=0}^{\infty} S(\lambda, h) = 1 + \sum_{m=1}^{\infty} (qX)^m \Delta(m),$$

so that

$$1 + \sum_{m=1}^{\infty} (qX)^m \Delta(m) = \frac{1 + (\overline{\#\sigma_{T'}(1)} - q^{n-3})qX - q^{n-1}X^2}{1 - q^n X^2}.$$

For general $\nu_{h\delta}$, the same formula gives

(6.3)

$$\begin{aligned} \alpha(h, X) &= 1 + \sum_{\lambda=1}^{\nu_{h\delta}} (qX)^\lambda \left(\sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^{(n-2)\lambda} \right) \\ &\quad + \sum_{\lambda=\nu_{h\delta}+1}^{\infty} (qX)^\lambda \sum_{m=\lambda-\nu_{h\delta}}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) \\ &= \sum_{\lambda=0}^{\nu_{h\delta}} (q^{n-1}X)^\lambda + \sum_{\lambda=1}^{\nu_{h\delta}} (qX)^\lambda \sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) \\ &\quad + \sum_{\lambda=\nu_{h\delta}+1}^{\infty} (qX)^\lambda \sum_{m=\lambda-\nu_{h\delta}}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m). \end{aligned}$$

The first sum in (6.3) is $\frac{1 - (q^{n-1}X)^{\nu_{h\delta}+1}}{1 - q^{n-1}X}$. The second sum is

$$\begin{aligned} &\sum_{m=1}^{\nu_{h\delta}} q^{-m(n-2)} \Delta(m) \sum_{\lambda=m}^{\nu_{h\delta}} (q^{n-1}X)^\lambda \\ &= \sum_{m=1}^{\nu_{h\delta}} q^{-m(n-2)} \Delta(m) (q^{n-1}X)^m \frac{1 - (q^{n-1}X)^{\nu_{h\delta}+1-m}}{1 - q^{n-1}X} \\ &= \sum_{m=1}^{\nu_{h\delta}} \Delta(m) (qX)^m \frac{1 - (q^{n-1}X)^{\nu_{h\delta}+1-m}}{1 - q^{n-1}X}. \end{aligned}$$

The third sum is

$$\begin{aligned} &\sum_{m=1}^{\nu_{h\delta}} q^{-m(n-2)} \Delta(m) \sum_{\lambda=\nu_{h\delta}+1}^{m+\nu_{h\delta}} (q^{n-1}X)^\lambda \\ &\quad + \sum_{m=\nu_{h\delta}+1}^{\infty} q^{-m(n-2)} \Delta(m) \sum_{\lambda=m}^{m+\nu_{h\delta}} (q^{n-1}X)^\lambda, \end{aligned}$$

which splits into

$$\begin{aligned} & \sum_{m=1}^{\nu_{h\mathfrak{d}}} q^{-m(n-2)} \Delta(m) (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1} \frac{1 - (q^{n-1} X)^m}{1 - q^{n-1} X} \\ &= \sum_{m=1}^{\nu_{h\mathfrak{d}}} q^{-m(n-2)} \Delta(m) (q^{n-1} X)^m \frac{(q^{n-1} X)^{\nu_{h\mathfrak{d}}+1-m} - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1} X} \\ &= \sum_{m=1}^{\nu_{h\mathfrak{d}}} \Delta(m) (qX)^m \left(\frac{1 - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1} X} - \frac{1 - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1-m}}{1 - q^{n-1} X} \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{m=\nu_{h\mathfrak{d}}+1}^{\infty} q^{-m(n-2)} \Delta(m) (q^{n-1} X)^m \frac{1 - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1} X} \\ &= \sum_{m=\nu_{h\mathfrak{d}}+1}^{\infty} \Delta(m) (qX)^m \frac{1 - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1} X}. \end{aligned}$$

The total is thus

$$\begin{aligned} \alpha(h, X) &= \left(1 + \sum_{m=1}^{\infty} \Delta(m) (qX)^m \right) \left(\frac{1 - (q^{n-1} X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1} X} \right) \\ &= \frac{1 + (\#\overline{\sigma_{T'}}(1) - q^{n-3})qX - q^{n-1}X^2}{1 - q^n X^2} G_{h,v}(X), \end{aligned}$$

which completes the proof of the proposition. □

For n even, observe that since

$$\det T' = -\det T, \theta(\mathfrak{p}) = \left(\frac{(-1)^{\frac{n}{2}-1} \det T'}{\mathfrak{p}} \right),$$

and the first factor becomes

$$\frac{1 + q^{\frac{n}{2}-2}(q-1)\theta(\mathfrak{p})qX - q^{n-1}X^2}{(1 + q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1 - q^{\frac{n}{2}}\theta(\mathfrak{p})X)} = \frac{1 - q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{1 - q^{\frac{n}{2}}\theta(\mathfrak{p})X}.$$

For n odd the first factor becomes $\frac{1 - q^{n-1}X^2}{1 - q^n X^2}$. Thus,

$$\alpha(h, X) = \begin{cases} \frac{1 - q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{1 - q^{\frac{n}{2}}\theta(\mathfrak{p})X} G_{h,v}(X) & \text{if } T[h] = 0, n \text{ even} \\ \frac{1 - q^{n-1}X^2}{1 - q^n X^2} G_{h,v}(X) & \text{if } T[h] = 0, n \text{ odd.} \end{cases}$$

The value of $\alpha(h, X)$ when $\nu(T[h']) = 0$.

PROPOSITION 6.8. *For $h \in L'$ such that $\nu(T[h']) = 0$, $\alpha(h, X)$ is a polynomial $H_{h,\nu}(X) \in \mathbb{Q}[X]$ of degree $< 2(\nu_{h\mathfrak{d}} + 1)$. If $\nu_{h\mathfrak{d}} = 0$ then*

$$\alpha(h, X) = 1 + \frac{1}{q-1} \left(q(\#\overline{\sigma_{T''}}(1) - q^{n-2}) - (\#\overline{\sigma}(1) - q^{n-1}) \right) X,$$

where $T'' = T|_W$, with $W = \{w \in L : T(w, h) = 0\}$.

Proof. We will compute $S'(\lambda, h)$ for all values of λ . For $\lambda \leq \nu_{h\mathfrak{d}}$, $S'(\lambda, h) = \#\overline{\sigma'(\lambda)}$ is clear. Suppose now that $\lambda > \nu_{h\mathfrak{d}} + 1$. Any $y \in \sigma(\lambda)$ takes the form $y = ah' + w$, where $a = \frac{T(y, h')}{T[h']}$, $w \in W$, $T[w] \equiv -a^2T[h'] \pmod{\mathfrak{p}^\lambda}$, and $\nu(y) = 0$ if and only if $\nu(w) = 0$. For $l \geq 0$, $l = \nu(T(y, h)) = \nu(T(ah', \pi^{\nu_h}h')) = \nu(a) + \nu_h$ if and only if $\nu(a) = l - \nu_h$. Thus by Proposition 6.6,

(6.4)

$$\begin{aligned} S'(\lambda, h) &= \#\{y \in \overline{\sigma'(\lambda)} : \nu(T(y, h)) \geq \lambda - \nu_{\mathfrak{d}}\} \\ &\quad - \frac{1}{q-1} \#\{y \in \overline{\sigma'(\lambda)} : \nu(T(y, h)) = \lambda - \nu_{\mathfrak{d}} - 1\} \\ &= \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a) \geq \lambda - \nu_{h\mathfrak{d}}} } \#\{w \in \overline{\sigma'_{T''}(\lambda, -a^2T[h'])}\} \\ &\quad - \frac{1}{q-1} \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a) = \lambda - \nu_{h\mathfrak{d}} - 1}} \#\{w \in \overline{\sigma'_{T''}(\lambda, -a^2T[h'])}\}. \end{aligned}$$

Since $\lambda > \nu_{h\mathfrak{d}} + 1$, $\nu(a^2) > 0$ in both sums. By Proposition 6.2, the set cardinalities depend only on $a^2T[h'] \pmod{\mathfrak{p}}$, which is 0, so for $\lambda > \nu_{h\mathfrak{d}} + 1$,

$$\begin{aligned} S'(\lambda, h) &= \#\overline{\sigma'_{T''}(\lambda)} \left(\#\{a \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(a) \geq \lambda - \nu_{h\mathfrak{d}}\} \right. \\ &\quad \left. - \frac{1}{q-1} \#\{a \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(a) = \lambda - \nu_{h\mathfrak{d}} - 1\} \right) \\ &= \#\overline{\sigma'_{T''}(\lambda)} \left(q^{\nu_{h\mathfrak{d}}} - \frac{1}{q-1} (q^{\nu_{h\mathfrak{d}}+1} - q^{\nu_{h\mathfrak{d}}}) \right) \\ &= 0. \end{aligned}$$

This bounds the degree, for if $\lambda \geq 2\nu_{h\mathfrak{d}} + 2$ then

$$S(\lambda, h) = \sum_{r=0}^{\nu_{h\mathfrak{d}}} q^{nr} S'(\lambda - 2r, \pi^{-r}h)$$

by repeated application of Proposition 6.3, and the summand is always zero since $\lambda > 2\nu_{h\mathfrak{d}} + 1$ implies $\lambda - 2r > \nu_{\pi^{-r}h, \mathfrak{d}} + 1$ for $r = 0, \dots, \nu_{h\mathfrak{d}}$.

The remaining case is $\lambda = \nu_{h\mathfrak{d}} + 1$. In this instance (6.4) becomes

$$S'(\nu_{h\mathfrak{d}} + 1, h) = q^{\nu_{h\mathfrak{d}}} \overline{\#\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} - \frac{1}{q-1} \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a)=0}} \overline{\#\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1, -a^2T[h'])}.$$

To simplify this expression, note that

$$\begin{aligned} \overline{\#\sigma'(\nu_{h\mathfrak{d}} + 1)} &= q^{\nu_{h\mathfrak{d}}} \overline{\#\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} \\ &\quad + \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a)=0}} \overline{\#\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1, -a^2T[h'])}, \end{aligned}$$

obtained from $\overline{\#\sigma'(\nu_{h\mathfrak{d}} + 1)} = S'(\nu_{h\mathfrak{d}} + 1, 0)$ and analysis of $S'(\nu_{h\mathfrak{d}} + 1, 0)$ similar to the argument above. Combining these gives

$$\begin{aligned} S'(\nu_{h\mathfrak{d}} + 1, h) &= \frac{1}{q-1} \left(q^{\nu_{h\mathfrak{d}}+1} \overline{\#\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} - \overline{\#\sigma'(\nu_{h\mathfrak{d}} + 1)} \right) \\ &= \frac{1}{q-1} \left(q^{\nu_{h\mathfrak{d}}+1} q^{(n-2)\nu_{h\mathfrak{d}}} \overline{\#\sigma'_{T''}(1)} - q^{(n-1)\nu_{h\mathfrak{d}}} \overline{\#\sigma'(1)} \right). \end{aligned}$$

The expression for $\nu_{h\mathfrak{d}} = 0$ follows since in this case the formulae for $S'(\lambda, h)$ give

$$\begin{aligned} \alpha(h, X) &= S(0, h) + X + S'(1, h)X \\ &= 1 + X + \frac{1}{q-1} \left(q \overline{\#\sigma'_{T''}(1)} - \overline{\#\sigma'(1)} \right) X \\ &= 1 + \frac{1}{q-1} \left(q(1 + \overline{\#\sigma'_{T''}(1)}) - (1 + \overline{\#\sigma'(1)}) \right) X \\ &= 1 + \frac{1}{q-1} \left(q \overline{\#\sigma_{T''}(1)} - \overline{\#\sigma(1)} \right) X, \end{aligned}$$

which completes the proof. □

DEFINITION. For n odd and h such that $\nu(T[h']) = 0$, define a quadratic character θ_h by $\theta_h(\mathfrak{p}) = \left(\frac{(-1)^{\frac{n-1}{2}} T[h'] \det T}{\mathfrak{p}} \right)$.

Since $\det T'' = T[h']^{-1} \det T$ and $\left(\frac{T[h']^{-1}}{\mathfrak{p}} \right) = \left(\frac{T[h']}{\mathfrak{p}} \right)$, when $\nu_{h\mathfrak{d}} = 0$ we get

$$\alpha(h, X) = \begin{cases} 1 + q^{\frac{n}{2}-1} \theta(\mathfrak{p})X & \text{if } \nu(T[h']) = 0, n \text{ even} \\ 1 + q^{\frac{n-1}{2}} \theta_h(\mathfrak{p})X = \frac{1 - q^{n-1} X^2}{1 - q^{\frac{n-1}{2}} \theta_h(\mathfrak{p})X} & \text{if } \nu(T[h']) = 0, n \text{ odd.} \end{cases}$$

The value of $\alpha(h, X)$ when $\nu(T[h']) > 0$.

LEMMA. For $y \in L, h \in L', \mu \in \mathbb{Z}$, the following equivalence holds:

$$Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1} L} \text{ for some } a \in \mathfrak{d}^{-1} \\ \Leftrightarrow (T(d, y) \in \mathfrak{p}^\mu \Rightarrow T(d, h) \in \mathfrak{p}^\mu \mathfrak{d}^{-1} \text{ for all } d \in L).$$

Proof. \Rightarrow : If $Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1} L}$ then $T(d, h) \equiv aT(d, y) \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ for all $d \in L$, hence $T(d, y) \in \mathfrak{p}^\mu \Rightarrow aT(d, y) \in \mathfrak{p}^\mu \mathfrak{d}^{-1} \Rightarrow T(d, h) \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$ for all $d \in L$.

\Leftarrow : If $(Ty)_i \in \mathfrak{p}^\mu$ then setting $d = e_i$ (the i^{th} basis vector) gives $T(d, y) = (Ty)_i \in \mathfrak{p}^\mu$, so $T(d, h) = (Th)_i \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$. At such i , $(Th)_i \equiv a(Ty)_i \equiv 0 \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ holds for any $a \in \mathfrak{d}^{-1}$.

If $(Ty)_i \notin \mathfrak{p}^\mu$, setting $d = \pi^{\mu-\nu(Ty)_i} e_i$ gives

$$T(d, y) = \pi^{\mu-\nu(Ty)_i} (Ty)_i \in \mathfrak{p}^\mu,$$

so $T(d, h) = \pi^{\mu-\nu(Ty)_i} (Th)_i \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$, showing $\nu(Th)_i \geq \nu(Ty)_i - \nu_{\mathfrak{d}}$. We may assume that $(Ty)_1$ has the smallest valuation among the $(Ty)_i$ and define $a = \frac{(Th)_1}{(Ty)_1} \in \mathfrak{d}^{-1}$. $(Th)_1 \equiv a(Ty)_1 \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ certainly holds. For $i \neq 1$ such that $(Ty)_i \notin \mathfrak{p}^\mu$, set

$$d = \pi^{\nu(Ty)_i} ((Ty)_1^{-1} e_1 - (Ty)_i^{-1} e_i) \in L.$$

$T(d, y) = 0 \in \mathfrak{p}^\mu$, hence

$$T(d, h) = \pi^{\nu(Ty)_i} \left(a - \frac{(Th)_i}{(Ty)_i} \right) \in \mathfrak{p}^\mu \mathfrak{d}^{-1},$$

so

$$\pi^{\nu(Ty)_i} a \equiv \frac{\pi^{\nu(Ty)_i}}{(Ty)_i} (Th)_i \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}},$$

i.e., $(Th)_i \equiv a(Ty)_i \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$.

The relation now holds at all i , showing that

$$Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}L}. \quad \square$$

LEMMA.

$$S(\lambda, h) = \sum_{y \in \tau(\lambda, h)} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right),$$

where

$$\tau(\lambda, h) = \left\{ y \in \overline{\sigma(\lambda)} : \nu(Th - aTy) \geq \lfloor \frac{\lambda}{2} \rfloor - \nu_{\mathfrak{d}} \text{ for some } a \in \mathfrak{d}^{-1} \right\}.$$

Proof. Let $\mu = \lfloor \frac{\lambda}{2} \rfloor$ and $\nu = \lambda - \mu$ so that $2\nu \geq \lambda$. For any $y \in \sigma(\lambda)$ and $d \in L$ we have $T[y + \pi^\nu d] \equiv 2\pi^\nu T(y, d) \pmod{\mathfrak{p}^\lambda}$, showing that $\sigma(\lambda) = \{ y + \pi^\nu d : y \in \sigma(\lambda), d \in L, T(y, d) \in \mathfrak{p}^\mu \}$. Projecting mod \mathfrak{p}^λ , $\overline{\sigma(\lambda)} = \{ y + \pi^\nu d : y \in \overline{\sigma(\lambda)}, d \in L/\mathfrak{p}^\lambda, T(y, d) \in \mathfrak{p}^\mu/\mathfrak{p}^\lambda \}$. To avoid redundancy, take only $y \in \sigma(\lambda) \pmod{\mathfrak{p}^\nu L}$. So

$$\begin{aligned} S(\lambda, h) &= \sum_{\substack{y \in \sigma(\lambda) \pmod{\mathfrak{p}^\nu L} \\ d \in L/\mathfrak{p}^\lambda \\ T(y, d) \in \mathfrak{p}^\mu/\mathfrak{p}^\lambda}} e_v \left(-\frac{T(y + \pi^\nu d, h)}{\pi^\lambda} \right) \\ &= \sum_y e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right) \sum_d e_v \left(-\frac{T(d, h)}{\pi^\mu} \right). \end{aligned}$$

The sum over d vanishes if there exists some $d \in L$ such that $T(y, d) \in \mathfrak{p}^\mu$ and $e_v \left(-\frac{T(d, h)}{\pi^\mu} \right) \neq 1$, since it is then a nontrivial character sum over a finite group. Such d exists if and only if $T(y, d) \in \mathfrak{p}^\mu \not\equiv T(d, h) \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$. So by the previous lemma, we may sum only over y such that $Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}L}$ for some

$a \in \mathfrak{d}^{-1}$, thus:

$$\begin{aligned}
 S(\lambda, h) &= \sum_{\substack{y+\pi^\nu d: \\ y \in \sigma(\lambda) \pmod{\mathfrak{p}^\nu L} \\ d \in L \pmod{\mathfrak{p}^\mu} \\ T(y, d) \in \mathfrak{p}^\mu / \mathfrak{p}^\lambda \\ Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1} L} \\ \text{(for some } a \in \mathfrak{d}^{-1}\text{)}}} e_v \left(-\frac{T(y + \pi^\nu d, h)}{\pi^\lambda} \right) \\
 &= \sum_{y \in \tau(\lambda, h)} e_v \left(-\frac{T(y, h)}{\pi^\lambda} \right).
 \end{aligned}$$

□

PROPOSITION 6.9. *If $\nu(T[h']) > 0$, $\alpha(h, X)$ is a polynomial $K_{h, \nu}(X) \in \mathbb{Q}[X]$ of degree less than $2(\nu' + 1 + 2\nu_{h\mathfrak{d}} + \nu_{\mathfrak{d}})$, where $\nu' = \nu(T[h'])$.*

Proof. We will prove $\tau(\lambda, h)$ is empty for $\lambda \geq 2(\nu' + 1 + 2\nu_{h\mathfrak{d}} + \nu_{\mathfrak{d}})$. Suppose $y \in \tau(\lambda, h)$. Then for some $a \in \mathfrak{d}^{-1}$, $Th - aTy \in \mathfrak{p}^{\lfloor \frac{\lambda}{2} \rfloor} \mathfrak{d}^{-1} L \subset \mathfrak{p}^{(\nu'+1+2\nu_{\mathfrak{d}})} L$, i.e., $Th \equiv aTy \pmod{\mathfrak{p}^{\nu'+1+2\nu_{\mathfrak{d}}} L}$. Multiplying by T^{-1} gives also $h \equiv ay \pmod{\mathfrak{p}^{\nu'+1+2\nu_{\mathfrak{d}}} L}$, so $\pi^{2\nu_h} T[h'] = T[h] \equiv a^2 T[y] \pmod{\mathfrak{p}^{\nu'+1+2\nu_{\mathfrak{d}}}}$. But since $y \in \tau(\lambda, h)$, $a^2 T[y] \in \mathfrak{p}^{\lambda} \mathfrak{d}^{-2} \subset \mathfrak{p}^{2(\nu'+1+2\nu_{h\mathfrak{d}})} \subset \mathfrak{p}^{\nu'+1+2\nu_{h\mathfrak{d}}}$, giving the contradiction $T[h'] \in \mathfrak{p}^{\nu'+1+2\nu_{\mathfrak{d}}}$. □

Summary. We gather the results of this chapter.

THEOREM 6.10. *For n even,*

$$\alpha_\nu(h_\nu, X) = \begin{cases} \left(1 - q_v^{\frac{n}{2}-1} \theta(\mathfrak{p}_\nu) X\right) \left(1 - q_v^{\frac{n}{2}} \theta(\mathfrak{p}_\nu) X\right)^{-1} & \text{if } h_\nu = 0 \\ \left(1 - q_v^{n-1} X\right)^{-1} & \\ \left(1 - q_v^{\frac{n}{2}-1} \theta(\mathfrak{p}_\nu) X\right) \left(1 - q_v^{\frac{n}{2}} \theta(\mathfrak{p}_\nu) X\right)^{-1} & \\ G_{h, \nu}(X) & \text{if } T[h_\nu] = 0 \\ \left(1 - q_v^{\frac{n}{2}-1} \theta(\mathfrak{p}_\nu) X\right) & \text{if } \nu(T[h'_\nu]) = 0, \nu_{h\mathfrak{d}} = 0 \\ H_{h, \nu}(X) & \text{if } \nu(T[h'_\nu]) = 0, \nu_{h\mathfrak{d}} > 0 \\ K_{h, \nu}(X) & \text{if } \nu(T[h'_\nu]) > 0. \end{cases}$$

For n odd,

$$\alpha_v(h_v, X) = \begin{cases} (1 - q_v^{n-1} X^2)(1 - q_v^n X^2)^{-1} & \text{if } h_v = 0 \\ (1 - q_v^{n-1} X)^{-1} & \text{if } T[h_v] = 0 \\ (1 - q_v^{n-1} X^2)(1 - q_v^n X^2)^{-1} G_{h,v}(X) & \text{if } \nu(T[h'_v]) = 0, \nu_{h\mathfrak{d}} = 0 \\ (1 - q_v^{n-1} X^2) \left(1 - q_v^{\frac{n-1}{2}} \theta_h(\mathfrak{p}_v) X\right)^{-1} & \text{if } \nu(T[h'_v]) = 0, \nu_{h\mathfrak{d}} > 0 \\ H_{h,v}(X) & \text{if } \nu(T[h'_v]) > 0. \\ K_{h,v}(X) & \end{cases}$$

Recalling that $a_v(h_v, s) = \alpha_v(h_v, X_v(s))$ for $v \in \mathfrak{f}$, $v \nmid \mathfrak{b}$, where from before $X_v(s) = \psi(\mathfrak{p}_v)^{-1} q_v^{-k-2s}$, and taking the product over such v gives,

THEOREM 6.11. For $z = (z_v) = (x_v + iy_v) \in \mathcal{H}^{\mathfrak{a}}$,

$$E(z, s; k, \psi, \mathfrak{b}) = (-1)^{dk} 2^{d(k+2s)} \sum_{h \in L'} a(h, y, s) \mathbf{e} \left(\sum_{v \in \mathfrak{a}} T^v(x_v, h_v) \right),$$

with

$$a(h, y, s) = N\mathfrak{d}^{-n/2} a_{\mathfrak{a}}(h, y, s) a_{\mathfrak{f}}(h, s),$$

where

$$a_{\mathfrak{a}}(h, y, s) = \prod_{v \in \mathfrak{a}} \xi(y_v, h_v; k + s, s; T^v);$$

for n even,

(6.5a)

$$a_{\mathfrak{f}}(h, s) = L_{\mathfrak{b}} \left(k + 2s + 1 - \frac{n}{2}, \theta\psi^{-1} \right)^{-1} \cdot \begin{cases} L_{\mathfrak{b}} \left(k + 2s - \frac{n}{2}, \theta\psi^{-1} \right) L_{\mathfrak{b}}(k + 2s - n + 1, \psi^{-1}) & \text{if } h = 0 \\ L_{\mathfrak{b}} \left(k + 2s - \frac{n}{2}, \theta\psi^{-1} \right) \prod_{v \nmid \mathfrak{b}: \nu_v(h) + \nu_v(\mathfrak{d}) > 0} G_{h,v}(X_v(s)) & \text{if } T[h] = 0 \\ \prod_{\substack{v \nmid \mathfrak{b}: \nu_v(T[h'_v]) = 0, \\ \nu_v(h) + \nu_v(\mathfrak{d}) > 0}} \frac{H_{h,v}(X_v(s))}{(1 - q_v^{\frac{n}{2}-1} \theta(\mathfrak{p}_v) X_v(s))} \cdot \prod_{v \nmid \mathfrak{b}: \nu_v(T[h'_v]) > 0} \frac{K_{h,v}(X_v(s))}{\left(1 - q_v^{\frac{n}{2}-1} \theta(\mathfrak{p}_v) X_v(s)\right)} & \text{if } T[h] \neq 0; \end{cases}$$

and for n odd,

(6.5b)

$$a_f(h, s) = L_{\mathfrak{b}}(2(k + 2s) - n + 1, \psi^{-2})^{-1} \begin{cases} L_{\mathfrak{b}}(2(k + 2s) - n, \psi^{-2})L_{\mathfrak{b}}(k + 2s - n + 1, \psi^{-1}) & \text{if } h = 0 \\ L_{\mathfrak{b}}(2(k + 2s) - n, \psi^{-2}) \prod_{\nu \dagger \mathfrak{b}: \nu_v(h) + \nu_v(\mathfrak{d}) > 0} G_{h,v}(X_v(s)) & \text{if } T[h] = 0 \\ L_{\mathfrak{b}\mathfrak{h}} \left(k + 2s - \frac{n-1}{2}, \theta_h \psi^{-1} \right) \cdot \prod_{\substack{\nu \dagger \mathfrak{b}: \nu_v(T[h'_v])=0, \\ \nu_v(h) + \nu_v(\mathfrak{d}) > 0}} \frac{H_{h,v}(X_v(s))}{(1 - q_v^{n-1} X_v(s)^2)} \\ \cdot \prod_{\nu \dagger \mathfrak{b}: \nu_v(T[h'_v]) > 0} \frac{K_{h,v}(X_v(s))}{(1 - q_v^{n-1} X_v(s)^2)} & \text{if } T[h] \neq 0. \end{cases}$$

Here $\mathfrak{h} = \prod_{\substack{\nu \dagger \mathfrak{b}: \nu_v(T[h'_v])=0, \\ \nu_v(h) + \nu_v(\mathfrak{d}) > 0}} \mathfrak{p}_v \prod_{\nu \dagger \mathfrak{b}: \nu_v(T[h'_v]) > 0} \mathfrak{p}_v$, θ and θ_h are the quadratic characters defined in this chapter, and $G_{h,v}$, $H_{h,v}$ and $K_{h,v}$ are the polynomials from Propositions 6.7, 6.8 and 6.9.

7. $E(z, s)$ at special values of s .

The order of $a(h, y, s)$ at $s = 0$. For a discussion of near holomorphy and arithmeticity of a class of functions containing $E(z, s)$ the reader is referred to [Sh86], [Sh87], [Bl90], [Blpp]. As a special case, we exhibit the Fourier expansion of $E(z, s)$ at $s = 0$.

DEFINITION. For $h \in L'$ such that $T[h] \neq 0$, define

$$p_h = \# \{ v \in \mathfrak{a} : h_v \in \mathcal{P}_v \},$$

$$q_h = \# \{ v \in \mathfrak{a} : -h_v \in \mathcal{P}_v \},$$

$$r_h = \# \{ v \in \mathfrak{a} : T^v[h_v] < 0 \}.$$

For nonzero $h \in L'$ with $T[h] = 0$, define

$$s_h = \# \{ v \in \mathfrak{a} : T^v(h_v, \varepsilon_v) > 0 \},$$

$$t_h = \# \{ v \in \mathfrak{a} : T^v(h_v, \varepsilon_v) < 0 \}.$$

Define $b = \# \{ v \in \mathfrak{f} : v \mid \mathfrak{b} \}$.

Observe that $p_h + q_h + r_h = s_h + t_h = d$, where $d = [F : \mathbb{Q}]$, and that $b > 0$.

PROPOSITION 7.1. *For n even and $k \geq n/2$, $L_b(k + 2s + 1 - n/2, \theta\psi^{-1})a(h, y, s) |_{s=0}$ has a zero of order at least*

$$\begin{cases} d - 1, & \text{if } h = 0 \text{ and } k = n/2 + 1, \psi = \theta \\ d, & \text{if } h = 0 \text{ otherwise} \\ d + t_h - 1, & \text{if } T[h] = 0 \text{ and } k = n/2 + 1, \psi = \theta \\ d, & \text{if } T[h] = 0 \text{ otherwise} \\ 2q_h + r_h, & \text{if } T[h] \neq 0. \end{cases}$$

For n odd and $k \geq (n+1)/2$, $L_b(2(k+2s)+1-n, \psi^{-2})a(h, y, s) |_{s=0}$ has a zero of order at least

$$\begin{cases} d - 1, & \text{if } h = 0 \text{ or } T[h] = 0 \text{ and } k = (n + 1)/2, \psi^2 = 1 \\ d, & \text{if } h = 0 \text{ or } T[h] = 0 \text{ otherwise} \\ q_h + r_h - 1, & \text{if } T[h] \neq 0 \text{ and } k = (n + 1)/2, \psi = \theta_h \\ q_h + r_h, & \text{if } T[h] \neq 0 \text{ otherwise.} \end{cases}$$

Proof. This is straightforward from examining the Γ - and L -factors that occur in $a(h, y, s) |_{s=0}$. For example, consider the case n even, $k \geq n/2$, $h = 0$. A d -fold product of the archimedean factor in (5.1) gives a zero of order $2d$ if $k \geq n$; d if $n/2 < k < n$; 0 if $k = n/2$. The term $L_b(k - n/2, \theta\psi^{-1})$ in (6.5a) gives a zero of order 0 if $k > n/2 + 1$ or $k = n/2 + 1, \psi \neq \theta$; -1 if $k = n/2 + 1, \psi = \theta$; $d - 1 + b \geq d$ if $k = n/2, \psi = \theta$; d if $k = n/2, \psi \neq \theta$. And the term $L_b(k + 1 - n, \theta\psi^{-1})$ in (6.5a) gives a zero of order 0 unless $k = n, \psi = 1$; -1 if $k = n, \psi = 1$. Combining these gives the result. The other cases are simpler. □

COROLLARY 7.2. *For n even and $k \geq n/2$, $L_b(k + 2s + 1 - n/2, \theta\psi^{-1})a(h, y, s) |_{s=0}$ is finite. It is nonzero only in the cases (a) $h \in \mathcal{P}^a$, (b) $F = \mathbb{Q}, k = n/2 + 1, \psi = \theta, T[h] = 0, T(h, \varepsilon) > 0$ or $h = 0$.*

For n odd and $k \geq (n + 1)/2$, excepting the case $k = (n + 1)/2, \psi = \theta_h$ for some h , $L_b(2(k + 2s) - n + 1, \psi^{-2})a(h, y, s) |_{s=0}$ is finite. It is nonzero only in the cases (a) $h \in \mathcal{P}^a$, (b) $F = \mathbb{Q}, k = (n + 1)/2, \psi^2 = 1, T[h] = 0$ or $h = 0$.

The Fourier expansion of $E(z, s)$ at $s = 0$. From Proposition 5.1 we obtain

$$a_{\mathbf{a}}(h, y, 0) = (-1)^{dk} 2^d \pi^{d(2k+1-\frac{n}{2})} \Gamma(k)^{-d} \Gamma(k+1-n/2)^{-d} \cdot |N(\det T)|^{-\frac{1}{2}} N(T[h])^{k-\frac{n}{2}}$$

$e(\sum_{v \in \mathbf{a}} T^v(iy_v, h_v))$ if $h \in \mathcal{P}^{\mathbf{a}}$. Thus for n even, $k \geq n/2$, excepting the case $F = \mathbb{Q}$, $k = n/2 + 1$, $\psi = \theta$, specializing to $s = 0$ gives the holomorphic function

$$(7.1) \quad L_{\mathbf{b}} \left(k + 2s + 1 - \frac{n}{2}, \theta\psi^{-1} \right) E(z, s; k, \psi, \mathbf{b}) \Big|_{s=0} \\ = \pi^{d(2k+1-\frac{n}{2})} |N(\det T)|^{-\frac{1}{2}} N\mathfrak{d}^{-\frac{n}{2}} 2^{d(k+1)} \Gamma(k)^{-d} \Gamma \left(k + 1 - \frac{n}{2} \right)^{-d} \\ \cdot \sum_{h \in L' \cap \mathcal{P}^{\mathbf{a}}} N(T[h])^{k-\frac{n}{2}} \prod_{\substack{\mathfrak{p}|\mathbf{b}: \nu_{\mathfrak{p}}(T[h'_v])=0, \\ \nu_{\mathfrak{p}}(h) + \nu_{\mathfrak{p}}(\mathfrak{d}) > 0}} \frac{H_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{\left(1 - \theta\psi^{-1}(\mathfrak{p}_v)q_v^{\frac{n}{2}-k-1} \right)} \\ \cdot \prod_{\mathfrak{p}|\mathbf{b}: \nu_{\mathfrak{p}}(T[h'_v]) > 0} \frac{K_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{\left(1 - \theta\psi^{-1}(\mathfrak{p}_v)q_v^{\frac{n}{2}-k-1} \right)} e \left(\sum_{v \in \mathbf{a}} T^v(z_v, h_v) \right),$$

with Fourier coefficients in $\pi^{d(2k+1-\frac{n}{2})} |N(\det T)|^{-\frac{1}{2}} \mathbb{Q}(\psi)$, where $\mathbb{Q}(\psi)$ is the extension of \mathbb{Q} generated by values of ψ .

In the case $F = \mathbb{Q}$, $k = n/2 + 1$, $\psi = \theta$ our function also has nonholomorphic terms at $s = 0$. Using Proposition 5.1 gives

$$(7.2) \quad \zeta_{\mathbf{b}}(2 + 2s) E \left(z, s; \frac{n}{2} + 1, \theta, \mathbf{b} \right) \Big|_{s=0} \\ = \pi^{\frac{n}{2}+1} |\det T|^{-\frac{1}{2}} \left(1 - \frac{n}{2} \right) \prod_{\mathfrak{p}|\mathbf{b}} (1 - p^{-1}) 2^{\frac{n}{2}-2} \\ \cdot \Gamma \left(\frac{n}{2} + 1 \right)^{-1} L_{\mathbf{b}} \left(2 - \frac{n}{2}, \theta \right) T[y]^{-1} \\ + \pi^{\frac{n}{2}+2} |\det T|^{-\frac{1}{2}} \prod_{\mathfrak{p}|\mathbf{b}} (1 - p^{-1}) 2^{\frac{n}{2}+1} \Gamma \left(\frac{n}{2} + 1 \right)^{-1} \\ \cdot \sum_{\substack{h \in L': T[h]=0, \mathfrak{p}|\mathbf{b}: \nu_{\mathfrak{p}}(h) > 0 \\ T(h, \varepsilon) > 0}} \prod_{\mathfrak{p}|\mathbf{b}} G_{h,p}(\theta(p)p^{1-\frac{n}{2}}) T[y]^{-1} T(y, h) e(T(z, h))$$

$$\begin{aligned}
 & + \pi^{\frac{n}{2}+3} |\det T|^{-\frac{1}{2}} 2^{\frac{n}{2}+2} \Gamma\left(\frac{n}{2} + 1\right)^{-1} \\
 & \cdot \sum_{h \in L' \cap \mathcal{P}} T[h] \prod_{\substack{\mathfrak{p} \nmid \mathfrak{b}: \nu_{\mathfrak{p}}(T[h'_{\mathfrak{p}}])=0, \\ \nu_{\mathfrak{p}}(h) > 0}} \frac{H_{h,\mathfrak{p}}(\theta(\mathfrak{p})p^{-\frac{n}{2}-1})}{(1 - p^{-2})} \\
 & \cdot \prod_{\mathfrak{p} \nmid \mathfrak{b}: \nu_{\mathfrak{p}}(T[h'_p]) > 0} \frac{K_{h,\mathfrak{p}}(\theta(\mathfrak{p})p^{-\frac{n}{2}-1})}{(1 - p^{-2})} \mathbf{e}(T(z, h)).
 \end{aligned}$$

Here the coefficient of $T[y]^{-1}$ in the $h = 0$ term is in $\pi^{\frac{n}{2}+1} |\det T|^{-\frac{1}{2}} \mathbb{Q}$ and is nonzero only if $n \equiv 2 \pmod{4}$; the coefficients of $T[y]^{-1} T(y, h) \mathbf{e}(T(z, h))$ in the $T[h] = 0, T(h, \varepsilon) > 0$ terms are in $\pi^{\frac{n}{2}+2} |\det T|^{-\frac{1}{2}} \mathbb{Q}$; and the Fourier coefficients of the holomorphic terms are in $\pi^{\frac{n}{2}+3} |\det T|^{-\frac{1}{2}} \mathbb{Q}$.

Similar calculations show that for n odd, $k \geq (n+1)/2$, excepting the case $F = \mathbb{Q}, k = (n+1)/2, \psi^2 = 1$, specializing to $s = 0$ gives the holomorphic function

$$\begin{aligned}
 (7.3) \quad & L_{\mathfrak{b}}(2(k+2s) + 1 - n, \psi^{-2}) E(z, s; k, \psi, \mathfrak{b}) \Big|_{s=0} \\
 & = \pi^{d(2k+1-\frac{n}{2})} \Gamma\left(k + 1 - \frac{n}{2}\right)^{-d} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} 2^{d(k+1)} \Gamma(k)^{-d} \\
 & \cdot \sum_{h \in L' \cap \mathcal{P}^{\mathfrak{a}}} N(T[h])^{k-\frac{n}{2}} L_{\mathfrak{b}\mathfrak{h}}\left(k - \frac{n-1}{2}, \theta_h \psi^{-1}\right) \\
 & \cdot \prod_{\substack{\mathfrak{v} \nmid \mathfrak{b}: \nu_{\mathfrak{v}}(T[h'_v])=0, \\ \nu_{\mathfrak{v}}(h) + \nu_{\mathfrak{v}}(\mathfrak{d}) > 0}} \frac{H_{h,\mathfrak{v}}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{(1 - \psi^{-2}(\mathfrak{p}_v)q_v^{n-2k-1})} \\
 & \cdot \prod_{\mathfrak{v} \nmid \mathfrak{b}: \nu_{\mathfrak{v}}(T[h'_v]) > 0} \frac{K_{h,\mathfrak{v}}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{(1 - \psi^{-2}(\mathfrak{p}_v)q_v^{n-2k-1})} \mathbf{e}\left(\sum_{\mathfrak{v} \in \mathfrak{a}} T^{\mathfrak{v}}(z_{\mathfrak{v}}, h_{\mathfrak{v}})\right).
 \end{aligned}$$

In this case the Fourier coefficients are in

$$\pi^{d(2k-\frac{n-1}{2})} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} \mathbb{Q}_{\mathfrak{ab}}(\psi),$$

where $\mathbb{Q}_{\mathfrak{ab}}$ denotes the maximal abelian extension of \mathbb{Q} in \mathbb{C} .

In the case $F = \mathbb{Q}, k = (n+1)/2, \psi^2 = 1, \psi \neq \theta_h$ for all h , our function again has nonholomorphic terms at $s = 0$. Let

$l = \lim_{s \rightarrow 0} L_{\mathfrak{b}}((n + 1)/2 - n + 1 + 2s, \psi)/2s$. Then

$$\begin{aligned}
 (7.4) \quad & \zeta_{\mathfrak{b}}(2 + 4s)E\left(z, s; \frac{n + 1}{2}, \psi, \mathfrak{b}\right) \Big|_{s=0} \\
 &= \pi^{\frac{n+1}{2}} |\det T|^{-\frac{1}{2}} \prod_{p|\mathfrak{b}} (1 - p^{-1}) (-1)^{\frac{n+1}{2}} 2^{\frac{n-1}{2}} \Gamma\left(\frac{n + 1}{2}\right)^{-1} \\
 &\quad \cdot \Gamma\left(\frac{n - 1}{2}\right)^{-1} \Gamma\left(1 - \frac{n}{2}\right)^{-1} lT[y]^{-\frac{1}{2}} \\
 &+ \pi^{\frac{n+1}{2}} |\det T|^{-\frac{1}{2}} \prod_{p|\mathfrak{b}} (1 - p^{-1}) 2^{\frac{n+1}{2}} \Gamma\left(\frac{n + 1}{2}\right)^{-1} \\
 &\quad \cdot \sum_{\substack{h \in L': T[h]=0, \\ T(h, \varepsilon) > 0}} \prod_{p \nmid \mathfrak{b}: \nu_p(h) > 0} G_{h,p}(\psi(p)p^{-\frac{n+1}{2}}) T[y]^{-\frac{1}{2}} T(y, h)^{\frac{1}{2}} e(T(z, h)) \\
 &+ \pi^{\frac{n+1}{2}+1} |\det T|^{-\frac{1}{2}} 2^{\frac{n+1}{2}+2} \Gamma\left(\frac{n + 1}{2}\right)^{-1} \\
 &\quad \cdot \sum_{h \in L' \cap \mathcal{P}} T[h]^{\frac{1}{2}} L_{\mathfrak{b}\mathfrak{h}}(1, \theta_h \psi) \prod_{\substack{p \nmid \mathfrak{b}: \nu_p(T[h'_p])=0, \\ \nu_p(h) > 0}} \frac{H_{h,p}(\psi(p)p^{-\frac{n+1}{2}})}{(1 - p^{-2})} \\
 &\quad \cdot \prod_{p \nmid \mathfrak{b}: \nu_p(T[h'_p]) > 0} \frac{K_{h,p}(\psi(p)p^{-\frac{n+1}{2}})}{(1 - p^{-2})} e(T(z, h)).
 \end{aligned}$$

The residue of $E(z, s)$ at special values of s . Analysis of (5.1) and (6.5) shows that for n even, $k = n/2 - 1$, $s = 1$, $L_{\mathfrak{b}}(k + 2s + 1 - n/2, \theta\psi^{-1})E(z, s; k, \psi, \mathfrak{b})$ is finite unless $\psi = \theta$, in which case it has a simple pole and

$$\begin{aligned}
 (7.5) \quad & \text{Res}_{s=1} \zeta_{\mathfrak{b}}(2s)E\left(z, s; \frac{n}{2} - 1, \theta, \mathfrak{b}\right) \\
 &= \pi^{d(\frac{n}{2}+1)} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} 2^{d\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)^{-d} \\
 &\quad \cdot \text{Res}_{\sigma=1} \zeta_{\mathfrak{b}}(\sigma) T[y]^{-d} \left\{ 2^{-d} L_{\mathfrak{b}}\left(2 - \frac{n}{2}, \theta\right) \right. \\
 &\quad \left. + \sum_{\substack{h \in L': T[h]=0, \\ T^v(h_v, \varepsilon_v) > 0, v \in \mathfrak{a}}} \prod_{v \nmid \mathfrak{b}: \nu_v(h) > 0} G_{h,v}(\theta(p_v)q_v^{-\frac{n}{2}-1}) e\left(\sum_{v \in \mathfrak{a}} T^v(z_v, h_v)\right) \right\}.
 \end{aligned}$$

Similarly for n odd, $k = (n - 1)/2$, $s = 1/2$, excluding the case $\psi = \theta_h$ for some h , $L_{\mathfrak{b}}(2(k + 2s) + 1 - n, \psi^{-2})E(z, s; k, \psi, \mathfrak{b})$ is finite unless $\psi^2 = 1$, in which case it has a simple pole and

$$\begin{aligned}
 (7.6) \quad & \text{Res}_{s=1/2} \zeta_{\mathfrak{b}}(4s) E \left(z, s; \frac{n-1}{2}, \psi, \mathfrak{b} \right) \\
 &= \pi^{d(\frac{n}{2}+1)} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} 2^{d\frac{n-1}{2}-2} \Gamma \left(\frac{n}{2} \right)^{-d} \text{Res}_{\sigma=1} \zeta_{\mathfrak{b}}(\sigma) \\
 &\quad \cdot T[y]^{-\frac{d}{2}} \left\{ 2^{-d} L_{\mathfrak{b}} \left(2 - \frac{n+1}{2}, \psi^{-1} \right) \right. \\
 &\quad \left. \sum_{\substack{h \in L': T[h]=0, \\ T^v(h_v, \varepsilon_v) > 0, v \in \mathfrak{a}}} \prod_{v: \mathfrak{b}_v(h_v > 0)} G_{h,v} \left(\psi(\mathfrak{p}_v) q_v^{-\frac{n+1}{2}} \right) e \left(\sum_{v \in \mathfrak{a}} T^v(z_v, h_v) \right) \right\}.
 \end{aligned}$$

In (7.5) and (7.6), multiplying the residue by $T[y]^{sd}$ gives a holomorphic function.

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has a simple pole and

$$\begin{aligned}
 & (7.5) \\
 & \text{Res}_{s=1} \zeta_{\mathfrak{b}}(2s) E \left(z, s; \frac{n}{2} - 1, \theta, \mathfrak{b} \right) \\
 & = \pi^{d(\frac{n}{2}+1)} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} 2^{d\frac{n}{2}-1} \Gamma \left(\frac{n}{2} \right)^{-d} \\
 & \cdot \text{Res}_{\sigma=1} \zeta_{\mathfrak{b}}(\sigma) T[y]^{-d} \left\{ 2^{-d} L_{\mathfrak{b}} \left(2 - \frac{n}{2}, \theta \right) \right. \\
 & \left. + \sum_{\substack{h \in L': T[h]=0, \\ T^v(h_v, \varepsilon_v) > 0, v \in \mathfrak{a}}} \prod_{v \nmid \mathfrak{b}: \nu_v(h) > 0} G_{h,v} \left(\theta(\mathfrak{p}_v) q_v^{-\frac{n}{2}-1} \right) e \left(\sum_{v \in \mathfrak{a}} T^v(z_v, h_v) \right) \right\}.
 \end{aligned}$$

Similarly for n odd, $k = (n - 1)/2$, $s = 1/2$, excluding the case $\psi = \theta_h$ for some h , $L_{\mathfrak{b}}(2(k + 2s) + 1 - n, \psi^{-2}) E(z, s; k, \psi, \mathfrak{b})$ is finite unless $\psi^2 = 1$, in which case it has a simple pole and

$$\begin{aligned}
 & (7.6) \\
 & \text{Res}_{s=1/2} \zeta_{\mathfrak{b}}(4s) E \left(z, s; \frac{n-1}{2}, \psi, \mathfrak{b} \right) \\
 & = \pi^{d(\frac{n}{2}+1)} |N(\det T)|^{-\frac{1}{2}} N \mathfrak{d}^{-\frac{n}{2}} 2^{d\frac{n-1}{2}-2} \Gamma \left(\frac{n}{2} \right)^{-d} \text{Res}_{\sigma=1} \zeta_{\mathfrak{b}}(\sigma) \\
 & \cdot T[y]^{-\frac{d}{2}} \left\{ 2^{-d} L_{\mathfrak{b}} \left(2 - \frac{n+1}{2}, \psi^{-1} \right) \right. \\
 & \left. \sum_{\substack{h \in L': T[h]=0, \\ T^v(h_v, \varepsilon_v) > 0, v \in \mathfrak{a}}} \prod_{v \nmid \mathfrak{b}: \nu_v(h) > 0} G_{h,v} \left(\psi(\mathfrak{p}_v) q_v^{-\frac{n+1}{2}} \right) e \left(\sum_{v \in \mathfrak{a}} T^v(z_v, h_v) \right) \right\}.
 \end{aligned}$$

In (7.5) and (7.6), multiplying the residue by $T[y]^{sd}$ gives a holomorphic function.

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