Pacific Journal of Mathematics

CHARACTERS OF BRAUER'S CENTRALIZER ALGEBRAS

ARUN RAM

Volume 169 No. 1 May 1995

PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 1, 1995

CHARACTERS OF BRAUER'S CENTRALIZER ALGEBRAS

ARUN RAM

Brauer's centralizer algebras are finite dimensional algebras with a distinguished basis. Each Brauer centralizer algebra contains the group algebra of a symmetric group as a subalgebra and the distinguished basis of the Brauer algebra contains the permutations as a subset. In view of this containment it is desirable to generalize as many known facts concerning the group algebra of the symmetric group to the Brauer algebras as possible. This paper studies the irreducible characters of the Brauer algebras in view of the distinguished basis. In particular we define an analogue of conjugacy classes, and derive Frobenius formulas for the characters of the Brauer algebras. Using the Frobenius formulas we derive formulas for the irreducible character of the Brauer algebras in terms of the irreducible characters of the symmetric groups and give a combinatorial rule for computing these irreducible characters.

Introduction. Classically, Frobenius [**Fr**] determined the characters of the symmetric group by exploiting the connection between the power symmetric functions and the Schur functions. Schur [**Sc1**, **Sc2**] later showed that this connection arises from the fact that the general linear group and the symmetric group each generate the full centralizer of each other on tensor space, now referred to as the Schur-Weyl duality. In his landmark book [**Wy**], Weyl used this duality as the principal algebraic tool for studying the representations of the classical groups. In 1937 R. Brauer [**Br**] gave a nice basis for the centralizer algebra of the action of the orthogonal and symplectic groups on tensor space.

In [**R1**] we gave a formula for the characters of the Hecke algebra of type A in the same spirit as the original formula of Frobenius for the characters of the symmetric group. This formula was then used to derive a combinatorial rule for computing the characters of the

Hecke algebras of type A which is a q extension of the Murnaghan-Nakayama rule for computing the irreducible characters of the symmetric group. The algebraic structure motivating this approach to the characters of the Hecke algebra is a Schur-Weyl type duality between the Hecke algebra and the quantum group $U_q(\mathfrak{sl}(n))$[**Ji**].

In this paper we extend the classical method of determining the characters of the symmetric group to the Brauer algebras. In particular we derive Frobenius type formulas and a combinatorial rule for computing the irreducible characters of Brauer's centralizer algebras. This paper is organized as follows. Section 1 summarizes the necessary facts concerning semisimple algebras and the representation theory of the classical groups. Section 2 gives the definition of the Brauer algebras and a brief description of their structure. Section 3 defines an analogue of conjugacy classes for the Brauer algebra. Note: The Brauer algebras are *not* group algebras, so, conjugacy classes are not, a priori, natural. Section 4 gives a description of the Schur-Weyl duality for the case of the orthogonal and symplectic groups and evaluates the trace functions that are the key to the determination of the irreducible characters of the Brauer algebras. Section 5 gives formulas for the irreducible characters of the Brauer algebras in terms of the irreducible characters of the symmetric groups. Finally, section 6 gives a combinatorial rule for computing the irreducible characters of the Brauer algebras.

This paper is taken from a portion of the author's dissertation [**R2**] at the University of California, San Diego. Many of the implications of a Schur-Weyl type duality which are discussed there are not treated here. In particular, analogues of orthogonality relations for irreducible characters of semisimple algebras which are not group algebras (for example Brauer's centralizer algebras) and the Frobenius characteristic map (see [**Mac**]) in a general setting. Several of the interrelations between the orthogonal and symplectic characters (see [**Pr**] and [**K-T**]) can be derived immediately via the Frobenius formulas for Brauer's centralizer algebras and furthermore the module theoretic derivation of this formula gives a natural setting for the representation theoretic interpretation of these derivations.

I would like to thank my advisors H. Wenzl and A. Garsia at University of California, San Diego for all of the encouragement, support and helpful discussions throughout this work. I would also

like to thank J. Remmel for helpful discussions.

1. Notations. $M_n(F)$ denotes the full matrix algebra of $n \times n$ matrices with entries in the field F. We shall say that an algebra A over F is semisimple if it is isomorphic to a direct sum of full matrix algebras over F, i.e.

$$A \cong \bigoplus_{\lambda} M_{n_\lambda}(F),$$

where λ are in some finite index set and n_λ are positive integers. Note that this is what is usually called a split semisimple algebra. An element $p \in A$, $p \neq 0$, is idempotent if $p^2 = p$. An idempotent p is minimal if p cannot be written as a sum $p = p_1 + p_2$ of idempotents such that $p_1 p_2 = p_2 p_1 = 0$. A partition of unity in algebra A is a set of minimal idempotents $p_i \in A$ such that $p_i p_j = p_j p_i = 0$, for $i \neq j$ and $\sum_i p_i = 1$. A character of A is an F-linear functional $\chi : A \to F$ such that

$$\chi(ab) = \chi(ba),$$

for all $a, b \in A$. If A is semisimple then every character of A is a linear combination of irreducible characters.

S_f shall denote the symmetric group on f symbols and FS_f its group algebra over F. $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{C}(x)$ the field of rational functions in a single variable x.

A partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is a finite sequence of decreasing integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$. The weight $|\lambda|$ of λ is the sum of its parts. We say that λ is a partition of f if $|\lambda| = f$ and write $\lambda \vdash f$. We shall often use the notation $\lambda = (0^{m_0} 1^{m_1} 2^{m_2} ...)$ where m_i is the number parts of λ equal to i. The conjugate of the partition $\lambda = (\lambda_1, ..., \lambda_n)$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_n)$ given by $\lambda'_i = \mathrm{Card}(\{j | \lambda_j \geq i\})$. A partition with all parts equal to 0 is called the empty partition and denoted by $\emptyset$. For partitions λ and μ, $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all i. We say that a partition λ is even if all its parts λ_i are even.

Facts from the representation theory of the classical groups. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}$, and let I be the $n \times n$ identity matrix. Let J be the $2n \times 2n$ matrix given by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

We use the following standard notations for the general linear, orthogonal, and symplectic groups:

$$\begin{aligned}
\mathrm{Gl}(n) &= \{g \in M_n(\mathbb{C}) | \det g \neq 0\}, \\
\mathrm{O}(n) &= \{g \in M_n(\mathbb{C}) | gg^t = I\}, \\
\mathrm{Sp}(2n) &= \{g \in M_{2n}(\mathbb{C}) | gJg^t = J\}.
\end{aligned}$$

For each partition $\lambda = (\lambda_1, ..., \lambda_n)$ define the following polynomials in $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}]$,

$$\begin{aligned}
sa_\lambda(x_1, ..., x_n) &= \frac{\left| x_i^{\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|}, \\
sb_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}) &= \frac{\left| x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)} \right|}{\left| x_i^{n-j+1/2} - x_i^{-(n-j+1/2)} \right|}, \\
sc_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}) &= \frac{\left| x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)} \right|}{\left| x_i^{n-j} + x_i^{-(n-j)} \right|}.
\end{aligned}$$

The Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ are nonnegative integers defined by the equation

$$sa_\lambda(x_1, ..., x_n) sa_\mu(x_1, ..., x_n) = \sum_\lambda c_{\lambda\mu}^\nu sa_\nu(x_1, ..., x_n).$$

The Littlewood-Richardson coefficients are defined for each triple of partitions of lengths less than or equal to n. ($c_{\lambda\mu}^\nu = 0$ if sa_ν does not appear in the expansion of $sa_\lambda sa_\mu$.)

The following identities, due to Littlewood [**Li**], hold in $\mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ ([**K-T**] contains an easily accessible proof),

(1.1)

$$\begin{aligned}
sa_\nu(x_1, x_1^{-1}, ..., x_n, x_n^{-1}, 1) &= \sum_{\lambda \subseteq \nu} \left(\sum_{\beta \text{ even}} c_{\lambda\beta}^\nu \right) sb_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}), \\
sa_\nu(x_1, x_1^{-1}, ..., x_n, x_n^{-1}) &= \sum_{\lambda \subseteq \nu} \left(\sum_{\beta' \text{ even}} c_{\lambda\beta}^\nu \right) sc_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}), \\
sa_\nu(x_1, x_1^{-1}, ..., x_n, x_n^{-1}) &= \sum_{\lambda \subseteq \nu} \left(\sum_{\beta \text{ even}} c_{\lambda\beta}^\nu \right) sd_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}).
\end{aligned}$$

Recall that a partition β is even if all its parts are even.

THEOREM 1.2 ([**Wy, Li**]). (a) *The irreducible polynomial representation of* $\mathrm{Gl}(n)$ *are indexed by partitions* λ *such that* $l(\lambda) \leq n$. *Let* $g \in \mathrm{Gl}(n)$ *and let* $x_1, \ldots, x_n$ *denote the eigenvalues of* g. *The character* s_λ *of the irreducible representation of* $\mathrm{Gl}(n)$ *corresponding to* λ *evaluated at* g *is given by*

$$s_\lambda(g) = sa_\lambda(x_1, \ldots, x_n).$$

(b) *The irreducible polynomial representations of* $\mathrm{O}(2n+1)$ *are indexed by partitions* λ *such that* $\lambda_1' + \lambda_2' \leq 2n+1$. *If* λ *is such that* $\lambda_1' + \lambda_2' \leq 2n+1$ *and* $l(\lambda) > n$, *let* $\tilde{\lambda}$ *be the partition given by*

$$\tilde{\lambda}_i' = \begin{cases} \lambda_i', & \text{for } i > 1 \\ 2n+1-\lambda_1', & \text{for } i=1. \end{cases}$$

Let $g \in \mathrm{O}(2n+1)$ *and suppose that* $\det(g) = 1$, *whence the eigenvalues of* g *will be in the form* $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1$. *Then the character* so_λ *of the irreducible representation of* $\mathrm{O}(2n+1)$ *corresponding to* λ *evaluated at* g *is given by*

$$so_\lambda(g) = \begin{cases} sb_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) \leq n, \\ sb_{\tilde{\lambda}}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) > n. \end{cases}$$

(c) *The irreducible polynomial representations of* $\mathrm{Sp}(2n)$ *are indexed by partitions* λ *such that* $l(\lambda) \leq n$. *Let* $g \in \mathrm{Sp}(2n)$ *and let* $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ *be the eigenvalues of* g. *Then the character* sp_λ *of the irreducible representation of* $\mathrm{Sp}(2n)$ *corresponding to* λ *evaluated at* g *is given by*

$$sp_\lambda = sc_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}).$$

(d) *The irreducible representations of* $\mathrm{O}(2n)$ *are indexed by partitions* λ *such that* $\lambda_1' + \lambda_2' \leq 2n$. *If* λ *is such that* $\lambda_1' + \lambda_2' \leq 2n$ *and* $l(\lambda) > n$, *let* $\tilde{\lambda}$ *be the partition given by*

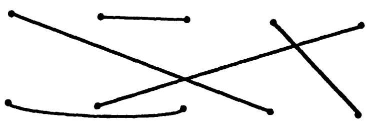
$$\tilde{\lambda}_i' = \begin{cases} \lambda_i', & \text{for } i > 1, \\ 2n - \lambda_1', & \text{for } i = 1. \end{cases}$$

Let $g \in \mathrm{O}(2n)$ and suppose that $\det(g) = 1$, whence the eigenvalues of g will be of the form $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. Then the character so_λ of the irreducible representation of $\mathrm{O}(2n)$ corresponding to λ is given by

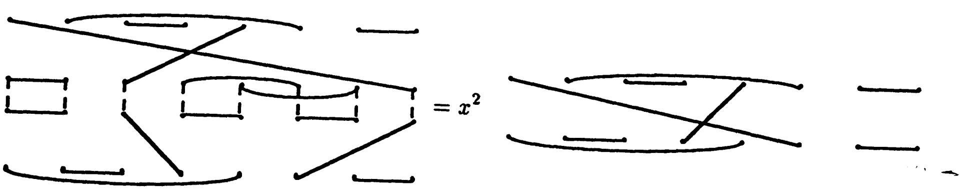
$$so_\lambda(g) = \begin{cases} sd_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) \leq n, \\ sd_{\tilde{\lambda}}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) > n. \end{cases}$$

2. The Brauer algebra. In this section we give the definition and the basic facts about the Brauer algebras necessary for our study of the characters of the Brauer algebra. Most of these facts appear in [**Wz1**].

A *diagram* on f dots is given by two rows of f dots each and f edges which connect the $2f$ dots in pairs. The following is a diagram on 5 dots.

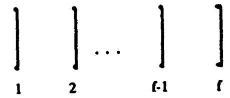


Let x be an indeterminate. Let d_1 and d_2 be two diagrams on f dots and let c denote the number of cycles created by placing d_1 directly above d_2 and attaching the lower dots of d_1 to the upper dots of d_2. The product $d_1 d_2$ is x^c times the diagram on f dots resulting from this attachment. For example

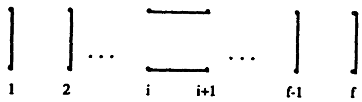
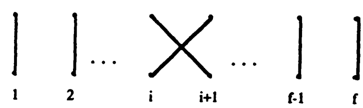


The Brauer algebra $D_f(x)$ is the $\mathbb{C}(x)$ span of the diagrams on f dots where the multiplication is given by the linear extension of the product of diagrams. The dimension of $D_f(x)$ is $(2f-1)(2f-3)\cdots 5 \cdot 3 \cdot 1$.

The diagram



on f dots is the identity element of $D_f(x)$ which we shall denote by 1. For each $0 < i < f$ let e_i and g_i denote the diagrams

 and 

respectively. Note that the g_i and the e_i, $1 \leq i \leq f-1$, generate $D_f(x)$.

Any edge connecting two dots that are either both in the upper row or both in the lower row is called a horizontal edge. The group algebra of the symmetric group $\mathbb{C}(x)S_f$ is a subalgebra of $D_f(x)$ in a natural way. Each permutation π of S_f, the symmetric group, is identified with the diagram on f dots which has edges connecting the ith dot of the lower row to the $\pi(i)$th dot of the upper row. In this way, any diagram on f dots which has all dots in the lower row connected to dots in the upper row (i.e. no horizontal edges) is viewed as a permutation in S_f and is invertible with inverse given by flipping the diagram from top to bottom. Note that the g_i, $1 \leq i \leq f-1$, generate the symmetric group.

There is a natural embedding of $D_m(x) \otimes D_n(x)$ into $D_{n+m}(x)$. If d is a diagram on m dots and d' is a diagram on n dots then $d \otimes d'$ corresponds to the diagram on $m+n$ dots given by placing d adjacent to d'. Let $d^{\otimes k}$ denote the diagram $d \otimes d \otimes \cdots \otimes d$ (k factors).

For each complex number α in $\mathbb{C}$ one defines a Brauer algebra $D_f(\alpha)$ over $\mathbb{C}$ as the linear span of the diagrams on f dots where the multiplication is given as above except with x replaced by α. $D_f(\alpha)$ is an algebra of dimension $1 \cdot 3 \cdot 5 \cdots (2f-1)$.

The following theorems concerning the structure of the Brauer algebras are given in [**Wz1**] and [**Wy**].

THEOREM 2.1 (H. Wenzl, [**Wz1**]). *$D_f(x)$ is a semisimple algebra over $\mathbb{C}(x)$.*

THEOREM 2.2. *The irreducible representations of $D_f(x)$ are in-*

dexed by partitions λ *of* $f-2k$, $k=0,1,\ldots,\left[\frac{f}{2}\right]$.

THEOREM 2.3 ([**Wz1**]). *$D_f(\alpha)$ is semisimple and has irreducible representations indexed by partitions λ of $f-2k$, $k=0,1,\ldots,\left[\frac{f}{2}\right]$, for all but a finite number of $\alpha \in \mathbb{C}$.*

Let p_i be a partition of unity in $D_f(x)$. Assume that each p_i is expressed in terms of the basis of diagrams on f dots. Each coefficient in this expansion is a rational function in x. In this way it makes sense to consider the specialization $p_i(\alpha)$, $\alpha \in \mathbb{C}$ given by setting $x=\alpha$. $p_i(\alpha)$ will be well defined and nonzero for all but a finite number of $\alpha \in \mathbb{C}$.

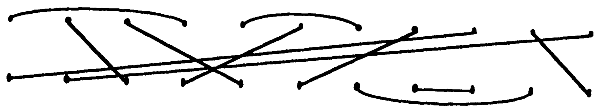
For all but a finite number of $\alpha \in \mathbb{C}$ we will have that $p_i(\alpha)$ form a partition of unity for $D_f(\alpha)$.

For each $\lambda \vdash f-2k$ we shall denote the irreducible character of $D_f(x)$ corresponding to λ by $\chi^\lambda_{(f,x)}$. Similarly for each α such that $D_f(\alpha)$ is semisimple and has irreducible representations indexed by $\lambda \vdash f-2k$ we denote the irreducible character of $D_f(\alpha)$ corresponding to λ by $\chi^\lambda_{(f,\alpha)}$. The following corollary follows from the above remarks concerning a partition of unity in $D_f(\alpha)$.

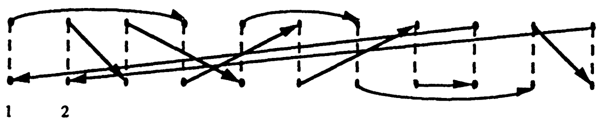
COROLLARY 2.4. *For all but a finite number of $\alpha \in \mathbb{C}$ the character $\chi^\lambda_{(f,\alpha)}$ of $D_f(\alpha)$ is given by evaluating the character $\chi^\lambda_{(f,x)}$ of $D_f(x)$ at $x=\alpha$.*

3. Characters of $D_f(x)$. To each diagram d on f dots we associate a partition $\tau(d)$, the type of d, determined in the following manner. Connect each dot in the upper row of the diagram d to the corresponding dot in the lower row by a dotted line. Beginning with an arbitrarily chosen dot of d follow the path determined by the edges and the dotted lines and assign each edge a direction as it is transversed. Returning to the original dot completes a cycle. If not all edges have been transversed and a cycle is completed choose a dot connected to an edge which has not yet been assigned a direction and continue to follow the edges and dotted lines. Do this until all edges have been assigned a direction. We call the resulting graph a directed form of d.

Assign to each cycle the absolute value of the difference between the number of edges in the cycle that are directed from top to bottom and the number of edges in the cycle that are directed from bottom to top (horizontal edges and dotted lines are ignored). Note that a cycle may be assigned the number 0. This sequence of numbers determines the partition $\tau(d)$. $\tau(d)$ is a partition of $f-2k$ for some integer $0 \leq k \leq \left[\frac{f}{2}\right]$. As an example, the diagram

$d =$ 

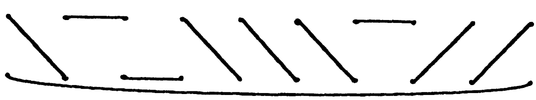
has directed form given by



The type of d, $\tau(d) = (21)$.

The type of a diagram d is analogous to the cycle type of a permutation. If d has no horizontal edges then the type of d is exactly the same as the cycle type of the permutation represented by the diagram d.

A cycle of length k is a diagram on k dots such that one dot in each column is connected to a dot in the next column (and a dot in the kth column is connected to a dot in the 1st column). For example



is a 10-cycle. Let e denote the diagram

$e =$

on 2-dots. Let γ_k denote the diagram on k dots given by the permutation

$$\gamma_k = \begin{pmatrix} 1 & 2 & 3 \cdots & k-1 & k \\ 2 & 3 & 4 \cdots & k & 1 \end{pmatrix}.$$

For each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ with all parts nonzero, let γ_μ denote the diagram $\gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_k}$.

If d and d' are diagrams on f dots, we say that d' is a conjugate of d if there exists some permutation π of S_f such that $\pi d \pi^{-1} = d'$. Note that $\pi d \pi^{-1}$ is the diagram given by rearranging both the upper dots of d and the lower dots of d according to the permutation π. One has the following easy facts.

(1) If d and d' are diagrams on f dots and d and d' are conjugate and χ is a character of $D_f(x)$ then $\chi(d) = \chi(d')$.

(2) Any two diagrams which are conjugate have the same type.

(3) If $d = d_1 \otimes d_2 \otimes \cdots \otimes d_k$ then d is conjugate to $d_{\pi(1)} \otimes d_{\pi(2)} \otimes \cdots \otimes d_{\pi(k)}$ for any permutation π of the k factors d_i.

(4) Every diagram d is conjugate to one which is a product $d' = c_1 \otimes c_2 \otimes \cdots \otimes c_k$ of cycles. To see this, let a permutation π be given so that $\pi(j) = i$ if, in the process of determining the type of the diagram d, the ith edge to be assigned a direction is pointing to a dot in the jth column. Then $d' = \pi d \pi^{-1}$ will be a product of cycles.

(5) Any cycle of length k with no horizontal edges is conjugate to γ_k.

THEOREM 3.1. *If d is a diagram on f dots and χ is a character of $D_f(x)$ then*

$$\chi(d) = \left(1/x^h\right) \chi\left(e^{\otimes k} \otimes \gamma_\mu\right)$$

where μ is the partition formed by nonzero parts of the type $\tau = (0^{m_0} 1^{m_1} 2^{m_2} \ldots)$ of the diagram d, and k and h are given by

$$k = (f - |\mu|)/2,$$

$$h = k - m_0.$$

Proof. Every diagram d is conjugate to a diagram

$$d' = c_1 \otimes c_2 \otimes \cdots,$$

which is a product of cycles c_i. Let c_j be a cycle in d' which is not e but that does have horizontal edges. Suppose that c_j has a horizontal edge connecting the ith and $(i+1)$st dot of the upper row of d'. Then $xd' = e_i d'$ and $d' e_i$ is conjugate to

$$d'' = c_1 \otimes \cdots \otimes c_{j-1} \otimes e \otimes c_j' \otimes c_{j+1} \otimes \cdots$$

where c_j' is a cycle and the length of the cycle c_j' is 2 less than the length of the cycle c_j. For any character χ of $D_f(x)$ we have that

$$(3.2)\,\chi(d) = \chi(d') = (1/x)\chi(e_i d') = (1/x)\chi(d' e_i) = (1/x)\chi(d'').$$

Note that the type of the cycle c_j' is the same as the type of the cycle c_j.

Repeat this process with d'' in place of d' until all cycles with horizontal edges are of the form e. Since any cycle of length r with no horizontal edges is conjugate to γ_k the resulting diagram is conjugate to a diagram

$$d_\tau = e^{\otimes k} \otimes \gamma_{\tau_1} \otimes \gamma_{\tau_2} \otimes \cdots$$

where the τ_i are the types of the cycles with nonzero type in d'. If μ is the partition determined by the nonzero τ_i then d_τ is conjugate to the diagram $e^{\otimes k} \otimes \gamma_\mu$. Since d_τ is a diagram on f dots, $k = (f - |\mu|)/2$.

Each reduction from d' to d'' introduces a factor of $1/x$ in the computation of the character and decreases the length of the cycle c_j by 2. Let $|c_j|$ denote the length of the cycle c_j in d. If type τ_j of c_j is not zero it takes $(|c_i| - \tau_j)/2$ reductions to reduce c_j to a cycle without horizontal edges. If $\tau_j = 0$ then it takes $(|c_j| - \tau_j)/2 - 1$ reductions to reduce c_j to be e. Summing over all cycles gives

$$\begin{aligned} h &= \sum_{\tau_j > 0} (|c_j| - \tau_j)/2 + \sum_{\tau_j = 0} (|c_j| - \tau_j)/2 - 1 \\ &= (1/2)\left(\sum_j |c_j| - \sum_j \tau_j\right) - m_0 \\ &= (1/2)(f - |\tau|) - m_0 \\ &= k - m_0. \end{aligned}$$

□

Theorem (3.1) shows that any character on $D_f(x)$ is completely determined by its values on diagrams of the form $e^{\otimes k} \otimes \gamma_\mu$ where μ is a partition of $f - 2k$, $0 \le k \le [f/2]$. From the structure of $D_f(x)$ we know that the irreducible characters of $D_f(x)$ are indexed by partitions λ of $f - 2k$, $0 \le k \le [f/2]$. This implies that the condition in Theorem (3.1) is not only a necessary condition but also a sufficient condition that a linear functional on $D_f(x)$ be a character. This gives the following corollary.

COROLLARY 3.3. *A linear functional $t : D_f(x) \to \mathbb{C}(x)$ is a character of $D_f(x)$ if and only if it satisfies the relations in Theorem (3.1) for all diagrams d on f dots.*

4. Schur-Weyl duality. Let V be a vector space over $\mathbb{C}$ with basis $v_1, v_2, ..., v_n$. For each of the classical groups G define an action of G on V by

$$Av_j = \sum_{i=1}^{n} v_i a_{ij}, \tag{4.1}$$

for each $A = \|a_{ij}\| \in G$. This action defines the standard or fundamental representation (ρ, V) of G. The vector space $V^{\otimes f} = V \otimes V \otimes \cdots \otimes V$, ($f$ factors) has a basis given by the elements $v_{i_1} v_{i_2} \cdots v_{i_f}$ (we omit the $\otimes$ signs between the v_{i_j} for brevity in notation). Define an action of G on $V^{\otimes f}$ by

$$Av_{i_1} v_{i_2} \cdots v_{i_f} = (Av_{i_1})(Av_{i_2}) \cdots (Av_{i_f}), \tag{4.2}$$

for all $A \in G$. We denote this representation by $(\rho^{\otimes f}, V^{\otimes f})$.

Define representations π_a, π_b, and π_c, of S_f, $D_f(n)$, and $D_f(-n)$ (n even), respectively, on $V^{\otimes f}$ as follows.

(a) Define an action of the symmetric group S_f on $V^{\otimes f}$ by defining, for all $\sigma \in S_f$,

$$v_{i_1} \cdots v_{i_f} \sigma = v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(f)}}.$$

This defines a representation π_a of S_f on $V^{\otimes f}$.

(b) Define operators e and g on $V^{\otimes 2}$ by

$$v_i v_j g = v_j v_i, \text{ and } v_i v_j e = \delta_{ij} \sum_{k=1}^{n} v_k v_k,$$

respectively. Then, for each $j = 1, 2, ..., f-1$, define the action of g_j and e_j on $V^{\otimes f}$ by

$$\begin{aligned} v_{i_1} \cdots v_{i_f} g_j &= v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} g) v_{i_{j+2}} \cdots v_{i_f}, \text{and} \\ v_{i_1} \cdots v_{i_f} e_j &= v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} e) v_{i_{j+2}} \cdots v_{i_f}, \end{aligned}$$

respectively. This defines a representation π_b of $D_f(n)$ on $V^{\otimes f}$, where $n = \dim V$.

(c) Suppose that $n = \dim V$ is even, $n = 2m$, and let $i' = m + i$, for $1 \leq i \leq m$. Let

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } j = i'; \\ -1 & \text{if } i = j'; \\ 0 & \text{otherwise.} \end{cases}$$

Define operators e and g on $V^{\otimes 2}$ by

$$v_i v_j g = -v_j v_i, \text{ and } v_i v_j e = -\epsilon_{ij} \sum_{k=1}^{m} v_k v_{k'} - v_{k'} v_k.$$

respectively. Then, for each $j = 1, 2, \ldots, f-1$, define the action of g_j and e_j on $V^{\otimes f}$ by

$$v_{i_1} \cdots v_{i_f} g_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} g) v_{i_{j+2}} \cdots v_{i_f}, \text{and}$$
$$v_{i_1} \cdots v_{i_f} e_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} e) v_{i_{j+2}} \cdots v_{i_f},$$

respectively. This defines a representation π_c of $D_f(-2m)$ on $V^{\otimes f}$.

THEOREM 4.3. (a) (Schur [**Sc1**, **Sc2**]) *Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(\mathrm{Gl}(n))$ in* $\mathrm{End}(V^{\otimes f})$. *Then $\mathcal{A}$ and $\pi_a(\mathbb{C}S_f)$ are the full centralizers of each other in* $\mathrm{End}(V^{\otimes f})$.

(b) (Brauer [**Br**]). *Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(\mathrm{O}(n))$ in* $\mathrm{End}(V^{\otimes f})$. *Then $\mathcal{A}$ and $\pi_b(D_f(n))$ are the full centralizers of each other in* $\mathrm{End}(V^{\otimes f})$.

(c) (Brauer [**Br**]). *Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(\mathrm{Sp}(2m))$ in* $\mathrm{End}(V^{\otimes f})$. *Then $\mathcal{A}$ and $\pi_c(D_f(-2m))$ are the full centralizers of each other in* $\mathrm{End}(V^{\otimes f})$.

Since $\pi_a(\mathbb{C}S_f)$ and $\rho^{\otimes f}(\mathrm{Gl}(n))$, $\pi_a \times \rho^{\otimes f}$ is a well defined representation of the group $S_f \times \mathrm{Gl}(n)$ on $V^{\otimes f}$. Let $D_f(n) \times \mathrm{O}(n)$ denote the $\mathbb{C}$-algebra consisting of all $\mathbb{C}$-linear combinations of pairs (d, A) where d is a diagram on f dots and $A \in \mathrm{O}(n)$. The multiplication in $D_f(n) \times \mathrm{O}(n)$ is the linear extension of componentwise multiplication of these pairs. Then, in view of part b) of Theorem (4.3), one has a well defined representation $\pi_b \times \rho^{\otimes f}$ of $D_f(n) \times \mathrm{O}(n)$ on $V^{\otimes f}$. Define $D_f(-2m) \times \mathrm{Sp}(2m)$ and a representation $\pi_c \times \rho^{\otimes f}$ of $D_f(-2m) \times \mathrm{Sp}(-2m)$ on $V^{\otimes f}$ analogously.

THEOREM 4.4 ([**Sc1, Sc2, Wy**]). (a) *Let S_λ denote the irreducible S_f module corresponding to λ and let U_λ denote the irreducible* $\mathrm{Gl}(n)$ *module corresponding to λ. Then, as $S_f \times \mathrm{Gl}(n)$ representations,*

$$V^{\otimes f} \cong \bigoplus_{\substack{\lambda \vdash f \\ \ell(\lambda) \leq n}} S_\lambda \otimes U_\lambda.$$

(b) *Let D_λ denote the irreducible $D_f(n)$ module corresponding to λ and let V_λ denote the irreducible* $\mathrm{O}(n)$ *module corresponding to λ. Then, as $D_f(n) \times \mathrm{O}(n)$ representations,*

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\lambda \vdash f-2k \\ \lambda_1' + \lambda_2' \leq n}} D_\lambda \otimes V_\lambda.$$

(c) *Let D_λ denote the irreducible $D_f(-2m)$ module corresponding to λ and let W_λ denote the irreducible* $\mathrm{Sp}(2m)$ *module corresponding to λ. Then, as $D_f(-2m) \times \mathrm{Sp}(2m)$ representations,*

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\lambda \vdash f-2k \\ \ell(\lambda) \leq m}} D_{\lambda'} \otimes W_\lambda.$$

REMARK. In the above Theorem we have chosen to index the irreducible representations of $D_f(n)$ and $D_f(-2m)$ in the same fashion as in [**Wz1**]. The indexing of representation of $\mathrm{O}(n)$ and $\mathrm{Sp}(2m)$ is as in [**Wy**] Chapter VII. This follows the usual convention. Note, however, that, using this labeling, $D_{\lambda'}$ gets paired with W_λ in part (c), where λ' is the conjugate of the partition λ.

For the cases (b) and (c) in Theorem (4.3) one has that ([**Wy**], [**Wz1**]) for $n > 2f$ the representation of $D_f(n)$ (resp. $D_f(-2m)$, $n = 2m$) on $V^{\otimes f}$ is a faithful representation of $D_f(n)$ (resp. $D_f(-2m)$). $D_f(n)$ is semisimple for $n > 2f$. In particular the irreducible character $\chi^\lambda_{(f,n)}$ of $D_f(n)$ is well defined for every partition λ of $f - 2k$, $k = 0, 1, 2, \ldots$.

The following corollary is obtained from Theorem (4.4) by taking traces.

COROLLARY 4.5. (a) *For $A \in \mathrm{Gl}(n)$ and $h \in \mathbb{C}S_f$,*

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_a(h)\right) = \sum_{\lambda \vdash f} \chi^\lambda_{S_f}(h) s_\lambda(A),$$

where $\chi^\lambda_{S_f}$ *denotes the irreducible character of* S_f *corresponding to* λ.

(b) *Let* $n > 2f$. *Then for* $A \in \mathrm{O}(n)$ *and* $h \in D_f(n)$,

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(h)\right) = \sum_{\lambda \vdash f-2k} \chi^\lambda_{(f,n)}(h) so_\lambda(A),$$

where $\chi^\lambda_{(f,n)}$ *denotes the irreducible character of* $D_f(n)$ *corresponding to* λ.

(c) *Let* $n = 2m > 2f$. *Then for* $A \in \mathrm{Sp}(2m)$ *and* $h \in D_f(-2m)$,

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_c(h)\right) = \sum_{\lambda \vdash f-2k} \chi^{\lambda'}_{(f,-2m)}(h) sp_\lambda(A),$$

where $\chi^\lambda_{(f,-2m)}$ *denotes the irreducible character of* $D_f(-2m)$ *corresponding to* λ.

Define the power symmetric functions as the following polynomials in $\mathbb{Z}[x_1, x_2, \dots, x_n]$. For each positive integer r define

$$p_r(x_1, x_2, \dots, x_n) = x_1^r + x_2^r + \cdots + x_n^r,$$

and for a partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ define

$$p_\mu(x_1, x_2, \dots, x_n) = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k}.$$

Part (a) of the following theorem is due to Schur [**Sc2**]. Stanley [**St**] has also noticed (part (b)) that the trace $\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(d)\right)$ should be a power symmetric function.

THEOREM 4.6. (a) *Let* $A \in \mathrm{Gl}(n)$ *and let* $\sigma \in S_f$. *Then*

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_a(\sigma)\right) = p_\mu(x_1, x_2, \dots, x_n),$$

where μ *is the type of the permutation* σ *and* $x_1, x_2, \dots, x_n$ *are the eigenvalues of* A.

(b) *Let* $A \in \mathrm{O}(n)$ *and let* d *be a diagram on* f *dots. Then*

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(d)\right) = n^{m_0} p_\mu(x_1, x_2, \dots, x_n),$$

where μ *is the partition given by the nonzero parts of the type of the diagram* d, m_0 *is the number of parts equal to 0 in the type of* d, *and* $x_1, x_2, \dots, x_n$ *are the eigenvalues of* A.

(c) *Let $A \in \mathrm{Sp}(2m)$ and let d be a diagram on f dots. Then*

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_c(d)\right) = (-2m)^{m_0}(-1)^{|\mu|-\ell(\mu)}p_\mu(x_1, x_2, \ldots, x_{2m}),$$

where μ is the partition given by the nonzero parts of the type of the diagram d, m_0 is the number of parts equal to 0 in the type of d, and $x_1, x_2, \ldots, x_{2m}$ are the eigenvalues of A.

Proof. (a) By continuity it is sufficient to assume that the eigenvalues of A are all distinct and thus we may assume that A is diagonal. The remainder of the proof follows by an easy computation.

(b) From Theorem (3.1) we have that

$$\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(d)\right) = \frac{1}{n^{k-m_0}}\,\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(e^{\otimes k} \otimes \gamma_\mu)\right),$$

where μ is the partition given by the nonzero parts in the type of the diagram d, m_0 is the number of 0 parts in the type d, and $k = (f - |\mu|)/2$. Let $e = e_1 \in D_2(x)$ and let $g \in \mathrm{O}(n)$. By the definition of the action of e we have

$$\mathrm{Tr}\left(\rho^{\otimes 2}(A)\pi_b(e)\right) = \sum_{1 \le i,j \le n} Av_iv_je|_{v_iv_j} = \sum_{i=1}^{n}\sum_{r=1}^{n} v_rv_r|_{v_iv_i} = n.$$

Since $A \in \mathrm{O}(n) \subseteq \mathrm{Gl}(n)$, part a) gives that

$$\mathrm{Tr}\left(\rho^{\otimes r}(A)\pi_b(\gamma_r)\right) = p_r(x_1, \ldots, x_n).$$

The remainder of the proof follows from the fact that if $d \in D_m(x)$ and $d' \in D_n(x)$ then

$$\mathrm{Tr}\left(\rho^{\otimes(n+m)}(A)\pi_b(d \otimes d')\right) = \mathrm{Tr}\left(\rho^{\otimes m}(A)\pi_b(d)\right)\mathrm{Tr}\left(\rho^{\otimes n}(A)\pi_b(d')\right).$$

We have

$$\begin{aligned}\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(d)\right) &= \frac{1}{n^{k-m_0}}\,\mathrm{Tr}\left(\rho^{\otimes f}(A)\pi_b(e^{\otimes k} \otimes \gamma_\mu)\right)\\ &= \frac{1}{n^{k-m_0}} n^k p_\mu(x_1, \ldots, x_n).\end{aligned}$$

(c) The proof is exactly as in the orthogonal case except that one has that for $A \in \mathrm{Sp}(2m)$ and $e = e_1 \in D_2(-2m)$,

$$\begin{aligned}\mathrm{Tr}\left(\rho^{\otimes 2}(A)\pi_c(e)\right) &= \sum_{1 \le i,j \le 2m} Av_iv_je|_{v_iv_j}\\ &= \sum_{i,j} -\epsilon_{ij}\sum_{k=1}^{m} v_kv_k' - v_k'v_k|_{v_iv_j} = -2m,\end{aligned}$$

by (4.6), and that, by part (a),

$$\mathrm{Tr}\left(\rho^{\otimes r}(A)\pi_c(\gamma_r)\right) = (-1)^{r-1}p_r(x_1, ..., x_{2m}). \qquad \square$$

5. The irreducible characters of $D_f(x)$. Our developments in the previous section give us all necessary tools to derive a formula for the irreducible characters of $D_f(x)$ in terms of the characters of the symmetric group S_f.

THEOREM 5.1. *Let λ be a partition of $f - 2k$ and let d be a diagram on f dots of type $\mu \vdash f-2h$. Then the irreducible character of $D_f(x)$ corresponding to λ is given by*

$$\chi^{\lambda}_{(f,x)}(d) = x^h \sum_{\substack{\nu \vdash f-2h \\ \nu \subseteq \lambda}} \left(\sum_{\beta} c^{\nu}_{\lambda\beta}\right) \chi^{\nu}_{S_{f-2h}}(\gamma_\mu).$$

Proof. Let $n = 2m+1$, $n \geq 2j$, and let $d = e^{\otimes h} \otimes \gamma_\mu$.Combining Theorem (4.6)(b), Theorem (4.5)(b), and Theorem (1.2)(b), we have

$$(1) \quad n^h p_\mu(x_1, x_1^{-1}, ..., x_m, x_m^{-1}, 1) = \sum_{\lambda \vdash f-2k} \chi^{\lambda}_{(f,n)}(d) sb_\lambda(x_1, x_1^{-1}, ..., x_m, x_m^{-1}),$$

and similarly by Theorem (4.6)(a), Theorem (4.5)(a), and Theorem (1.2)(a),

$$(2) \quad n^h p_\mu(x_1, x_1^{-1}, ..., x_m, x_m^{-1}, 1) = n^h \sum_{\lambda \vdash f-2h} \chi^{\lambda}_{S_{f-2h}}(\gamma_\mu) sa_\nu(x_1, x_1^{-1}, ..., x_m, x_m^{-1}, 1).$$

The branching rule (1.1) gives

(3)

$$sa_\nu(x_1, x_1^{-1}, ..., x_m, x_m^{-1}, 1) = \sum_{\lambda \subseteq \nu} \left(\sum_{\beta \text{ even}} c^{\nu}_{\lambda\beta}\right) sb_\lambda(x_1, x_1^{-1}, ..., x_m, x_m^{-1}).$$

Setting (1) and (2) equal and using (3) to expand the sa_λ in terms of the sb_λ,

$$\begin{aligned} n^h p_\mu &= \sum_{\lambda} \chi^{\lambda}_{(f,n)}(d) sb_\lambda \\ &= n^h \sum_{\nu \vdash f-2h} \chi^{\nu}_{S_{f-2h}}(\gamma_\mu) \sum_{\lambda \subseteq \nu} \left(\sum_{\beta \text{ even}} c^{\nu}_{\lambda\beta}\right) sb_\lambda. \end{aligned}$$

Since the sb_λ are algebraically independent $(n > 2f)$ we can equate coefficients of sb_λ to get that

$$\chi^\lambda_{(f,n)}(d) = n^h \sum_{\substack{\nu \vdash f-2h \\ \nu \supseteq \lambda}} \left(\sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_{f-2h}}(\gamma_\mu). \tag{4}$$

This identifies the irreducible character $\chi^\lambda_{(f,n)}$ of $D_f(n)$ for all odd $n > 2f$.

Let

$$ch(x) = x^h \sum_{\substack{\nu \vdash f-2h \\ \nu \supseteq \lambda}} \left(\sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_{f-2h}}(\gamma_\mu).$$

Then,

$$ch(n) = \chi^\lambda_{(f,n)}(d) = \chi^\lambda_{(f,x)}(d)|_{x=n}$$

for infinite number of $n \in \mathbb{Z}$, where the first equality follows from (4) and the second from Theorem (2.4). Since both $ch(x)$ and $\chi^\lambda_{(f,x)}(d)$ are rational functions in x and they are equal at an infinite number of points they must be equal everywhere. □

COROLLARY 5.2. *If $|\lambda| < f$ and d is a diagram on f dots of the form $e^{\otimes h} \otimes \gamma_\mu$ with $|\mu| < f$ then*

$$\chi^\lambda_{(f,x)}(d) = x\chi^\lambda_{(f-2,x)}(d'),$$

where $d' = e^{\otimes h-1} \otimes \gamma_\mu$.

Proof. Let $|\mu| = k$. Then

$$\begin{aligned}
\chi^\lambda_{(f,x)}(d) &= x^h \sum_{\substack{\nu \vdash k \\ \nu \subseteq \lambda}} \left(\sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_k}(\gamma_\mu) \\
&= x \left(x^{h-1} \sum_{\substack{\nu \vdash k \\ \nu \subseteq \lambda}} \left(\sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_k}(\gamma_\mu) \right) \\
&= x\chi^\lambda_{(f-2,x)}(d').
\end{aligned}$$

□

COROLLARY 5.3. *If $\lambda \vdash f$ and d is a diagram on f dots of the form $e^{\otimes h} \otimes \gamma_\mu$ with $|\mu| < f$ then*

$$\chi^\lambda_{(f,x)}(d) = 0.$$

Proof. Since λ is a partition of f and $|\mu| < f$, we have $c^\mu_{\lambda\beta} = 0$ for all β. □

COROLLARY 5.4. *If $\lambda \vdash f$ and if d is a diagram on f dots of the form γ_μ with $\mu \vdash f$, then*

$$\chi^\lambda_{(f,x)}(d) = \chi^\lambda_{S_f}(\gamma_\mu).$$

Proof.

$$c^\nu_{\lambda\beta} = \begin{cases} 1, & \text{if } \lambda = \nu \text{ and } \beta = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

giving that

$$\chi^\lambda_{(f,x)} = x^0 \sum_{\substack{\nu \vdash f-2h \\ \nu \subseteq \lambda}} \left(\sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_f}(\gamma_\mu) = \chi^\lambda S_f(\gamma_\mu). \qquad \square$$

Let Ξ_f denote the character table of $D_f(x)$, i.e. Ξ_f is a matrix with rows and columns indexed by partitions of $f - 2k$, $0 \leq k \leq [f/2]$, and the entry in the λth row and the μth column of Ξ_f is $\chi^\lambda_{(f,x)}(e^{\otimes h} \otimes \gamma_\mu)$ where $h = (f - |\mu|)/2$. We can summarize the results of Corollaries (5.2)-(5.4) by observing that the character table Ξ_f of $D_f(x)$ can be given in the form

$$\Xi_f = \begin{pmatrix} x\Xi_{f-2} & * \\ 0 & \Xi_{S_f} \end{pmatrix},$$

where Ξ_{f-2} is the character table of $D_{f-2}(x)$ and Ξ_{S_f} is the character table of the symmetric group S_f. More specifically, the character table of $D_f(x)$ can be given in block upper triangular form where the diagonal blocks are of the form $x^k \Xi_{S_{f-2k}}$, $0 \leq k \leq [f/2]$.

6. A combinatorial rule for computing the characters of $D_f(x)$. In this section we analyze the combinatorics of formula (1) in the proof of Theorem (5.1). We begin by reviewing the basic properties of alternating and symmetric functions for the hyperoctahedral group. Note that these basic results hold for any Weyl group, see [**Bou**]. We shall treat only the special case of the hyperoctahedral group in the following.

Define the hyperoctahedral group B_n as the group of $n \times n$ matrices $w = (w_{ij})$ such that

(1) $w_{ij} \in \{0, 1, -1\}$, for each $1 \leq i, j \leq n$, and

(2) the matrix $(|w_{ij}|)$ is a permutation matrix.

The symmetric group S_n is a subgroup of B_n. For each $w \in B_n$ define the sign of w by $\epsilon(w) = \det(w)$. As $B_n \subset M_n(\mathbb{C})$, there is a natural action of elements of B_n on elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$. For each $\alpha \in \mathbb{Z}^n$ let x^α denote the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Define an action of elements of B_n on monomials in the variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ by defining $wx^\alpha = x^{w\alpha}$.

Let $1/2 + \mathbb{Z}$ denote the set $\{1/2 + p, p \in \mathbb{Z}\}$ of half-integers and set

$$\delta = (n - 1/2, n - 3/2, \ldots, 3/2, 1/2).$$

For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in (1/2 + \mathbb{Z})^n$ define

$$b_\alpha = \sum_{w \in B_n} \epsilon(w) w x^\alpha.$$

Then, if λ is a partition $\ell(\lambda) \leq n$,

$$sb_\lambda = \frac{b_{\lambda+\delta}}{b_\delta}. \tag{6.1}$$

The polynomial b_α is skew symmetric under the action of B_n, i.e.,

$$w(b_\alpha) = \epsilon(w) b_\alpha = b_{w\alpha}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (1/2 + \mathbb{Z})^n$. Let m be the number of $\alpha_i < 0$ in α and let $|\alpha|$ denote the vector $(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|)$. Let $\mathrm{Re}(|\alpha|)$ denote the sequence given by rearranging the parts of $|\alpha|$ in decreasing order and let π denote the permutation such that $\pi(|\alpha|) = \mathrm{Re}(|\alpha|)$. Let $\lambda = \mathrm{Re}(|\alpha|) - \delta$. Since b_α is skew symmetric under the action of B_n, $b_{\mathrm{Re}(|\alpha|)} = 0$ unless $|\alpha_1|, |\alpha_2|, ..., |\alpha_n|$ are different. If the $|\alpha_i|$ are all positive and distinct then λ is a partition. We have that

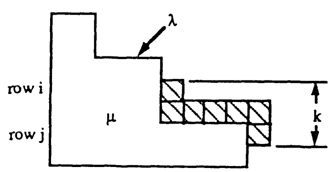
$$(6.2) \qquad b_\alpha = (-1)^m b_{|\alpha|} = (-1)^m \epsilon(\pi) b_{\mathrm{Re}}(|\alpha|) = \begin{cases} (-1)^m \epsilon(\pi) b_{\lambda+\delta}, & \text{if } \lambda \text{ is a partition,} \\ 0, & \text{otherwise.} \end{cases}$$

In the standard fashion (see [**Mac**]) we shall associate to each sequence of positive integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ a diagram consisting of n rows of boxes such that row i contains α_i boxes. Let e_i denote the vector (0,...,0,1,0,...,0) where the 1 appears in the ith entry. Let $\mu = (\mu_1, \mu_2, ..., \mu_n)$ be a partition. Let

$$\lambda = \mathrm{Re}(|\mu + \delta + re_i|) - \delta.$$

We say that the sequence λ is given by adding r boxes along the boundary of the diagram of μ, beginning in row i and continuing in rows $< i$.

(6.3)

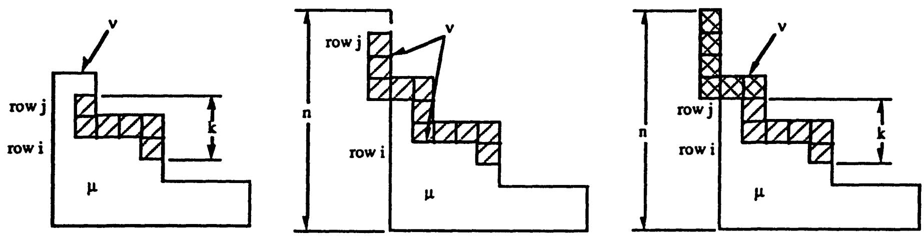


Let

$$\nu = \mathrm{Re}(|\mu + \delta - re_i|) - \delta.$$

We say that the sequence ν is given by removing a slinky of length r from μ beginning at the row i. Pictorially, the diagram of ν is given by removing r boxes from the boundary of the diagram of μ, beginning with the last box in row i and continuing in rows $> i$ as in diagram (6.4)(a). It may happen that r is large enough that not all r boxes are removed before reaching the $\ell(\mu)$th row. In this case one proceeds by continuing to remove boxes from an imaginary wall of height n adjacent to λ. In this case ν will be of the form $(*, *, ..., *, -1, -1, ..., -1, 0, ..., 0)$ where the $*$ entries are positive integers. Pictorally ν is given as in diagram (6.4)(b). If, in the process of removing boxes, the height $n-1$ in the imaginary wall is reached and still $r-1$ boxes have not been removed, then one begins placing boxes, first in the holes in the wall, then along the boundary of the shape, until a total of $r-1$ boxes have either removed or placed, diagram (6.4)(c).

(6.4)

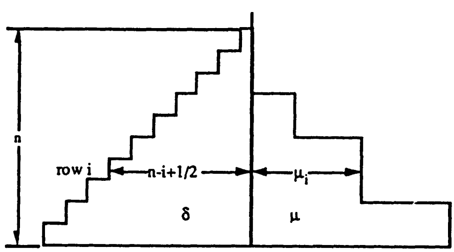


(a) (b) (c)

It will be clear that the sequence ν given by removing a slinky of length r at row i is given by this diagram from the proof of Lemma (6.6).

Represent the monomial $x^{\mu+\delta}$ by the diagram

(6.5)



Numbering the rows from the bottom to top, row i contains $\delta_i = n - i + 1/2$ boxes to the left of the vertical bar and μ_i boxes to the right of the vertical bar. One can view the action of B_n on monomials as an action of B_n on these diagrams. Let s_i denote the transposition $(i, i+1)$ and let s_0 denote the element of B_n given by the matrix (w_{ij}), where $w_{ij} = 0$ for $1 \le i \le n-1$ and $w_{nn} = -1$.

LEMMA 6.6. *Let μ be a partition and $r > 0$.*

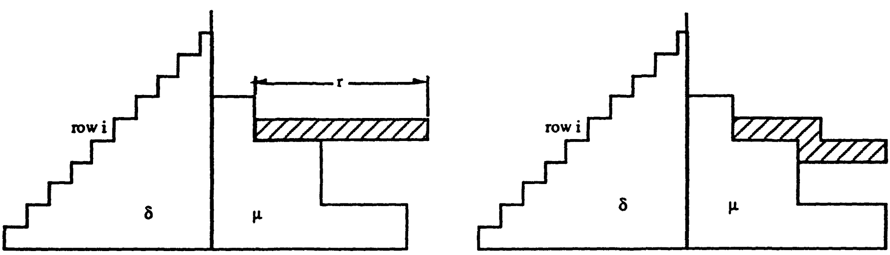
(a) *Let λ be the sequence given by adding a slinky of length r to μ at row i. Let k be the number of rows spanned by the slinky. Then*

$$\sum_{w\in B_n} \epsilon(w) w x_i^r x^{\mu+\delta} = \begin{cases} (-1)^{k-1} b_{\lambda+\delta}, & \textit{if } \lambda \textit{ is a partition,} \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *Let ν be the sequence given by removing a slinky of length r from μ beginning at row i. Let k be the number of rows in the slinky. Then*

$$\sum_{w\in B_n} \epsilon(w) w x_i^{-r} x^{\mu+\delta} = \begin{cases} (-1)^{k-1} b_{\nu+\delta}, & \textit{if } \nu \textit{ is a partition,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. (a) The diagram representing the monomials $x_i^r x^{\mu+\delta}$ and $s_{i-1} x_i^r x^{\mu+\delta}$ looks like

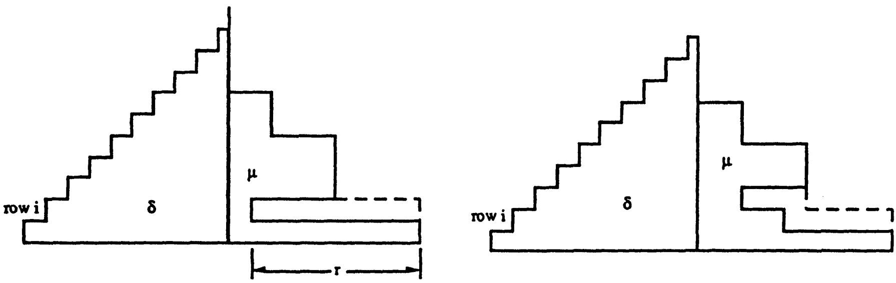


In some sense the factors from x_i^r have "slinkyed" one row down the shape of ν. Let $1 \le j \le i$ be the greatest $j \le i$ such that in the diagram of $s_j s_{j+1} \cdots s_{i-2} s_{i-1} x_i^r x^{\mu+\delta}$ the number of boxes in the j row is less than or equal to the number of boxes in the $j+1$st row. Pictorally j is such that the diagram of $s_j s_{j+1} \cdots s_{i-2} s_{i-1} x_i^r x^{\mu+\delta}$ looks like that in (6.3) and the factors from x_i^r have slinkyed down the shape of μ as far as possible.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be given by adding a slinky of length r to μ beginning at row i, as in (6.3). Setting $\pi = s_j \cdots s_{i-2} s_{i-1}$, we have that $\epsilon(\pi) = i - j = k - 1$, where k is the number of rows in the slinky, and that $x^{\lambda+\delta} = \pi x_i^r x^{\mu+\delta}$. Then

(6.7)
$$\sum_{w\in B_n} \epsilon(w) w x_i^r x^{\mu+\delta} = \sum_{w\in B_n} \epsilon(w) w x^{\mu+\delta+r\epsilon_i} = \epsilon(\pi) \sum_{w\in B_n} \epsilon(w) w x^{\lambda+\delta}$$
$$= \begin{cases} (-1)^{k-1} b_{\lambda+\delta} & \text{if } \lambda \text{ is a partition,} \\ 0 & \text{otherwise.} \end{cases}$$

(b) $x_i^{-r} x^{\mu+\delta} = x^{\mu+\delta-re_i}$ and $s_i x_i^{-r} x^{\mu+\delta}$ have diagrams of the form



In this case rather than adding a slinky of length r we are removing a slinky of length r beginning in row i. Let λ be the sequence given by removing a slinky of length r from μ beginning at row i. If λ is as in (6.4)(a) or (6.4)(b) let $\pi = s_j s_{j-1} \cdots s_i$. If λ is as in (6.4)(c) than let $\pi = s_j s_{j+1} \cdots s_{n-1} s_0 s_{n-1} \cdots s_i$. Then, in each case, we shall have that
$$x^{\lambda+\delta} = \pi x_i^{-r} x^{\mu+\delta}.$$

Let $k = |j-i|+1$, so that k is the number of rows spanned by the slinky. The result follows as in (6.7). □

REMARK. Notice that in the case of Lemma (6.6)(b), if n is large ($n > r + \ell(\mu)$), then one has that $\sum_{w\in B_n} \epsilon(w) w x_i^{-r} x^{\mu+\delta} = 0$ unless the sequence ν given by removing a slinky of r boxes from μ is either

(1) as in figure (6.4)(a), or

(2) r is odd and $n-1-i+1 = (r-1)/2$, in which case $\nu = \mu$.

These are the only cases for which ν will be a partition.

Fix n. Let λ and μ be partitions. Then we say that λ differs from μ by an r slinky if λ is given by either adding or removing a slinky of length r from μ.

THEOREM 6.8. *Let μ be a partition. Then*

$$p_r sb_\mu = sb_\mu + \sum_\lambda (-1)^{k(\lambda)-1} sb_\lambda,$$

where the sum is over all partitions λ such that λ differs from μ by an r-slinky and $k(\lambda)$ is a number of rows in this slinky. This expansion is independent of n for $n > r + \ell(\mu)$.

Proof.

$$\begin{aligned} &p_r(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1) b_{\mu+\delta} \\ &\quad = \left(1 + \sum_{i=1}^n x_i^r + x_i^{-r}\right)\left(\sum_{w\in B_n} \epsilon(w) w x^{\mu+\delta}\right) \\ &\quad = b_{\mu+\delta} + \sum_{i=1}^n \left(\sum_{w\in B_n} \epsilon(w) w x_i^r x^{\mu+\delta} + \sum_{w\in B_n} \epsilon(w) w x_i^{-r} x^{\mu+\delta}\right). \end{aligned}$$

Using Lemma (6.6) we have that

$$p_r b_{\mu+\delta} = b_{\mu+\delta} + \sum_\lambda (-1)^{k(\lambda)-1} b_{\lambda+\delta},$$

where the sum is over all λ that differ from μ by a slinky of length r and $k(\lambda)$ is a number of rows in this slinky. The result follows by dividing by b_δ.

The fact that this expansion is independent of n for large n follows from the remark following Lemma (6.6). □

REMARK. There is a slight subtlety in the definition of when λ differs from μ by an r-slinky. Let us restate this in the language of border strips (connected skew diagrams with no 2×2 blocks of boxes), see [**Mac**, §3, Ex. 11]. Assume n is large $n > r + \ell(\mu)$. Then λ differs from μ by an r-slinky if either

(1) $\mu \subseteq \lambda$ and λ/μ is a border strip of length r,

(2) $\lambda \subseteq \mu$ and μ/λ is a border strip of length r, or

(3) r is odd and $\lambda = \mu$.

Let $\mu = (\mu_1, \mu_2, ..., \mu_n)$ be a partition. Define a μ-slinky tableau of shape λ to be a sequence of partitions

$$T = (\emptyset, \lambda^{(1)}, ..., \lambda^{(k)} = \lambda)$$

such that for each i either $\lambda^{(i)} = \lambda^{(i+1)}$ or $\lambda^{(i+1)}$ differs from $\lambda^{(i)}$ by a μ_i slinky.

THEOREM 6.9. *Let d be a diagram on f dots of the form $e^{\otimes h} \otimes \gamma_\mu$. Then*

$$\chi^\lambda_{(f,x)}(d) = x^h \sum_T wt(T),$$

where the sum is over all μ-slinky tableaux T of shape λ and

$$wt(T) = \prod_{\substack{\text{slinkies} \\ \text{in } T}} (-1)^{\#\text{ of rows in slinky}-1},$$

where the product is over all slinkys in T.

Proof. By Corollary (5.2)

$$\chi^\lambda_{(f,x)}(e^{\otimes h} \otimes \gamma_\mu) = \chi^\lambda_{(f-2h,x)}(\gamma_\mu).$$

thus it is sufficient to prove the theorem for $h = 0$.

Let $h = 0$ and let $m = 2n + 1 > 2f + 1$ be odd. Then from the proof of Theorem (5.1) one knows that

$$\chi^\lambda_{(f,x)} = \chi^\lambda_{(f,m)},$$

and further that

$$m^h p_\mu(x_1, x_1^{-1}, ..., x_n, x_n^{-1}, 1) = \sum_{\lambda \vdash f-2k} \chi^\lambda_{(f,m)}(\gamma_\mu) sb_\lambda(x_1, x_1^{-1}, ..., x_n, x_n^{-1}).$$

Thus, $\chi^\lambda_{(f,x)}$ is given by the coefficient of sb_λ in the expansion of $p_\mu(x_1, x_1^{-1}, ..., x_n, x_n^{-1}, 1)$. This coefficient is given by repeated application of Theorem (6.8). In view of the fact that n is large $(n > f \geq |\mu|)$ this expansion is independent of n. The theorem follows. □

REMARK. J. Stembridge [**Ste**] has given a combinatorial rule for computing the characters of the hyperoctahedral group B_n which involves placing and removing slinkys in much the same fashion as for the Brauer algebra. Although this may seem to be merely coincidence it seems that there is a deeper connection between the hyperoctahedral group and the Brauer algebra which is also reflected in the work of Hanlon and Wales [**HW1-2**].

REFERENCES

[Bou] N. Bourbaki, *Groupes et Algebres de Lie*, Chap. 4-6. Paris: Hermann 1960.

[Br] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. Math., **38** (1937), 854-872.

[Fr] F.G. Frobenius, *Über die Charactere der symmetrischen Gruppe*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1900), 516-534; reprinted in *Gessamelte Abhandlungen* **3**, 148-166.

[HW1] P. Hanlon and D.B. Wales, *On the decomposition of Brauer's centralizer algebras*, J. Algebra, **121** (1989), 409-445.

[HW2] ———, *Eigenvalues connected with Brauer's centralizer algebras*, J. Algebra, **121** (1989), 446-476.

[Ji] M. Jimbo, *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys., **11** (1986), 247-252.

[KT] K. Koike and I. Terada, *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, J. Algebra, **107** (1987), 466-511.

[Li] D.E. Littlewood, *Theory of Group Characters*, Oxford University Press, (1940).

[Mac] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford (1979).

[Pr] R. Proctor, *Interconnections between symplectic and orthogonal characters*, Contemp. Math., **88** (1989), 145-162.

[R1] A. Ram, *A frobenius formula for the characters of the Hecke algebras*, Invent. Math., **106** (1991), 461-488.

[R2] ———, *Representation theory and character theory of centralizer algebras*, Dissertation, University of California, San Diego, 1991.

[Sc1] I. Schur, *Über eine Klasse von Matrizen, die sich einer gegeben Matrix zuordnen lassen*, Dissertation, 1901; reprinted in *Gessamelte Abhandlungen* **1**, 1-72.

[Sc2] ———, *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*, Sitzungsberichte der Königlich Preussischen Akademie der

Wissenschaften zu Berlin (1927), 58-75; reprinted in *Gessamelte Abhandlungen* **3**, 68-85.

[St] R. Stanley, Letter to Phil Hanlon, 1985.

[Ste] J. Stembridge, unpublished notes, 1987.

[Wy] H. Weyl, *Classical Groups*, Princeton University Press, 1946.

[Wz1] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. of Math., **128** (1988), 173-193.

[Wz2] ———, *Quantum groups and subfactors of type B, C and D*, Comm. Math. Phys., **133** (1990), 1-49.

Received November 1, 1992 and in revised form July 16, 1993. Work partially supported by an NSF grant at the University of California, San Diego. The author is currently partially supported by an NSF postdoctoral fellowship.

UNIVERSITY OF WISCONSIN
MADISON, WI 53706
E-mail address: ram@math.wisc.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded by E. F. Beckenbach (1906-1982) and F. Wolf (1904-1989)

Manuscripts must be prepared in accordance with the instructions provided on the inside back cover.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $215.00 a year (10 issues). Special rate: $108.00 a year to individual members of supporting institutions.

Subscriptions, orders for back issues published within the last three years, and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION
This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.

PACIFIC JOURNAL OF MATHEMATICS

Volume 169 No. 1 May 1995
