LP-BOUNDEDNESS OF THE HILBERT TRANSFORM AND MAXIMAL FUNCTION ALONG FLAT CURVES IN \( \mathbb{R}^n \)

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**L^p-Boundedness of the Hilbert Transform and Maximal Function Along Flat Curves in R^n**

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We consider the Hilbert transform and maximal function associated to a curve \( \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \) in \( \mathbb{R}^n \). It is well-known that for a plane convex curve \( \Gamma(t) = (t, \gamma(t)) \) these operators are bounded on \( L^p, 1 < p < \infty \), if \( \gamma' \) doubles. We give an \( n \)-dimensional analogue, \( n \geq 2 \), of this result.

1. Introduction. Let \( \Gamma : \mathbb{R} \to \mathbb{R}^n \) be a curve in \( \mathbb{R}^n, \ n \geq 2, \) with \( \Gamma(0) = 0 \). We define the associated Hilbert transform, \( \mathcal{H}_\Gamma \) and maximal function \( \mathcal{M}_\Gamma \) by

\[
\mathcal{H}_\Gamma f(x) = \text{p. v.} \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t}
\]

and

\[
\mathcal{M}_\Gamma f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \Gamma(t))| \, dt,
\]

respectively. We use p. v. to indicate that we are taking a principal value integral.

There has been considerable interest in finding conditions on \( \Gamma \) which give \( L^2(\mathbb{R}^n) \)-boundedness or \( L^p(\mathbb{R}^n) \)-boundedness, \( 1 < p < \infty \), of \( \mathcal{H}_\Gamma \) and \( \mathcal{M}_\Gamma \), when \( \Gamma \) is permitted to be flat (i.e. vanish to infinite order) at the origin; the case of well-curved \( \Gamma \) was dealt with in the 1970's, see for example [7].

The aim of this paper is to give an \( n \)-dimensional analogue of the following well-known theorem for plane curves.

**Theorem 1.1.** [1]. Let \( \Gamma : \mathbb{R} \to \mathbb{R}^2, \Gamma(t) = (t, \gamma(t)) \) be a convex curve such that \( \gamma \in C^2(0, \infty) \) is either even or odd and \( \gamma(0) = \gamma'(0) = 0 \). Suppose that \( \exists 1 < \lambda < \infty \) such that \( \forall t \in (0, \infty) \)

\[
(1) \quad \gamma'(\lambda t) \geq 2\gamma'(t).
\]
Then
\[ \| \mathcal{H}_f \|_p \leq C \| f \|_p, \quad 1 < p < \infty. \]

Conditions such as (1) are known as doubling conditions; in this case we say that \( \gamma' \) doubles.

In \( \mathbb{R}^n \) we shall consider curves \( \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \) which are of class \( C^n(0, \infty) \) and such that \( \Gamma(0) = 0 \). The convexity hypothesis for plane curves we replace by the "convexity" hypothesis used in the \( n \)-dimensional results of [6] and [4].

So we define determinants \( D_j, \ j = 1, \ldots, n \) by

\[
D_j = \det \begin{pmatrix}
1 & \gamma'_2 & \cdots & \gamma'_j \\
0 & \gamma''_2 & \cdots & \gamma''_j \\
\vdots & \vdots & \ddots & \vdots \\
0 & \gamma^{(j)}_2 & \cdots & \gamma^{(j)}_j
\end{pmatrix}
\]

and say that \( \Gamma \) is "convex" if

\[
(2) \quad D_j(t) > 0, \quad j = 2, \ldots, n, t \in (0, \infty).
\]

We also introduce the determinants \( N_j, \ j = 1, \ldots, n \), given by

\[
N_j = \det \begin{pmatrix}
t & \gamma_2 & \cdots & \gamma_j \\
1 & \gamma'_2 & \cdots & \gamma'_j \\
\vdots & \vdots & \ddots & \vdots \\
0 & \gamma^{(j-1)}_2 & \cdots & \gamma^{(j-1)}_j
\end{pmatrix},
\]

and as in [6] define functions \( h_j, \ j = 1, \ldots, n \), by

\[
(3) \quad h_j(t) = \frac{N_j(t)}{D_{j-1}(t)},
\]

where we take \( D_0 \equiv 1 \).

In order to state our theorem we also introduce the differential operators \( L_k \), of [6], defined by

\[
(4) \quad L_{k+1}f = \frac{h_k}{h'_{k+1}} (L_k f)', \quad k = 1, \ldots, n - 1.
\]
It is also useful to have the following formula, proven via a Sylvester determinant identity in [6]:

\[
L_k f(t) = \frac{E_k f(t)}{D_k(t)}, \quad k = 1, \ldots, n,
\]

where

\[
E_k f(t) = \det \begin{pmatrix}
1 & \gamma_2'(t) & \cdots & \gamma_{k-1}'(t) & f'(t) \\
0 & \gamma_2''(t) & \cdots & \gamma_{k-1}''(t) & f''(t) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \gamma_2^{(k)}(t) & \cdots & \gamma_{k-1}^{(k)}(t) & f^{(k)}(t)
\end{pmatrix}.
\]

From this we can see, immediately, that

\[
L_k \gamma_j = 0, \quad j = 1, \ldots, k - 1; \quad k = 1, \ldots, n
\]
\[
L_k \gamma_k = 1, \quad k = 1, \ldots, n.
\]

Our result is the following.

**Theorem 1.2.** Let \( \Gamma : \mathbb{R} \to \mathbb{R}^n \), \( \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \), \( n \geq 2 \), be an odd curve in \( \mathbb{R}^n \), of class \( C^n(0, \infty) \) such that \( \Gamma(0) = 0 \) and (2) is satisfied. Suppose that there exists \( A \in \text{GL}(n, \mathbb{R}) \) such that, with \( \tilde{\Gamma}(t) = (t, \tilde{\gamma}_2(t), \ldots, \tilde{\gamma}_n(t)) := A\Gamma(t) \), \( \tilde{\Gamma} \) also satisfies (2) and

\[
\lim_{t \to 0} L_j \tilde{\gamma}_k(t) = 0 \quad j = 1, \ldots, n - 1, k = j + 1, \ldots, n.
\]

Suppose also that there exists \( 1 < \lambda < \infty \) such that, \( \forall t \in (0, \infty) \),

\[
L_k \tilde{\gamma}_{k+1}(\lambda t) \geq 2L_k \tilde{\gamma}_{k+1}(t), \quad k = 1, \ldots, n - 1.
\]

Then

\[
\|H_\Gamma f\|_p \leq C\|f\|_p, \quad 1 < p < \infty.
\]

**Remarks.** (a) Since \( L^p \)-boundedness of \( H_\Gamma \) and of \( M_\Gamma \) is a \( \text{GL}(n, \mathbb{R}) \) invariant property, in the proof we shall assume, without loss of generality, that the initial curve \( \Gamma \) satisfies (8) and (9).

(b) For \( n = 2 \) our theorem is precisely Theorem 1.1.
(c) It is easily checked that the "convexity" hypothesis, (2), is equivalent to requiring that
\[(L_k\gamma_{k+1})' > 0, \quad k = 1, \ldots, n - 1.\]
Thus, for the class of "convex" curves our conditions are natural analogues of the \(\gamma'\) doubling condition for plane convex curves (i.e. those for which \((L_1\gamma)' > 0\)).

(d) The condition that \(\Gamma\) be odd is convenient but not essential; it may be replaced by other conditions on \(\Gamma\) giving suitable compatibility of the two halves \(\Gamma(t), \ t > 0\) and \(\Gamma(t), \ t < 0\). For example each \(\gamma_k, k = 2, \ldots, n\) may be either even or odd; this will be clear from the proof.

(e) The role of (8) is to impose a certain ordering of the components of the curve. Further, it follows easily from Lemma 3 of [6] (see Lemma 3.1) that each \(L_j\gamma_k\) has at most \(k - j\) zeros and at most \(k - j - 1\) changes of monotonicity on \((0, \infty)\); the normalization conditions (8) force the \(L_j\gamma_k\) to be positive and increasing, thus much simplifying matters.

We note that if \(\lim_{t \to 0} L_j\gamma_k(t)\) exists for all \(1 \leq j \leq k - 1 \leq n - 1\), then we can find an \(A \in \text{GL}(n, \mathbb{R})\) such that \(\widetilde{\Gamma} = A\Gamma\) satisfies (8). To see this we first define an operator \(\mathcal{L}\) by
\[
\mathcal{L}\Gamma(t) = \begin{pmatrix}
L_1\gamma_1(t) & L_2\gamma_1(t) & \cdots & L_n\gamma_1(t) \\
L_1\gamma_2(t) & L_2\gamma_2(t) & \cdots & L_n\gamma_2(t) \\
\vdots & \vdots & \ddots & \vdots \\
L_1\gamma_n(t) & L_2\gamma_n(t) & \cdots & L_n\gamma_n(t)
\end{pmatrix}
\]
using (6) and (7).

It is easily shown that if \(A \in T_-\), the subgroup of \(\text{GL}(n, \mathbb{R})\) consisting of lower triangular matrices with 1 in the top left-hand corner and positive diagonal entries, then \(A\) preserves "convexity", i.e. if
Γ satisfies (2) then so does AΓ. Moreover, an easy calculation using (5) shows that if A ∈ T_ and has diagonal entries all equal to 1 then
\[ \mathcal{L}(A\Gamma) = A(\mathcal{L}\Gamma). \]

We now let \( A = (\lim_{t \to 0} \mathcal{L}\Gamma(t))^{-1} \), where \( \lim_{t \to 0} \mathcal{L}\Gamma(t) \) denotes the matrix with entries \( \lim_{t \to 0} L_j \gamma_k(t) \). Then \( \tilde{\Gamma} = A\Gamma \) is “convex” and \( \lim_{t \to 0} \mathcal{L}\tilde{\Gamma}(t) \) is the identity matrix, from which we see that \( \lim_{t \to 0} L_j \tilde{\gamma}_k(t) = 0, \quad j = 1, \ldots, n - 1; \quad k = j + 1, \ldots, n. \)

Curves for which we do not have the existence of \( \lim_{t \to 0} L_j \gamma_k(t) \) for all \( 1 \leq j \leq k - 1 \leq n - 1 \) may still satisfy the hypotheses of our theorem. Consider, for example the “convex” curve in \( \mathbb{R}^3 \), \( \Gamma(t) = (t, t^3, -t^2) \); in this case we have \( L_2 \gamma_3(t) = -\frac{1}{3t} \). However taking \( A \) to be
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\]
we obtain the curve \( \tilde{\Gamma}(t) = (t, t^2, t^3) \), which clearly satisfies the hypotheses (8) and (9).

(f) Theorem 1.1, after a technical adjustment to condition (1), may also be seen to hold for curves which are not \( C^2(0, \infty) \) but convex and piecewise-linear. We say that a piecewise-linear \( \gamma \) curve is convex if
\[
\frac{\gamma(c) - \gamma(b)}{c - b} \geq \frac{\gamma(b) - \gamma(a)}{b - a}, \quad 0 \leq a < b < c.
\]

Our method of proof of Theorem 1.2 allows us to extract the following result for piecewise-linear curves in \( \mathbb{R}^n \).

**Corollary 1.3.** Let \( \Gamma : \mathbb{R} \to \mathbb{R}^n \), \( \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \) be an odd curve such that \( \Gamma(0) = 0 \) and each \( \gamma_k, \ k = 2, \ldots, n, \) is convex and piecewise-linear on \( [\lambda^j, \lambda^{j+1}], \ j \in \mathbb{Z}, \) some \( \lambda > 1. \) Suppose
\[
\frac{\gamma_k(\lambda^{j+1}) - \gamma_k(\lambda^j)}{\gamma_{k-1}(\lambda^{j+1}) - \gamma_{k-1}(\lambda^j)} \geq 2 \frac{\gamma_k(\lambda^j) - \gamma_k(\lambda^{j-1})}{\gamma_{k-1}(\lambda^j) - \gamma_{k-1}(\lambda^{j-1})}, \]
for \( j \in \mathbb{Z}, \ k = 2, \ldots, n. \)

Then
\[
\|\mathcal{H}_\Gamma f\|_p \leq C\|f\|_p, \quad 1 < p < \infty.
\]
2. Sketch of Proof. We define measures $\mu_k, \sigma_k$ on the curve $\Gamma$ by
\[
\int f \, d\mu_k = \frac{1}{\lambda^k(\lambda - 1)} \int_{\lambda^k}^{\lambda^{k+1}} f(\Gamma(t)) \, dt \quad \text{and}
\int f \, d\sigma_k = \int_{\lambda^k \leq |t| \leq \lambda^{k+1}} f(\Gamma(t)) \frac{dt}{t},
\]
respectively. Then we have the associated Fourier multipliers
\[
\hat{\mu}_k(\zeta) = \frac{1}{\lambda^k(\lambda - 1)} \int_{\lambda^k}^{\lambda^{k+1}} e^{i\zeta \cdot \Gamma(t)} \, dt
\]
and
\[
\hat{\sigma}_k(\zeta) = \int_{\lambda^k \leq |t| \leq \lambda^{k+1}} e^{i\zeta \cdot \Gamma(t)} \frac{dt}{t}.
\]
We adopt the standard approach of decomposing $\mathcal{H}_\Gamma$ as
\[
\mathcal{H}_\Gamma f = \sum_k \sigma_k * f
\]
and majorizing $\mathcal{M}_\Gamma$ by
\[
\mathcal{M}_\Gamma f \leq C \sup_k |\mu_k * f|.
\]
From [4] the following theorem is easily extracted.

**Theorem 2.1.** Let $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^n$, $\Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t))$ be an odd curve in $\mathbb{R}^n$, $\Gamma(0) = 0$. Suppose $\exists$ a family of dilation matrices $\{A_k\} \subseteq \text{GL}(n, \mathbb{R})$ such that
\[
(a) \exists \alpha \text{ such that } \left\| A_{k+1}^{-1} A_k \right\| \leq \alpha < 1
\]
\[
(b) A_{k+1}^{-1} \text{ supp } \mu_k \subseteq \text{ fixed ball}
\]
\[
(c) |\hat{\mu}_k(\zeta)| \leq C |A_k^* \zeta|^{-\varepsilon} \text{ for some } \varepsilon > 0.
\]
Then
\[
\left\| \left\| \sup_k |\mu_k * f| \right\|_p \right\|_p \leq C \|f\|_p
\]
\[
\|\mathcal{H}_\Gamma f\|_p \leq C \|f\|_p, \quad 1 < p < \infty.
\]
In (8a) we use \( \| \cdot \| \) to denote the operator (matrix) norm. We note that the conditions of the theorem do not involve \( \sigma_k \). This is because, in view of the cancellation property,

\[
\int d\sigma_k = 0
\]

and the fact that \( \Gamma \) is odd, (12b) and (12c) give also analogous statements for \( \sigma_k \). Without the assumption that \( \Gamma \) is odd we require also that

\[
A_{k+1}^{-1} \text{ supp } \sigma_k \subseteq \text{ fixed ball}
\]

and

\[
|\hat{\sigma}_k(\zeta)| \leq C|A_k^*\zeta|^{-\epsilon} \text{ for some } \epsilon > 0.
\]

Condition (12a) is known as Rivière's condition and enables a Calderón-Zygmund theory with respect to balls \( \{A_jB\} \), for \( B \) the unit ball in \( \mathbb{R}^n \), and thence an "annular" Littlewood-Paley decomposition to be developed.

Conditions (12b) and (12c) give decay estimates for \( \hat{\mu}_k \) (and \( \hat{\sigma}_k \)) which may be combined with the Littlewood-Paley theory, along with a bootstrapping argument, to give the result. In [4] the authors find conditions on \( \Gamma \) under which (12c) holds, (12a) and (12b) being easily satisfied with an appropriate choice of the dilation matrices.

Our approach is to consider, for each \( k \), the points \( \zeta \in \mathbb{R}^n \) where (12c) may fail and to develop a conical Littlewood-Paley decomposition to deal with these "bad" \( \zeta \), in the spirit of [1] or [5].

In Section 3 we shall give some essential properties of "convex" curves and define our choice of dilation matrices \( \{A_k\} \). In Section 4 we consider the set of \( \zeta \in \mathbb{R}^n \) where the required decay estimates for \( \hat{\mu}_k \), \( \hat{\sigma}_k \) may fail and show that these \( \zeta \) are contained in a cone \( C_k \). Next we give conditions on \( \Gamma \), of which there are \( \frac{1}{2}n(n - 1) \), under which these \( C_k \) form a Littlewood-Paley decomposition and show how they may be reduced to the \( n - 1 \) conditions, (9), in the statement of our theorem. Finally in Section 5 we indicate how to combine the conical Littlewood-Paley theory of Section 4 with the "annular" Littlewood-Paley theory of Theorem 2.1 to complete the proof.

3. "Convexity" and dilation matrices. Most of the consequences of "convexity" that we shall need are dealt with in [6].
First, from Lemma 2 of [6] we know that for a “convex” curve we have, for \( k = 2, \ldots, n, \ t \in (0, \infty) \)

\[
(13) \quad h_k(t) > 0 \quad \text{and} \quad h'_k(t) > 0.
\]

The tool we have for estimating oscillatory integrals such as \( \hat{\mu}_k \) is Van der Corput’s lemma; in order to be able to use this we need to know that \( \zeta, \Gamma' \) has a bounded number of changes of monotonicity on each \([\lambda^k, \lambda^{k+1})\). This is given in Lemma 3 of [6].

**Lemma 3.1.** ([6, Lemma 3]). Let \( \Gamma \in C^n(0, \infty) \) be a “convex” curve in \( \mathbb{R}^n \), \( \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \) such that \( \Gamma(0) = 0 \). Then for \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n \), \( L_n(\zeta, \Gamma) = \zeta_n \) and for \( j = 1, 2, \ldots, n, L_j(\zeta, \Gamma) \) has at most \( n - j \) zeros in \((0, \infty)\), provided \( \zeta_n \neq 0 \).

The proof of this in [6] establishes the identity (5) mentioned previously, the result then following easily. We shall also need the following:

**Lemma 3.2.** Let \( \Gamma \in C^n(0, \infty), \ \Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) \), \( \Gamma : \mathbb{R} \rightarrow \mathbb{R}^n \) be a “convex” curve in \( \mathbb{R}^n \), satisfying also (8), i.e.

\[
\lim_{t \to 0} L_k \gamma_{j+1}(t) = 0, \quad j = k, \ldots, n - 1, \ k = 1, \ldots, n - 1.
\]

Then for \( t \in (0, \infty) \)

\[
(14) \quad (L_k \gamma_j)'(t) > 0 \quad \text{and} \quad (L_k \gamma_j)(t) > 0,
\]

\[
k = 1, \ldots, n - 1, \ j = k + 1, \ldots, n.
\]

In particular \( \gamma_j'' > 0, \ j = 2, \ldots, n. \)

**Proof.** We recall that, for \( k = 1, \ldots, n - 1, \)

\[
L_{k+1} f = \frac{h_k}{h'_{k+1}} (L_k f)'.
\]

So by (7) we have, for \( k = 1, \ldots, n - 1, \ t \in (0, \infty) \)

\[
(L_k \gamma_{k+1})(t) = \frac{h'_{k+1}(t)}{h_k(t)} > 0,
\]
using (13). Then (8) gives us also
\[ L_k \gamma_{k+1}(t) > 0, \quad k = 1, \ldots, n-1, \quad t \in (0, \infty). \]

We now fix \( j \in \{k + 1, \ldots, n\} \) and suppose that for some \( k \in \{1, \ldots, j\} \), \( t \in (0, \infty) \),
\[ (L_k \gamma_j)'(t) > 0 \quad \text{and} \quad L_k \gamma_j(t) > 0. \]

Then, for \( t \in (0, \infty) \),
\[ (L_{k-1} \gamma_j)'(t) = \frac{h_k'(t)}{h_{k-1}(t)} L_k \gamma_j(t) > 0, \]
using again (13). We also have \( L_{k-1} \gamma_j(t) > 0, \ t \in (0, \infty), \) using (8).

The result now follows by induction. \( \square \)

**Corollary 3.3.** Let \( \Gamma \) be as in the lemma. Suppose also that
\( \Gamma(0) = 0, \gamma_k(0) = 0, k = 2, \ldots, n. \) Then for \( k = 2, \ldots, n \)
(a) \( \gamma_k' \) is increasing and non-negative on \((0, \infty)\)
(b) \( \gamma_k \) is increasing and non-negative on \((0, \infty)\)
(c) \( \gamma_k(\lambda^{j+1}) \geq \lambda \gamma_j(\lambda^j), \quad \forall j \in \mathbb{Z}. \)

*Proof.* Immediate from Lemma 3.2. \( \square \)

**Lemma 3.4.** Let \( \Gamma \) be as in Lemma 3.2. Then, for \( t \in (0, \infty) \),
\[ \left( \frac{L_k \gamma_{j+1}}{L_k \gamma_j} \right)'(t) > 0, \quad \forall j = k, \ldots, n-1, \quad k = 1, \ldots, n-1. \]

*Proof.* We proceed by induction. Let \( k \in \{1, \ldots, n-1\} \) be fixed. Then
\[ \left( \frac{L_k \gamma_{k+1}}{L_k \gamma_k} \right)' = (L_k \gamma_{k+1})' = \frac{h_{k+1}'}{h_k} > 0. \]
Now we suppose that
\[ \left( \frac{L_m \gamma_{k+1}}{L_m \gamma_k} \right)' > 0, \quad \text{for some} \quad m \in \{2, \ldots, k\}. \]

Then
\[ \left( \frac{L_m \gamma_{k+1}}{L_m \gamma_k} \right)' = \left( \frac{(L_{m-1} \gamma_{k+1})'}{(L_{m-1} \gamma_k)'} \right)' > 0. \]
So by the Second Mean Value Theorem, if \( \varepsilon \in (0, t) \),
\[
\frac{L_{m-1}\gamma_{k+1}(t) - L_{m-1}\gamma_{k+1}(\varepsilon)}{L_{m-1}\gamma_k(t) - L_{m-1}\gamma_k(\varepsilon)} = \frac{(L_{m-1}\gamma_{k+1})'(\eta)}{(L_{m-1}\gamma_k)'(\eta)},
\]
for some \( \eta \in (0, t) \). Then, by (15) and (8),
\[
(16) \quad \frac{L_{m-1}\gamma_{k+1}(t)}{L_{m-1}\gamma_k(t)} < \frac{(L_{m-1}\gamma_{k+1})'(t)}{(L_{m-1}\gamma_k)'(t)}.
\]
Hence, using (16) and (14),
\[
\left( \frac{L_{m-1}\gamma_{k+1}}{L_{m-1}\gamma_k} \right)' = \frac{(L_{m-1}\gamma_k)'}{(L_{m-1}\gamma_k)} \left\{ \left( \frac{L_{m-1}\gamma_{k+1}}{(L_{m-1}\gamma_k)'} \right) - \frac{L_{m-1}\gamma_{k+1}}{L_{m-1}\gamma_k} \right\} > 0.
\]
Thus, by induction, for each fixed \( k \in \{1, \ldots, n-1\} \) we have
\[
\left( \frac{L_m\gamma_{k+1}}{L_m\gamma_k} \right)' > 0, \quad \forall m = 1, \ldots, k.
\]
\[\square\]

We now turn to defining our dilation matrices \( \{A_k\} \). The choice of these is motivated by the fact that we are looking for a theory which admits piecewise-linear curves; we want, therefore, the \( A_k \) to have entries involving at most 1 derivative of \( \gamma_k \), \( k = 2, \ldots, n \).

We define the diagonal matrix \( A \) by
\[
A(t) = \begin{pmatrix}
t & 0 & \cdots & 0 \\
0 & \gamma_2(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_n(t)
\end{pmatrix}
\]
and put \( A_j = A(\lambda^j) \), \( j \in \mathbb{Z} \).

That these matrices satisfy (12a) and (12b) is trivial, using Corollary 3.3.

4. A conical Littlewood-Paley decomposition. We wish to consider the \( \zeta \in \mathbb{R}^n \) where we cannot expect (12c) to hold. By Lemma 3.1c) we know that \( \zeta, \Gamma' \) has at most \( (n - 2) \) changes of monotonicity in \((0, \infty)\), thence must have a bounded number of changes of monotonicity in each interval \([\lambda^k, \lambda^{k+1})\).
So, by Van der Corput's lemma, if

\[ |\zeta \Gamma'(t)| \geq \frac{C}{\lambda^k} |A_k \zeta| \quad \forall t \in [\lambda^k, \lambda^{k+1}), \]

then

\[ |\mu_k(\zeta)| \leq C |A_k \zeta|^{-1}. \]

We consider, therefore, the set of \( \zeta \) where (17) may fail, i.e.

\[ \bigcup_{t \in [\lambda^k, \lambda^{k+1})} C^t_k, \]

where

\[ C^t_k := \left\{ \zeta \in \mathbb{R}^n : |\zeta \Gamma'(t)| < \frac{\varepsilon}{\lambda^k} |A_k \zeta| \right\}. \]

Here \( \varepsilon > 0 \) may be as small as we like.

**Proposition 4.1.** (a) Let \( \Gamma \) be a "convex" \( C^n(0, \infty) \) curve in \( \mathbb{R}^n \). Then \( \exists \) cones \( C_k \) such that

\[ \bigcup_{t \in [\lambda^k, \lambda^{k+1})} C^t_k \subseteq C_k := \bigcup_{m=1}^{n-1} (C_{km} \cup \tilde{C}_{km}) \]

where

\[ C_{km} = \left\{ \zeta : \sum_{j=m}^{n} \zeta_j L_m \gamma_j(\lambda^k) < \varepsilon \sum_{j=m}^{n} |\zeta_j| L_m \gamma_j(\lambda^k) \right\} \]

and \( \zeta \in C_{km} \iff -\zeta \in \tilde{C}_{km} \).

(b) Let \( \Gamma \) be piecewise-linear on \([\lambda^k, \lambda^{k+1}], k \in \mathbb{Z} \) and \( \gamma_j \) convex, \( j = 2, \ldots, n \). Then

\[ \bigcup_{t \in [\lambda^k, \lambda^{k+1})} C^t_k \subseteq C_{k1} \cup \tilde{C}_{k1}, \]

where \( C_{k1}, \tilde{C}_{k1} \) are as defined in (a).

**Proof.** Let \( \zeta \in \bigcup_{t \in [\lambda^k, \lambda^{k+1})} C^t_k \). We suppose first that \( \zeta \Gamma' \) is monotone-increasing on \([\lambda^k, \lambda^{k+1}) \). Then \( \forall t \in [\lambda^k, \lambda^{k+1}) \),

\[ L_1(\zeta \Gamma')(\lambda^k) = \zeta \Gamma'(\lambda^k) \leq \zeta \Gamma'(t) \leq \zeta \Gamma'(\lambda^{k+1}) = L_1(\zeta \Gamma')(\lambda^{k+1}). \]
Hence

\[(18) \quad L_1(\zeta \cdot \Gamma)(\lambda^k) < \frac{\varepsilon}{\lambda^k} |A_k^* \zeta| \]

and

\[(19) \quad L_1(\zeta \cdot \Gamma)(\lambda^{k+1}) > -\frac{\varepsilon}{\lambda^k} |A_k^* \zeta|. \]

By Corollary 3.3 and the definition of the \( A_k \) we have

\[\frac{1}{\lambda^k} |A_k^* \zeta| \leq \sum_{j=1}^{n} \gamma_j'(\lambda^k) |\zeta_j| \leq \sum_{j=1}^{n} \gamma_j'(\lambda^{k+1}) |\zeta_j|, \]

which, together with (18) and (19), gives

\[\sum_{j=1}^{n} \zeta_j L_1 \gamma_j(\lambda^k) = L_1(\zeta \cdot \Gamma)(\lambda^k) \]

\[< \varepsilon \sum_{j=1}^{n} |\zeta_j| \gamma_j'(\lambda^k) = \varepsilon \sum_{j=1}^{n} |\zeta_j| L_1 \gamma_j(\lambda^k) \]

and

\[\sum_{j=1}^{n} \zeta_j L_1 \gamma_j(\lambda^{k+1}) = L_1(\zeta \cdot \Gamma)(\lambda^{k+1}) \]

\[> -\varepsilon \sum_{j=1}^{n} |\zeta_j| \gamma_j'(\lambda^{k+1}) = -\varepsilon \sum_{j=1}^{n} |\zeta_j| L_1 \gamma_j(\lambda^{k+1}). \]

Thus, if \( \zeta \cdot \Gamma' \) is monotone-increasing on \([\lambda^k, \lambda^{k+1}]\), then \( \zeta \in C_{k1} \). Similarly, if \( \zeta \cdot \Gamma' \) is monotone-decreasing on \([\lambda^k, \lambda^{k+1}]\), then \( \zeta \in \bar{C}_{k1} \).

We note here that if \( \Gamma \) is piecewise-linear on \([\lambda^k, \lambda^{k+1}]\), then \( \zeta \cdot \Gamma' \) is constant on \((\lambda^k, \lambda^{k+1})\); by a suitable definition of \( \zeta \cdot \Gamma'(\lambda^k) \) we may take \( \zeta \cdot \Gamma' \) to be constant on \([\lambda^k, \lambda^{k+1}]\) and thus (b) is proven.

We now suppose that \( \zeta \cdot \Gamma'(t) \) is not monotone on \([\lambda^k, \lambda^{k+1}]\). Then \( \exists \ t_0 \in [\lambda^k, \lambda^{k+1}] \) such that \( \zeta \cdot \Gamma''(t_0) = 0 \). Then

\[L_2(\zeta \cdot \Gamma)(t_0) = \frac{h_1}{h'_2} \zeta \cdot \Gamma''(t_0) = 0. \]

If then \( L_2(\zeta \cdot \Gamma) \) is monotone-increasing on \([\lambda^k, \lambda^{k+1}]\),

\[L_2(\zeta \cdot \Gamma)(\lambda^k) \leq 0 = L_2(\zeta \cdot \Gamma)(t_0) \leq L_2(\zeta \cdot \Gamma)(\lambda^{k+1}). \]
and so \( \zeta \in C_{k_2} \); similarly \( \zeta \in \tilde{C}_{k_2} \) if \( L_2(\zeta \cdot \Gamma) \) is monotone-decreasing on \([\lambda^k, \lambda^{k+1}]\). If \( L_2(\zeta \cdot \Gamma) \) is not monotone on \([\lambda^k, \lambda^{k+1}]\), then \( \exists \ t_1 \in [\lambda^k, \lambda^{k+1}] \) such that \( (L_2(\zeta \cdot \Gamma))(t_1) = 0 \), from which we obtain \( L_3(\zeta \cdot \Gamma)(t_1) = 0 \) and so if \( L_3(\zeta \cdot \Gamma) \) is monotone on \([\lambda^k, \lambda^{k+1}]\), we obtain \( \zeta \in C_{k_3} \cup \tilde{C}_{k_3} \). We repeat this process iteratively. By Lemma 3.1 \( L_n(\zeta \cdot \Gamma)(t) = \zeta_n \) so it follows that \( L_{n-1}(\zeta \cdot \Gamma) \) must be monotone on \([\lambda^k, \lambda^{k+1}]\) and hence the final possibility is that \( \zeta \in C_{k(n-1)} \cup \tilde{C}_{k(n-1)} \).

We now wish to find conditions on \( \Gamma \) under which these cones give a Littlewood-Paley decomposition for \( L^p(\mathbb{R}^n) \). The next result, in the same spirit as the lacunary Littlewood-Paley decomposition of [5], leads to the choice of these conditions. First we give our definition of lacunarity.

**Definition 4.2.** Let \( \{E_k(n, \varepsilon)\} \) be a family of cones in \( \mathbb{R}^n \) given by

\[
E_k(n, \varepsilon) = \left\{ \zeta \in \mathbb{R}^n : \sum_{j=1}^{n} \alpha_j^k \zeta_j < \varepsilon \sum_{j=1}^{n} \alpha_j^k |\zeta_j| \quad \text{and} \quad \sum_{j=1}^{n} \alpha_{k+1}^j \zeta_j > -\varepsilon \sum_{j=1}^{n} \alpha_{k+1}^j |\zeta_j| \right\},
\]

where \( \alpha_j^k \) are positive reals, \( j = 1, \ldots, n \), \( k \in \mathbb{Z} \) and \( \varepsilon > 0 \) is small. If

\[
\frac{\alpha_{k+1}^j}{\alpha_k^j} \geq 2 \frac{\alpha_{k+1}^{j-1}}{\alpha_k^{j-1}}, \quad \forall \ k \in \mathbb{Z}, \ j = 2, \ldots, n,
\]

then the \( E_k(n, \varepsilon) \) are said to be lacunary.

We define “smoothed-out” characteristic functions \( \Psi_{k,n}^{\varepsilon} \) of the cones \( E_k(n, \varepsilon) \) as follows.

Let \( \Psi_{n,\varepsilon} \) be a \( C^\infty \) function away from 0, homogeneous of degree zero such that

\[
\Psi_{n,\varepsilon}(\zeta) = \begin{cases} 
1 & \sum_{j=1}^{n} \zeta_j < \varepsilon \sum_{j=1}^{n} |\zeta_j| \\
0 & \sum_{j=1}^{n} \zeta_j > -2\varepsilon \sum_{j=1}^{n} |\zeta_j|,
\end{cases}
\]

and put

\[
\Psi_{k,n}^{\varepsilon}(\zeta) = \Psi_{n,\varepsilon}(\alpha_k^1 \zeta_1, \ldots, \alpha_k^n \zeta_n) \Psi_{n,\varepsilon}(-\alpha_{k+1}^1 \zeta_1, \ldots, -\alpha_{k+1}^n \zeta_n).
\]
Associated to these $\Psi_k^{n,\varepsilon}$ are operators $T_k$ given by

$$(T_k f)(\zeta) = \Psi_k^{n,\varepsilon}(\zeta) \hat{f}(\zeta), \quad k \in \mathbb{Z}.$$  

**Theorem 4.3.** If \( \{\mathcal{E}(n, \varepsilon)\} \) is a lacunary family of cones in \( \mathbb{R}^n \) then

$$\left\| \left( \sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p, \quad 1 < p < \infty.$$  

**Proof.** It suffices to show that $\sum_k \pm \Psi_k^{n,\varepsilon}$ is a multiplier for $L^p(\mathbb{R}^n)$, $1 < p < \infty$, independently of the choice of $\pm$; the result then follows by a standard Rademacher function argument. We use the formulation of the Marcinkiewicz multiplier theorem given in [2]. So, we let $\phi^n$ be a $C_0^\infty(\mathbb{R}^n)$ function such that $0 \leq \phi^n \leq 1$ and

$$\phi^n(\zeta) = \begin{cases} 1 & 1 \leq |\zeta_j| \leq 2, \ j = 1, \ldots, n \\ 0 & \text{off} \ \frac{1}{2} \leq |\zeta_j| \leq 4, \ j = 1, \ldots, n, \end{cases}$$

and define $L^2_\alpha(\mathbb{R}^n)$ to be the $n$-parameter Sobolev space given by

$$L^2_\alpha(\mathbb{R}^n) = \left\{ g : \|g\|^2_{L^2_\alpha} = \int |\hat{g}(\zeta)|^2 \prod_{i=1}^n (1 + \zeta_i^2)\alpha d\zeta < \infty \right\}.$$  

Then, by Theorem A of [2], it suffices to show that

$$(21) \quad \sup_{i_1, \ldots, i_n} \left\| \sum_k \pm \Psi_k^{n,\varepsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \phi^n(\zeta) \right\|_{L^2_\alpha(\mathbb{R}^n)} < \infty,$$

for some $\alpha > \frac{1}{2}$.

We show (21) for $\alpha = 1$ and for convenience take $\varepsilon = \frac{1}{2^{2n}}$. Our proof is by induction on $n$; the argument for $n = 2$ is contained in the inductive step and therefore we omit it.

Suppose, therefore, that

$$\sup_{i_1, \ldots, i_{n-1}} \left\| \sum_k \pm \Psi_k^{n-1,\varepsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_{n-1}} \zeta_{n-1}) \phi^{n-1}(\zeta) \right\|_{L^2_\alpha(\mathbb{R}^{n-1})} < \infty,$$

with $\varepsilon = \frac{1}{2^{2n-2}}$, under the hypothesis that

$$(22) \quad \frac{\alpha_k^{j+1}}{\alpha_k^j} > 2 \frac{\alpha_k^{j-1}}{\alpha_k^{j-1}}, \quad \forall \ k \in \mathbb{Z}, \ j = 2, \ldots, n - 1$$
and consider
\[ \sup_{i_1, \ldots, i_n} \left\| \sum_k \pm \Psi_k^{n, \epsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \phi^n(\zeta) \right\|_{L^2(\mathbb{R}^n)}, \]
assuming that (22) now holds also for \( j = n \).

We suppose that, for some \( k, \Psi_k^{n, \epsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \phi^n(\zeta) \neq 0 \), i.e.
\[
\sum_{j=1}^n \alpha_k^j 2^{i_j} \zeta_j < 2 \varepsilon \sum_{j=1}^n \alpha_k^j 2^{i_j} |\zeta_j| \\
\sum_{j=1}^n \alpha_{k+1}^j 2^{i_j} \zeta_j > -2 \varepsilon \sum_{j=1}^n \alpha_{k+1}^j 2^{i_j} |\zeta_j|
\]
and
\[ \frac{1}{2} \leq |\zeta_j| \leq 4, \quad j = 1, \ldots, n. \]

Case 1. Suppose that for some \( j_0 \in \{1, \ldots, n\} \)
\[ \alpha_k^{j_0} 2^{i_{j_0}} |\zeta_{j_0}| \leq \frac{1}{2^{2n}} \sum_{j=1}^n \alpha_k^j 2^{i_j} |\zeta_j| \]
and
\[ \alpha_{k+1}^{j_0} 2^{i_{j_0}} |\zeta_{j_0}| \leq \frac{1}{2^{2n}} \sum_{j=1}^n \alpha_{k+1}^j 2^{i_j} |\zeta_j|. \]
In this instance we find that
\[ \Psi_k^{n, \epsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \neq 0 \]
implies
\[ \Psi_{k+1}^{n-1, \epsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_{j_0}-1} \zeta_{j_0-1}, 2^{i_{j_0}+1} \zeta_{j_0+1}, \ldots, 2^{i_n} \zeta_n) \neq 0. \]
Taking \( 2^{i_{j_0}} = 1 \), which we may by homogeneity of \( \Psi_k^{n, \epsilon} \), the problem is reduced to the \((n - 1)\)-dimensional case and we are done, by the inductive hypothesis.

Case 2. Suppose that for all \( j \in \{1, \ldots, n\} \) either
\[ \alpha_k^j 2^{i_j} |\zeta_j| \geq \frac{1}{2^{2n}} \sum_{j=1}^n \alpha_k^j 2^{i_j} |\zeta_j| \tag{23} \]
or
\[ \alpha_{k+1}^j 2^{i_j} |\zeta_j| \geq \frac{1}{2^{2n}} \sum_{j=1}^n \alpha_{k+1}^j 2^{i_j} |\zeta_j|. \tag{24} \]
Let us suppose that
\[ \alpha_k^{n} 2^{i_n} |\zeta_n| \geq \frac{1}{2^n} \sum_{j=1}^{n} \alpha_k^j 2^{i_j} |\zeta_j|. \]

Then by the lacunarity conditions (20) we have
\[ \alpha_{k+m}^{n} 2^{i_n} |\zeta_n| \geq 2 \sum_{j=1}^{n-1} \alpha_{k+m}^j 2^{i_j} |\zeta_j| \quad \forall \ m \geq N, \text{ say.} \]

Then if \( \zeta_n > 0 \) we find
\[ \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j} \zeta_j \geq \alpha_{k+m}^{n} 2^{i_n} |\zeta_n| - \sum_{j=1}^{n-1} \alpha_{k+m}^j 2^{i_j} |\zeta_j| \]
\[ \geq \frac{1}{3} \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j} |\zeta_j|, \]

whilst if \( \zeta_n < 0 \) we have
\[ \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j} \zeta_j \leq -\frac{1}{3} \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j} |\zeta_j|. \]

Thus
\[ (25) \quad \Psi_{k+m}^{n,\varepsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) = 0 \quad \forall \ m \geq N. \]

If we assume that (24) holds for \( j = n \) then the same argument follows. Further, \( \forall \ \zeta \text{ with } \Psi_k^{n,\varepsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \neq 0 \), for each \( j_0 \in \{1, \ldots, n\} \), we have either
\[ \alpha_{k_0}^{i_{j_0}} 2^{i_{j_0}} |\zeta_{j_0}| \leq \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \sum_{j \neq j_0} \alpha_k^j 2^{i_j} |\zeta_j| \]

or
\[ \alpha_{k+1}^{i_{j_0}} 2^{i_{j_0}} |\zeta_{j_0}| \leq \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \sum_{j \neq j_0} \alpha_{k+1}^j 2^{i_j} |\zeta_j|. \]

This, together with (23), (24), lacunarity and \( \phi^n(\zeta) \neq 0 \) gives that \( 2^{i_j} \sim 1 \forall j = 1, \ldots, n \). So using (25) we obtain
\[ \sup_{t_1, \ldots, t_n} \left| \sum_k \pm \Psi_k^{n,\varepsilon}(2^{i_1} \zeta_1, \ldots, 2^{i_n} \zeta_n) \phi^n(\zeta) \right| < \infty. \]
It is trivial to check that differentiating with respect to any $\zeta_j$ causes no problem. This concludes the proof.

Let us now see how may apply Theorem 4.3 to our cones $C_k$. It is clear that if we have a Littlewood-Paley theory for each $\{C_{km}\}$, $\{\tilde{C}_{km}\}$, $m = 1, \ldots, n - 1$, where we consider $C_{km}, \tilde{C}_{km}$ as cones in $\mathbb{R}^{n-m+1}$, then this will suffice to give a Littlewood-Paley theory for the $C_k$. We define now

$$\Phi_{km}(\zeta) = \Psi_k^{n,e}(0, \ldots, 0, \zeta_m, L_m\gamma_{m+1}(\lambda^k)\zeta_{m+1}, \ldots, L_m\gamma_n(\lambda^k)\zeta_n) \times \Psi_k^{n,e}(0, \ldots, 0, -\zeta_m, -L_m\gamma_{m+1}(\lambda^{k+1})\zeta_{m+1}, \ldots, -L_m\gamma_n(\lambda^{k+1})\zeta_n)$$

and put

$$\Phi_k(\zeta) = \sum_{m=1}^{n-1} \Phi_{km}(\zeta);$$

we associate to $\Phi_k$ the operator $S_k$ given by

$$\overline{(S_k f)}(\zeta) = \Phi_k(\zeta) \hat{f}(\zeta).$$

**Proposition 4.4.** If

$$\frac{L_m\gamma_{j+1}(\lambda^{k+1})}{L_m\gamma_j(\lambda^{k+1})} \geq 2 \frac{L_m\gamma_{j+1}(\lambda^k)}{L_m\gamma_j(\lambda^k)},$$

$\forall k \in \mathbb{Z}, j = m, \ldots, n - 1; m = 1, \ldots, n - 1$, then

$$\left\| \left( \sum_k |S_k f|^2 \right)^{1/2} \right\|_p \leq C\|f\|_p, \quad 1 < p < \infty.$$

**Proof.** If, for fixed $m$,

$$\frac{L_m\gamma_{j+1}(\lambda^{k+1})}{L_m\gamma_j(\lambda^{k+1})} \geq 2 \frac{L_m\gamma_{j+1}(\lambda^k)}{L_m\gamma_j(\lambda^k)} \quad \forall k \in \mathbb{Z}, j = m, \ldots, n$$

then the family of cones $\{C_{km}\}$, and hence also $\{\tilde{C}_{km}\}$, may be considered as lacunary in $\mathbb{R}^{n-m+1}$, i.e. $\sum_k \pm \Phi_{km}(\zeta)$ is a multiplier in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Thus if (27) is satisfied we have that $\sum_k \sum_{m=1}^{n-1} \pm \Phi_{km}(\zeta)$ is a multiplier for $L^p(\mathbb{R}^n)$. This gives the result.
Thus, assuming (27), the cones $C_k$ give a Littlewood-Paley decomposition of $\mathbb{R}^n$. Let us now see how the $\frac{1}{2}n(n - 1)$ conditions of (27) relate to the conditions in the statement of our theorem, i.e. (9).

**Lemma 4.5.** Let $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ be a "convex" curve such that $\Gamma \in C^1(0, \infty)$, $\Gamma(0) = 0$ and

\begin{equation}
\lim_{t \to 0} L_m \gamma_{k+1}(t) = 0 \text{ for } m = 1, \ldots, n - 1, \quad k = m, \ldots, n - 1.
\end{equation}

Suppose $\exists 1 < \lambda < \infty$ such that

\begin{equation}
L_k \gamma_{k+1}(\lambda t) \geq 2L_k \gamma_{k+1}(t) \quad k = 1, \ldots, n - 1.
\end{equation}

Then $\exists 1 < \mu < \infty$ such that

\begin{equation}
\frac{L_m \gamma_{k+1}(\mu t)}{L_m \gamma_k(\mu t)} \geq 2 \frac{L_m \gamma_{k+1}(t)}{L_m \gamma_k(t)},
\end{equation}

$m = 1, \ldots, n - 1, \quad k = m, \ldots, n - 1$.

**Proof.** Fix $k$. Clearly, by hypothesis (29), (30) holds for $m = k$, with $\mu = \lambda$. We now suppose that (30) holds for $m = j$ and show that it is then also true for $m = j - 1$. Now

\begin{equation}
L_{j-1} \gamma_k(\lambda t) = \frac{L_{j-1} \gamma_k(\lambda t)}{L_{j-1} \gamma_{k-1}(\lambda t)} \cdot \frac{L_{j-1} \gamma_{k-1}(\lambda t)}{L_{j-1} \gamma_{k-2}(\lambda t)} \cdots \frac{L_{j-1} \gamma_j(\lambda t)}{L_{j-1} \gamma_k(\lambda t)}
\geq 2L_{j-1} \gamma_k(t),
\end{equation}

by Lemma 3.4 and (29). Then

\begin{equation}
\frac{L_{j-1} \gamma_{k+1}(\mu^3 t)}{L_{j-1} \gamma_k(\mu^3 t)} \geq \frac{1}{2} \cdot \frac{L_{j-1} \gamma_{k+1}(\mu^3 t) - L_{j-1} \gamma_{k+1}(\mu^2 t)}{L_{j-1} \gamma_k(\mu^3 t) - L_{j-1} \gamma_k(\mu^2 t)}
\geq \frac{1}{2} \cdot \frac{(L_{j-1} \gamma_{k+1})'(\mu^2 t)}{(L_{j-1} \gamma_k)'(\mu^2 t)},
\end{equation}

by Lemma 3.4 and (29). Then
by the Second Mean Value Theorem and Lemma 3.4. Thus

\[
\frac{L_{j-1}\gamma_{k+1}(\mu^3 t)}{L_{j-1}\gamma_k(\mu^3 t)} \geq \frac{1}{2} \cdot \frac{L_j\gamma_{k+1}(\mu^2 t)}{L_j\gamma_k(\mu^2 t)}
\]

\[
\geq 2 \frac{L_j\gamma_{k+1}(t)}{L_j\gamma_k(t)}, \text{ by inductive hypothesis}
\]

\[
= 2 \frac{(L_{j-1}\gamma_{k+1})'(t)}{(L_{j-1}\gamma_k)'(t)}
\]

\[
\geq 2 \frac{L_{j-1}\gamma_{k+1}(t)}{L_{j-1}\gamma_k(t)},
\]

by Second Mean Value Theorem, Lemma 3.4 and hypothesis (28).

Lemma 4.5 and Proposition 4.4 together give us

**Proposition 4.6.** Let \( \Gamma : \mathbb{R} \rightarrow \mathbb{R}^n \) be a "convex" curve such that \( \Gamma \in C^n(0, \infty), \Gamma(0) = 0 \) and \( \lim_{t \to 0} L_j\gamma_k(t) = 0, \forall j = 1, \ldots, n - 1; k = j + 1, \ldots, n \). Suppose that \( \exists 1 < \lambda < \infty \) such that

\[ L_k\gamma_{k+1}(\lambda t) \geq 2L_k\gamma_{k+1}(t), \quad k = 1, \ldots, n - 1. \]

Then

\[ \left\| \sum_k |S_kf|^2 \right\|^{1/2}_p \leq C \|f\|_p. \]

In view of Proposition 4.1 (b), which defines the cones for a piecewise-linear curve, we also have a corresponding result for piecewise-linear curves if we replace the hypotheses of Proposition 4.6 with those of Corollary 1.3.

**5. Proof of Theorem 1.2.** We now have a family of dilation matrices \( \{A_k\} \) satisfying

(31) \( \exists \alpha \) such that \( \|A_{k+1}^{-1}A_k\| \leq \alpha < 1 \)

(32) \( A_{k+1}^{-1} \supp \mu_k \subseteq \) fixed ball
and a family of cones \( \{C_k\} \) with associated operators \( S_k \) given by (26) satisfying the Littlewood-Paley inequality

\[
\left\| \left( \sum_k |S_k f|^2 \right)^{1/2} \right\|_p \leq C\|f\|_p
\]  

and such that

\[
\zeta \notin C_k \implies |\mu_k(\zeta)| \leq C|A_k^*\zeta|^{-1}.
\]

We let \( f = S_k f + (I - S_k)f, \ k \in \mathbb{Z} \), and consider first \( \sup_k |\mu_k * f| \). We use the standard technique of combining a bootstrapping argument with the Littlewood-Paley theory to obtain an \( L^p \)-result, starting with just the \( L^2 \)-result. Now,

\[
\sup_k |\mu_k * f| \leq \sup_k |\mu_k * S_k f| + \sup_k |\mu_k * (I - S_k)f|
\]

\[
= A + B.
\]

By (33), Plancherel's theorem and the fact that the \( \mu_k \) have unit mass we immediately have

\[
A \leq C\|f\|_2.
\]

For \( B \) we use comparison of \( \mu_k \) with a measure \( \nu_k \) given by

\[
\nu_k(x) = \rho \left( A_{k+1}^{-1} x \right) \det A_{k+1}^{-1},
\]

where \( \rho \in C_0^\infty, \ 0 \leq \rho \leq 1 \) and \( \int \rho = 1 \). It is easily verified that \( \sup_k |\nu_k * f| \) is majorized by the Hardy-Littlewood maximal operator associated to balls \( A_k B \), where \( B \) is the unit ball in \( \mathbb{R}^n \), and thus, by [3], Proposition 2.2,

\[
\left\| \sup_k |\nu_k * f| \right\|_p \leq C\|f\|_p, \quad 1 < p < \infty.
\]

Then

\[
B \leq \left\| \sup_k |(\mu_k - \nu_k) * (I - S_k)f| \right\|_2 + \left\| \sup_k |\nu_k * f| \right\|_2
\]

\[
+ \left\| \sup_k |\nu_k * S_k f| \right\|_2.
\]
Now, by the same argument as used for $A$, 
\[ \left\| \sup_k |\nu_k * S_k f| \right\|_2 \leq C\|f\|_2, \]
so it remains to show that 
\[ \left\| \sup_k (\mu_k - \nu_k) * (I - S_k) f \right\|_2 \leq C\|f\|_2. \]

Taking into account (34) the proof of this is essentially contained in the proof of Proposition 5.1, [3]. To pass from the $L^2$-result to the $L^p$-result we have the following analogue of Proposition 5.1, [3].

**PROPOSITION 5.1.** Suppose 
\[ \left\| \sup_k |\mu_k * f| \right\|_{\tilde{p}} \leq C\|f\|_{\tilde{p}}, \text{ for some } 1 < \tilde{p} \leq 2. \]

Then 
\[ \left\| \sup_k |\mu_k * f| \right\|_p \leq C\|f\|_p \quad \forall \ p > \frac{2\tilde{p}}{\tilde{p} + 1}. \]

**Proof.** First we note that, under the hypothesis of the proposition, 
\[ \left( \sum_k |\mu_k * f|^2 \right)^{1/2} \leq C \left( \sum_k |f_k|^2 \right)^{1/2}, \]
\[ \forall \frac{2\tilde{p}}{\tilde{p} + 1} < p < \frac{2\tilde{p}}{\tilde{p} - 1}, \] exactly as in [3]. Then
\[ \left\| \sup_k |\mu_k * f| \right\|_p \leq \left\| \left( \sum_k |\mu_k * S_k f|^2 \right)^{1/2} \right\|_p + \left\| \left( \sum_k |\nu_k * S_k f|^2 \right)^{1/2} \right\|_p \]
\[ + \left\| \sup_k |\nu_k * f| \right\|_p + \left\| \sup_k (\mu_k - \nu_k) * (I - S_k) f \right\|_p \]
\[ = A + B + D + E. \]

Now (36) together with (33) gives suitable bounds for $A$ and $B$, $\forall \frac{2\tilde{p}}{\tilde{p} + 1} < p < \frac{2\tilde{p}}{\tilde{p} - 1}$, whilst $D \leq C\|f\|_p$, $\forall 1 < p < \infty$, by (35). It remains,
therefore, to bound $E$. Again the proof that $E \leq C\|f\|_p, \forall \frac{2p}{p+1} < p$
is essentially contained in Proposition 5.1, [3].

Proposition 5.1 completes the proof of $L^p$-boundedness of $\sup_k |\mu_k^* f|$ and thence of $\mathcal{M}_\Gamma$. Noting that from (33) we may also obtain

$$\left\| \sum_k S_k f_k \right\|_p \leq C \left\| \left( \sum_k |S_k f_k|^2 \right)^{1/2} \right\|_p, \quad 1 < p < \infty,$$

we may now deduce the result for $\mathcal{H}_\Gamma$ from that for $\mathcal{M}_\Gamma$, following the argument in [3].

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