CHARACTERS OF BRAUER’S CENTRALIZER ALGEBRAS

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Brauer's centralizer algebras are finite dimensional algebras with a distinguished basis. Each Brauer centralizer algebra contains the group algebra of a symmetric group as a subalgebra and the distinguished basis of the Brauer algebra contains the permutations as a subset. In view of this containment it is desirable to generalize as many known facts concerning the group algebra of the symmetric group to the Brauer algebras as possible. This paper studies the irreducible characters of the Brauer algebras in view of the distinguished basis. In particular we define an analogue of conjugacy classes, and derive Frobenius formulas for the characters of the Brauer algebras. Using the Frobenius formulas we derive formulas for the irreducible character of the Brauer algebras in terms of the irreducible characters of the symmetric groups and give a combinatorial rule for computing these irreducible characters.

Introduction. Classically, Frobenius [Fr] determined the characters of the symmetric group by exploiting the connection between the power symmetric functions and the Schur functions. Schur [Sc1, Sc2] later showed that this connection arises from the fact that the general linear group and the symmetric group each generate the full centralizer of each other on tensor space, now referred to as the Schur-Weyl duality. In his landmark book [Wy], Weyl used this duality as the principal algebraic tool for studying the representations of the classical groups. In 1937 R. Brauer [Br] gave a nice basis for the centralizer algebra of the action of the orthogonal and symplectic groups on tensor space.

In [R1] we gave a formula for the characters of the Hecke algebra of type A in the same spirit as the original formula of Frobenius for the characters of the symmetric group. This formula was then used to derive a combinatorial rule for computing the characters of the
Hecke algebras of type $A$ which is a $q$ extension of the Murnaghan-Nakayama rule for computing the irreducible characters of the symmetric group. The algebraic structure motivating this approach to the characters of the Hecke algebra is a Schur-Weyl type duality between the Hecke algebra and the quantum group $U_q(sl(n))[Ji]$.

In this paper we extend the classical method of determining the characters of the symmetric group to the Brauer algebras. In particular we derive Frobenius type formulas and a combinatorial rule for computing the irreducible characters of Brauer's centralizer algebras. This paper is organized as follows. Section 1 summarizes the necessary facts concerning semisimple algebras and the representation theory of the classical groups. Section 2 gives the definition of the Brauer algebras and a brief description of their structure. Section 3 defines an analogue of conjugacy classes for the Brauer algebra. Note: The Brauer algebras are not group algebras, so, conjugacy classes are not, a priori, natural. Section 4 gives a description of the Schur-Weyl duality for the case of the orthogonal and symplectic groups and evaluates the trace functions that are the key to the determination of the irreducible characters of the Brauer algebras. Section 5 gives formulas for the irreducible characters of the Brauer algebras in terms of the irreducible characters of the symmetric groups. Finally, section 6 gives a combinatorial rule for computing the irreducible characters of the Brauer algebras.

This paper is taken from a portion of the author's dissertation [R2] at the University of California, San Diego. Many of the implications of a Schur-Weyl type duality which are discussed there are not treated here. In particular, analogues of orthogonality relations for irreducible characters of semisimple algebras which are not group algebras (for example Brauer's centralizer algebras) and the Frobenius characteristic map (see [Mac]) in a general setting. Several of the interrelations between the orthogonal and symplectic characters (see [Pr] and [K-T]) can be derived immediately via the Frobenius formulas for Brauer's centralizer algebras and furthermore the module theoretic derivation of this formula gives a natural setting for the representation theoretic interpretation of these derivations.

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like to thank J. Remmel for helpful discussions.

1. Notations. $M_n(F)$ denotes the full matrix algebra of $n \times n$ matrices with entries in the field $F$. We shall say that an algebra $A$ over $F$ is semisimple if it is isomorphic to a direct sum of full matrix algebras over $F$, i.e.

$$A \cong \bigoplus_{\lambda} M_{n_{\lambda}}(F),$$

where $\lambda$ are in some finite index set and $n_{\lambda}$ are positive integers. Note that this is what is usually called a split semisimple algebra. An element $p \in A$, $p \neq 0$, is idempotent if $p^2 = p$. An idempotent $p$ is minimal if $p$ cannot be written as a sum $p = p_1 + p_2$ of idempotents such that $p_1p_2 = p_2p_1 = 0$. A partition of unity in algebra $A$ is a set of minimal idempotents $p_i \in A$ such that $p_ip_j = p_jp_i = 0$, for $i \neq j$ and $\sum_i p_i = 1$. A character of $A$ is an $F$-linear functional $\chi : A \to F$ such that

$$\chi(ab) = \chi(ba),$$

for all $a, b \in A$. If $A$ is semisimple then every character of $A$ is a linear combination of irreducible characters.

$S_f$ shall denote the symmetric group on $f$ symbols and $FS_f$ its group algebra over $F$. $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{C}(x)$ the field of rational functions in a single variable $x$.

A partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is a finite sequence of decreasing integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$. The weight $|\lambda|$ of $\lambda$ is the sum of its parts. We say that $\lambda$ is a partition of $f$ if $|\lambda| = f$ and write $\lambda \vdash f$. We shall often use the notation $\lambda = (0^{m_0}1^{m_1}2^{m_2}...)$ where $m_i$ is the number parts of $\lambda$ equal to $i$. The conjugate of the partition $\lambda = (\lambda_1, ..., \lambda_n)$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_n)$ given by $\lambda'_i = \text{Card} \{ j | \lambda_j \geq i \}$. A partition with all parts equal to 0 is called the empty partition and denoted by $\emptyset$. For partitions $\lambda$ and $\mu$, $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all $i$. We say that a partition $\lambda$ is even if all its parts $\lambda_i$ are even.

Facts from the representation theory of the classical groups. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}$, and let $I$ be the $n \times n$ identity matrix. Let $J$ be the $2n \times 2n$ matrix given by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$
We use the following standard notations for the general linear, orthogonal, and symplectic groups:

\[
\begin{align*}
\text{Gl}(n) &= \{ g \in M_n(\mathbb{C}) | \det g \neq 0 \}, \\
\text{O}(n) &= \{ g \in M_n(\mathbb{C}) | g g^t = I \}, \\
\text{Sp}(2n) &= \{ g \in M_{2n}(\mathbb{C}) | g J g^t = J \}.
\end{align*}
\]

For each partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) define the following polynomials in \( \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}] \),

\[
\begin{align*}
s_a(\lambda)(x_1, \ldots, x_n) &= \frac{|x_i^{\lambda_j + n-j}|}{|x_i^{n-j}|}, \\
s_b(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) &= \frac{|x_i^{\lambda_j + n-j + 1/2} - x_i^{-(\lambda_j + n-j + 1/2)}|}{|x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}|}, \\
s_c(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) &= \frac{|x_i^{\lambda_j + n-j} + x_i^{-(\lambda_j + n-j)}|}{|x_i^{n-j} + x_i^{-(n-j)}|}.
\end{align*}
\]

The Littlewood-Richardson coefficients \( c_{\lambda \mu}^\nu \) are nonnegative integers defined by the equation

\[
s_a(\lambda)(x_1, \ldots, x_n)s_a(\mu)(x_1, \ldots, x_n) = \sum_{\lambda} c_{\lambda \mu}^\nu s_a(\nu)(x_1, \ldots, x_n).
\]

The Littlewood-Richardson coefficients are defined for each triple of partitions of lengths less than or equal to \( n \). \( c_{\lambda \mu}^\nu = 0 \) if \( s_a(\nu) \) does not appear in the expansion of \( s_a(\lambda)s_a(\mu) \).

The following identities, due to Littlewood [Li], hold in \( \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) ([K-T] contains an easily accessible proof),

\[
(1.1)
\]

\[
\begin{align*}
s_a(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1) &= \sum_{\lambda \subseteq \nu} \left( \sum_{\beta \text{ even}} c_{\lambda \beta}^\nu \right) s_b(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), \\
s_a(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) &= \sum_{\lambda \subseteq \nu} \left( \sum_{\beta' \text{ even}} c_{\lambda \beta}^\nu \right) s_c(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), \\
s_a(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) &= \sum_{\lambda \subseteq \nu} \left( \sum_{\beta \text{ even}} c_{\lambda \beta}^\nu \right) s_d(\lambda)(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}).
\end{align*}
\]
Recall that a partition $\beta$ is even if all its parts are even.

**Theorem 1.2 ([Wy, Li])**. (a) The irreducible polynomial representation of $\text{Gl}(n)$ are indexed by partitions $\lambda$ such that $l(\lambda) \leq n$. Let $g \in \text{Gl}(n)$ and let $x_1, \ldots, x_n$ denote the eigenvalues of $g$. The character $s_\lambda$ of the irreducible representation of $\text{Gl}(n)$ corresponding to $\lambda$ evaluated at $g$ is given by

$$s_\lambda(g) = sa_\lambda(x_1, \ldots, x_n).$$

(b) The irreducible polynomial representations of $\text{O}(2n + 1)$ are indexed by partitions $\lambda$ such that $\lambda_1 + \lambda_2 \leq 2n + 1$. If $\lambda$ is such that $\lambda_1 + \lambda_2 \leq 2n + 1$ and $l(\lambda) > n$, let $\tilde{\lambda}$ be the partition given by

$$\tilde{\lambda}_i' = \begin{cases} 
\lambda_i', & \text{for } i > 1 \\
2n + 1 - \lambda_1', & \text{for } i = 1.
\end{cases}$$

Let $g \in \text{O}(2n + 1)$ and suppose that $\det(g) = 1$, whence the eigenvalues of $g$ will be in the form $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1$. Then the character $s_\lambda$ of the irreducible representation of $\text{O}(2n + 1)$ corresponding to $\lambda$ evaluated at $g$ is given by

$$s_\lambda(g) = \begin{cases} 
sb_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } l(\lambda) \leq n, \\
s\tilde{b}_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } l(\lambda) > n.\end{cases}$$

(c) The irreducible polynomial representations of $\text{Sp}(2n)$ are indexed by partitions $\lambda$ such that $l(\lambda) \leq n$. Let $g \in \text{Sp}(2n)$ and let $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ be the eigenvalues of $g$. Then the character $s_\lambda$ of the irreducible representation of $\text{Sp}(2n)$ corresponding to $\lambda$ evaluated at $g$ is given by

$$s_\lambda = sc_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}).$$

(d) The irreducible representations of $\text{O}(2n)$ are indexed by partitions $\lambda$ such that $\lambda_1 + \lambda_2 \leq 2n$. If $\lambda$ is such that $\lambda_1 + \lambda_2 \leq 2n$ and $l(\lambda) > n$, let $\tilde{\lambda}$ be the partition given by

$$\tilde{\lambda}_i' = \begin{cases} 
\lambda_i', & \text{for } i > 1, \\
2n - \lambda_1', & \text{for } i = 1.
\end{cases}$$
Let \( g \in O(2n) \) and suppose that \( \det(g) = 1 \), whence the eigenvalues of \( g \) will be of the form \( x_1, x_1^{-1}, \ldots, x_n, x_n^{-1} \). Then the character \( \text{so}_\lambda \) of the irreducible representation of \( O(2n) \) corresponding to \( \lambda \) is given by

\[
\text{so}_\lambda(g) = \begin{cases} 
sd_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) \leq n, \\
sd_{\lambda'}(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } \ell(\lambda) > n.
\end{cases}
\]

2. The Brauer algebra. In this section we give the definition and the basic facts about the Brauer algebras necessary for our study of the characters of the Brauer algebra. Most of these facts appear in [Wz1].

A diagram on \( f \) dots is given by two rows of \( f \) dots each and \( f \) edges which connect the \( 2f \) dots in pairs. The following is a diagram on 5 dots.

Let \( x \) be an indeterminate. Let \( d_1 \) and \( d_2 \) be two diagrams on \( f \) dots and let \( c \) denote the number of cycles created by placing \( d_1 \) directly above \( d_2 \) and attaching the lower dots of \( d_1 \) to the upper dots of \( d_2 \). The product \( d_1d_2 \) is \( x^c \) times the diagram on \( f \) dots resulting from this attachment. For example

The Brauer algebra \( D_f(x) \) is the \( \mathbb{C}(x) \) span of the diagrams on \( f \) dots where the multiplication is given by the linear extension of the product of diagrams. The dimension of \( D_f(x) \) is \( (2f - 1)(2f - 3) \cdots 5 \cdot 3 \cdot 1 \).
The diagram

on $f$ dots is the identity element of $D_f(x)$ which we shall denote by 1. For each $0 < i < f$ let $e_i$ and $g_i$ denote the diagrams

respectively. Note that the $g_i$ and the $e_i$, $1 \leq i \leq f - 1$, generate $D_f(x)$.

Any edge connecting two dots that are either both in the upper row or both in the lower row is called a horizontal edge. The group algebra of the symmetric group $C(x)S_f$ is a subalgebra of $D_f(x)$ in a natural way. Each permutation $\pi$ of $S_f$, the symmetric group, is identified with the diagram on $f$ dots which has edges connecting the $i$th dot of the lower row to the $\pi(i)$th dot of the upper row. In this way, any diagram on $f$ dots which has all dots in the lower row connected to dots in the upper row (i.e. no horizontal edges) is viewed as a permutation in $S_f$ and is invertible with inverse given by flipping the diagram from top to bottom. Note that the $g_i$, $1 \leq i \leq f - 1$, generate the symmetric group.

There is a natural embedding of $D_m(x) \otimes D_n(x)$ into $D_{m+n}(x)$. If $d$ is a diagram on $m$ dots and $d'$ is a diagram on $n$ dots then $d \otimes d'$ corresponds to the diagram on $m + n$ dots given by placing $d$ adjacent to $d'$. Let $d^{\otimes k}$ denote the diagram $d \otimes d \otimes \cdots \otimes d$ ($k$ factors).

For each complex number $\alpha$ in $\mathbb{C}$ one defines a Brauer algebra $D_f(\alpha)$ over $\mathbb{C}$ as the linear span of the diagrams on $f$ dots where the multiplication is given as above except with $x$ replaced by $\alpha$. $D_f(\alpha)$ is an algebra of dimension $1 \cdot 3 \cdot 5 \cdots (2f - 1)$.

The following theorems concerning the structure of the Brauer algebras are given in [Wz1] and [Wy].

**Theorem 2.1** (H. Wenzl, [Wz1]). $D_f(x)$ is a semisimple algebra over $\mathbb{C}(x)$.

**Theorem 2.2.** The irreducible representations of $D_f(x)$ are in-
indexed by partitions \( \lambda \) of \( f - 2k \), \( k = 0, 1, \ldots, \left\lfloor \frac{f}{2} \right\rfloor \).

**Theorem 2.3 ([Wz1])**. \( D_f(\alpha) \) is semisimple and has irreducible representations indexed by partitions \( \lambda \) of \( f - 2k \), \( k = 0, 1, \ldots, \left\lfloor \frac{f}{2} \right\rfloor \), for all but a finite number of \( \alpha \in \mathbb{C} \).

Let \( p_i \) be a partition of unity in \( D_f(x) \). Assume that each \( p_i \) is expressed in terms of the basis of diagrams on \( f \) dots. Each coefficient in this expansion is a rational function in \( x \). In this way it makes sense to consider the specialization \( p_i(\alpha) \), \( \alpha \in \mathbb{C} \) given by setting \( x = \alpha \). \( p_i(\alpha) \) will be well defined and nonzero for all but a finite number of \( \alpha \in \mathbb{C} \).

For all but a finite number of \( \alpha \in \mathbb{C} \) we will have that \( p_i(\alpha) \) form a partition of unity for \( D_f(\alpha) \).

For each \( \lambda \vdash f - 2k \) we shall denote the irreducible character of \( D_f(x) \) corresponding to \( \lambda \) by \( \chi_{(f,x)}^\lambda \). Similarly for each \( \alpha \) such that \( D_f(\alpha) \) is semisimple and has irreducible representations indexed by \( \lambda \vdash f - 2k \) we denote the irreducible character of \( D_f(\alpha) \) corresponding to \( \lambda \) by \( \chi_{(f,\alpha)}^\lambda \). The following corollary follows from the above remarks concerning a partition of unity in \( D_f(\alpha) \).

**Corollary 2.4.** For all but a finite number of \( \alpha \in \mathbb{C} \) the character \( \chi_{(f,\alpha)}^\lambda \) of \( D_f(\alpha) \) is given by evaluating the character \( \chi_{(f,x)}^\lambda \) of \( D_f(x) \) at \( x = \alpha \).

**3. Characters of** \( D_f(x) \). To each diagram \( d \) on \( f \) dots we associate a partition \( \tau(d) \), the type of \( d \), determined in the following manner. Connect each dot in the upper row of the diagram \( d \) to the corresponding dot in the lower row by a dotted line. Beginning with an arbitrarily chosen dot of \( d \) follow the path determined by the edges and the dotted lines and assign each edge a direction as it is transversed. Returning to the original dot completes a cycle. If not all edges have been transversed and a cycle is completed choose a dot connected to an edge which has not yet been assigned a direction and continue to follow the edges and dotted lines. Do this until all edges have been assigned a direction. We call the resulting graph a directed form of \( d \).

Assign to each cycle the absolute value of the difference between the number of edges in the cycle that are directed from top to bottom and the number of edges in the cycle that are directed from
bottom to top (horizontal edges and dotted lines are ignored). Note that a cycle may be assigned the number 0. This sequence of numbers determines the partition $\tau(d)$. $\tau(d)$ is a partition of $f - 2k$ for some integer $0 \leq k \leq \left\lfloor \frac{f}{2} \right\rfloor$. As an example, the diagram

![Diagram 1](image1)

has directed form given by

![Diagram 2](image2)

The type of $d$, $\tau(d) = (21)$.

The type of a diagram $d$ is analogous to the cycle type of a permutation. If $d$ has no horizontal edges then the type of $d$ is exactly the same as the cycle type of the permutation represented by the diagram $d$.

A cycle of length $k$ is a diagram on $k$ dots such that one dot in each column is connected to a dot in the next column (and a dot in the $k$th column is connected to a dot in the 1st column). For example

![Diagram 3](image3)

is a 10-cycle. Let $e$ denote the diagram

![Diagram 4](image4)

on 2-dots. Let $\gamma_k$ denote the diagram on $k$ dots given by the permutation

$$
\gamma_k = \begin{pmatrix} 1 & 2 & 3 & \cdots & k - 1 & k \\
2 & 3 & 4 & \cdots & k & 1
\end{pmatrix}.
$$

For each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ with all parts nonzero, let $\gamma_\mu$ denote the diagram $\gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_k}$.
If \( d \) and \( d' \) are diagrams on \( f \) dots, we say that \( d' \) is a conjugate of \( d \) if there exists some permutation \( \pi \) of \( S_f \) such that \( \pi d \pi^{-1} = d' \). Note that \( \pi d \pi^{-1} \) is the diagram given by rearranging both the upper dots of \( d \) and the lower dots of \( d \) according to the permutation \( \pi \).

One has the following easy facts.

1. If \( d \) and \( d' \) are diagrams on \( f \) dots and \( d \) and \( d' \) are conjugate and \( \chi \) is a character of \( D_f(x) \) then \( \chi(d) = \chi(d') \).
2. Any two diagrams which are conjugate have the same type.
3. If \( d = d_1 \otimes d_2 \otimes \cdots \otimes d_k \) then \( d \) is conjugate to \( d_{\pi(1)} \otimes d_{\pi(2)} \otimes \cdots \otimes d_{\pi(k)} \) for any permutation \( \pi \) of the \( k \) factors \( d_i \).
4. Every diagram \( d \) is conjugate to one which is a product \( d'' = c_1 \otimes c_2 \otimes \cdots \otimes c_k \) of cycles. To see this, let a permutation \( \pi' \) be given so that \( \pi'(j) = \mu(i) \) if, in the process of determining the type of the diagram \( d \), the \( i \)th edge to be assigned a direction is pointing to a dot in the \( j \)th column. Then \( d'' = \pi' d_{\pi'}^{-1} \) will be a product of cycles.
5. Any cycle of length \( k \) with no horizontal edges is conjugate to \( \gamma_k \).

**Theorem 3.1.** If \( d \) is a diagram on \( f \) dots and \( \chi \) is a character of \( D_f(x) \) then

\[
\chi(d) = \left( 1/x^h \right) \chi \left( e^{0k} \otimes \gamma_\mu \right)
\]

where \( \mu \) is the partition formed by nonzero parts of the type \( \tau = (0^{m_0}1^{m_1}2^{m_2}...) \) of the diagram \( d \), and \( k \) and \( h \) are given by

\[
k = (f - |\mu|)/2,
\]
\[
h = k - m_0.
\]

**Proof.** Every diagram \( d \) is conjugate to a diagram

\[
d' = c_1 \otimes c_2 \otimes \cdots,
\]

which is a product of cycles \( c_i \). Let \( c_j \) be a cycle in \( d' \) which is not \( e \) but that does have horizontal edges. Suppose that \( c_j \) has a horizontal edge connecting the \( i \)th and \( (i + 1) \)st dot of the upper row of \( d' \). Then \( xd' = e_i d' \) and \( d' e_i \) is conjugate to

\[
d'' = c_1 \otimes \cdots \otimes c_{j-1} \otimes e \otimes c'_j \otimes c_{j+1} \otimes \cdots
\]
where \( c'_j \) is a cycle and the length of the cycle \( c'_j \) is 2 less than the length of the cycle \( c_j \). For any character \( \chi \) of \( D_f(x) \) we have that

\[
(3.2) \quad \chi(d) = \chi(d') = (1/x) \chi(e_i d') = (1/x) \chi(d' e_i) = (1/x) \chi(d'').
\]

Note that the type of the cycle \( c'_j \) is the same as the type of the cycle \( c_j \).

Repeat this process with \( d'' \) in place of \( d' \) until all cycles with horizontal edges are of the form \( e \). Since any cycle of length \( r \) with no horizontal edges is conjugate to \( \gamma_k \) the resulting diagram is conjugate to a diagram

\[
d_\tau = e^{\otimes k} \otimes \gamma_{\tau_1} \otimes \gamma_{\tau_2} \otimes \cdots
\]

where the \( \tau_i \) are the types of the cycles with nonzero type in \( d' \). If \( \mu \) is the partition determined by the nonzero \( \tau_i \) then \( d_\tau \) is conjugate to the diagram \( e^{\otimes k} \otimes \gamma_{\mu} \). Since \( d_\tau \) is a diagram on \( f \) dots, \( k = (f - |\mu|)/2 \).

Each reduction from \( d' \) to \( d'' \) introduces a factor of \( 1/x \) in the computation of the character and decreases the length of the cycle \( c_j \) by 2. Let \( |c_j| \) denote the length of the cycle \( c_j \) in \( d \). If type \( \tau_j \) of \( c_j \) is not zero it takes \((|c_j| - \tau_j)/2\) reductions to reduce \( c_j \) to a cycle without horizontal edges. If \( \tau_j = 0 \) then it takes \((|c_j| - \tau_j)/2 - 1\) reductions to reduce \( c_j \) to be \( e \). Summing over all cycles gives

\[
h = \sum_{\tau_j > 0} (|c_j| - \tau_j)/2 + \sum_{\tau_j = 0} (|c_j| - \tau_j)/2 - 1
\]

\[
= (1/2) \left( \sum_j |c_j| - \sum_j \tau_j \right) - m_0
\]

\[
= (1/2)(f - |\tau|) - m_0
\]

\[
= k - m_0.
\]

Theorem (3.1) shows that any character on \( D_f(x) \) is completely determined by its values on diagrams of the form \( e^{\otimes k} \otimes \gamma_{\mu} \) where \( \mu \) is a partition of \( f - 2k \), \( 0 \leq k \leq \lfloor f/2 \rfloor \). From the structure of \( D_f(x) \) we know that the irreducible characters of \( D_f(x) \) are indexed by partitions \( \lambda \) of \( f - 2k \), \( 0 \leq k \leq \lfloor f/2 \rfloor \). This implies that the condition in Theorem (3.1) is not only a necessary condition but also a sufficient condition that a linear functional on \( D_f(x) \) be a character. This gives the following corollary.
**Corollary 3.3.** A linear functional \( t : D_f(x) \to \mathbb{C}(x) \) is a character of \( D_f(x) \) if and only if it satisfies the relations in Theorem (3.1) for all diagrams \( d \) on \( f \) dots.

**4. Schur-Weyl duality.** Let \( V \) be a vector space over \( \mathbb{C} \) with basis \( v_1, v_2, \ldots, v_n \). For each of the classical groups \( G \) define an action of \( G \) on \( V \) by

\[
Av_j = \sum_{i=1}^{n} v_i a_{ij},
\]

for each \( A = \|a_{ij}\| \in G \). This action defines the standard or fundamental representation \((\rho, V)\) of \( G \). The vector space \( V^\otimes f = V \otimes V \otimes \cdots \otimes V \), \((f \text{\ factors})\) has a basis given by the elements \( v_{i_1} v_{i_2} \cdots v_{i_f} \) (we omit the \( \otimes \) signs between the \( v_{i_j} \) for brevity in notation). Define an action of \( G \) on \( V^\otimes f \) by

\[
Av_{i_1} v_{i_2} \cdots v_{i_f} = (Av_{i_1})(Av_{i_2}) \cdots (Av_{i_f}),
\]

for all \( A \in G \). We denote this representation by \((\rho^\otimes f, V^\otimes f)\).

Define representations \( \pi_a, \pi_b, \) and \( \pi_c \), of \( S_f, D_f(n), \) and \( D_f(-n) \) \((n \text{ even})\), respectively, on \( V^\otimes f \) as follows.

(a) Define an action of the symmetric group \( S_f \) on \( V^\otimes f \) by defining, for all \( \sigma \in S_f \),

\[
v_{i_1} \cdots v_{i_f} \sigma = v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(f)}}.
\]

This defines a representation \( \pi_a \) of \( S_f \) on \( V^\otimes f \).

(b) Define operators \( e \) and \( g \) on \( V^\otimes 2 \) by

\[
v_i v_j g = v_j v_i, \text{ and } v_i v_j e = \delta_{ij} \sum_{k=1}^{n} v_k v_k,
\]

respectively. Then, for each \( j = 1, 2, \ldots, f - 1 \), define the action of \( g_j \) and \( e_j \) on \( V^\otimes f \) by

\[
v_{i_1} \cdots v_{i_f} g_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} g) v_{i_{j+2}} \cdots v_{i_f}, \text{ and}
\]
\[
v_{i_1} \cdots v_{i_f} e_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} e) v_{i_{j+2}} \cdots v_{i_f},
\]

respectively. This defines a representation \( \pi_b \) of \( D_f(n) \) on \( V^\otimes f \), where \( n = \dim V \).
(c) Suppose that $n = \dim V$ is even, $n = 2m$, and let $i' = m + z$, for $1 \leq i \leq m$. Let

$$
\epsilon_{ij} = \begin{cases} 
1 & \text{if } j = i'; \\
-1 & \text{if } i = j'; \\
0 & \text{otherwise.}
\end{cases}
$$

Define operators $e$ and $g$ on $V \otimes^2$ by

$$
v_i v_j g = -v_j v_i, \text{ and } v_i v_j e = -\epsilon_{ij} \sum_{k=1}^{m} v_k v_{k'} - v_{k'} v_k.
$$

respectively. Then, for each $j = 1, 2, ..., f - 1$, define the action of $g_j$ and $e_j$ on $V \otimes^f$ by

$$
v_{i_1} \cdots v_{i_j} g_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} g) v_{i_{j+2}} \cdots v_{i_f}, \text{ and }
$$

$$
v_{i_1} \cdots v_{i_j} e_j = v_{i_1} \cdots v_{i_{j-1}} (v_{i_j} v_{i_{j+1}} e) v_{i_{j+2}} \cdots v_{i_f},
$$

respectively. This defines a representation $\pi_c$ of $D_f(-2m)$ on $V \otimes^f$.

**Theorem 4.3.** (a) (Schur [Sc1, Sc2]) Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(Gl(n))$ in $\text{End}(V \otimes^f)$. Then $\mathcal{A}$ and $\pi_a(\mathcal{CS}_f)$ are the full centralizers of each other in $\text{End}(V \otimes^f)$.

(b) (Brauer [Br]). Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(O(n))$ in $\text{End}(V \otimes^f)$. Then $\mathcal{A}$ and $\pi_b(D_f(n))$ are the full centralizers of each other in $\text{End}(V \otimes^f)$.

(c) (Brauer [Br]). Let $\mathcal{A}$ be the algebra generated by $\rho^{\otimes f}(Sp(2m))$ in $\text{End}(V \otimes^f)$. Then $\mathcal{A}$ and $\pi_c(D_f(-2m))$ are the full centralizers of each other in $\text{End}(V \otimes^f)$.

Since $\pi_a(\mathcal{CS}_f)$ and $\rho^{\otimes f}(Gl(n))$, $\pi_a \times \rho^{\otimes f}$ is a well defined representation of the group $S_f \times Gl(n)$ on $V \otimes^f$. Let $D_f(n) \times O(n)$ denote the $\mathbb{C}$-algebra consisting of all $\mathbb{C}$-linear combinations of pairs $(d, A)$ where $d$ is a diagram on $f$ dots and $A \in O(n)$. The multiplication in $D_f(n) \times O(n)$ is the linear extension of componentwise multiplication of these pairs. Then, in view of part b) of Theorem (4.3), one has a well defined representation $\pi_b \times \rho^{\otimes f}$ of $D_f(n) \times O(n)$ on $V \otimes^f$. Define $D_f(-2m) \times Sp(2m)$ and a representation $\pi_c \times \rho^{\otimes f}$ of $D_f(-2m) \times Sp(-2m)$ on $V \otimes^f$ analogously.
Theorem 4.4 ([Sc1, Sc2, Wy]). (a) Let $S_\lambda$ denote the irreducible $S_f$ module corresponding to $\lambda$ and let $U_\lambda$ denote the irreducible $\text{Gl}(n)$ module corresponding to $\lambda$. Then, as $S_f \times \text{Gl}(n)$ representations,

$$V^{\otimes f} \cong \bigoplus_{\ell(\lambda) \leq n} S_\lambda \otimes U_\lambda.$$  

(b) Let $D_\lambda$ denote the irreducible $D_f(n)$ module corresponding to $\lambda$ and let $V_\lambda$ denote the irreducible $O(n)$ module corresponding to $\lambda$. Then, as $D_f(n) \times O(n)$ representations,

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\lambda_1'-f-2k, \lambda_2' + \lambda_2' \leq n} D_\lambda' \otimes V_\lambda.$$  

(c) Let $D_\lambda$ denote the irreducible $D_f(-2m)$ module corresponding to $\lambda$ and let $W_\lambda$ denote the irreducible $\text{Sp}(2m)$ module corresponding to $\lambda$. Then, as $D_f(-2m) \times \text{Sp}(2m)$ representations,

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\lambda_1'-f-2k, \ell(\lambda) \leq m} D_\lambda' \otimes W_\lambda.$$  

Remark. In the above Theorem we have chosen to index the irreducible representations of $D_f(n)$ and $D_f(-2m)$ in the same fashion as in [Wz1]. The indexing of representation of $O(n)$ and $\text{Sp}(2m)$ is as in [Wy] Chapter VII. This follows the usual convention. Note, however, that, using this labeling, $D_\lambda$ gets paired with $W_\lambda$ in part (c), where $\lambda'$ is the conjugate of the partition $\lambda$.

For the cases (b) and (c) in Theorem (4.3) one has that ([Wy], [Wz1]) for $n > 2f$ the representation of $D_f(n)$ (resp. $D_f(-2m)$, $n = 2m$) on $V^{\otimes f}$ is a faithful representation of $D_f(n)$ (resp. $D_f(-2m)$). $D_f(n)$ is semisimple for $n > 2f$. In particular the irreducible character $\chi^{(f,n)}_d$ of $D_f(n)$ is well defined for every partition $\lambda$ of $f - 2k$, $k = 0, 1, 2, \ldots$.

The following corollary is obtained from Theorem (4.4) by taking traces.

Corollary 4.5. (a) For $A \in \text{Gl}(n)$ and $h \in \mathbb{C}S_f$,

$$\text{Tr} \left( \rho^{\otimes f}(A) \pi_a(h) \right) = \sum_{\lambda' \vdash f} \chi^{\lambda}_{S_f}(h)s_{\lambda}(A),$$
where \( \chi^\lambda_{S_f} \) denotes the irreducible character of \( S_f \) corresponding to \( \lambda \).

(b) Let \( n > 2f \). Then for \( A \in O(n) \) and \( h \in D_f(n) \),
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_b(h) \right) = \sum_{\lambda \vdash f} \chi^\lambda_{(f,n)}(h)so_\lambda(A),
\]
where \( \chi^\lambda_{(f,n)} \) denotes the irreducible character of \( D_f(n) \) corresponding to \( \lambda \).

(c) Let \( n = 2m > 2f \). Then for \( A \in \text{Sp}(2m) \) and \( h \in D_f(-2m) \),
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_c(h) \right) = \sum_{\lambda \vdash f} \chi^\lambda_{(f,-2m)}(h)sp_\lambda(A),
\]
where \( \chi^\lambda_{(f,-2m)} \) denotes the irreducible character of \( D_f(-2m) \) corresponding to \( \lambda \).

Define the power symmetric functions as the following polynomials in \( \mathbb{Z}[x_1, x_2, ..., x_n] \). For each positive integer \( r \) define
\[
p_r(x_1, x_2, ..., x_n) = x_1^r + x_2^r + \cdots + x_n^r,
\]
and for a partition \( \mu = (\mu_1, \mu_2, ..., \mu_k) \) define
\[
p_\mu(x_1, x_2, ..., x_n) = p_{\mu_1}p_{\mu_2} \cdots p_{\mu_k}.
\]

Part (a) of the following theorem is due to Schur [Sc2]. Stanley [St] has also noticed (part (b)) that the trace \( \text{Tr} \left( \rho^{\otimes f}(A) \pi_b(d) \right) \) should be a power symmetric function.

**Theorem 4.6.** (a) Let \( A \in \text{Gl}(n) \) and let \( \sigma \in S_f \). Then
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_a(\sigma) \right) = p_\mu(x_1, x_2, ..., x_n),
\]
where \( \mu \) is the type of the permutation \( \sigma \) and \( x_1, x_2, ..., x_n \) are the eigenvalues of \( A \).

(b) Let \( A \in O(n) \) and let \( d \) be a diagram on \( f \) dots. Then
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_b(d) \right) = n^{m_0}p_\mu(x_1, x_2, ..., x_n),
\]
where \( \mu \) is the partition given by the nonzero parts of the type of the diagram \( d \), \( m_0 \) is the number of parts equal to 0 in the type of \( d \), and \( x_1, x_2, ..., x_n \) are the eigenvalues of \( A \).
(c) Let $A \in \text{Sp}(2m)$ and let $d$ be a diagram on $f$ dots. Then
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_c(d) \right) = (-2m)^{m_0} (-1)^{|\mu| - \ell(\mu)} p_\mu(x_1, x_2, \ldots, x_{2m}),
\]
where $\mu$ is the partition given by the nonzero parts of the type of the diagram $d$, $m_0$ is the number of parts equal to 0 in the type of $d$, and $x_1, x_2, \ldots, x_{2m}$ are the eigenvalues of $A$.

**Proof.** (a) By continuity it is sufficient to assume that the eigenvalues of $A$ are all distinct and thus we may assume that $A$ is diagonal. The remainder of the proof follows by an easy computation.

(b) From Theorem (3.1) we have that
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_b(d) \right) = \frac{1}{n^{k-m_0}} \text{Tr} \left( \rho^{\otimes f}(A) \pi_b(e^{\otimes k} \otimes \gamma_\mu) \right),
\]
where $\mu$ is the partition given by the nonzero parts in the type of the diagram $d$, $m_0$ is the number of 0 parts in the type $d$, and $k = (f - |\mu|)/2$. Let $e = e_1 \in D_2(x)$ and let $g \in O(n)$. By the definition of the action of $e$ we have
\[
\text{Tr} \left( \rho^{\otimes 2}(A) \pi_b(e) \right) = \sum_{1 \leq i, j \leq n} A v_i v_j e |v_i v_j = \sum_{i=1}^n \sum_{r=1}^n v_r v_r |v_i v_i = n.
\]
Since $A \in O(n) \subseteq \text{Gl}(n)$, part a) gives that
\[
\text{Tr} \left( \rho^{\otimes r}(A) \pi_b(\gamma_r) \right) = p_r(x_1, \ldots, x_n).
\]
The remainder of the proof follows from the fact that if $d \in D_m(x)$ and $d' \in D_n(x)$ then
\[
\text{Tr} \left( \rho^{\otimes (m+n)}(A) \pi_b(d \otimes d') \right) = \text{Tr} \left( \rho^{\otimes m}(A) \pi_b(d) \right) \text{Tr} \left( \rho^{\otimes n}(A) \pi_b(d') \right).
\]
We have
\[
\text{Tr} \left( \rho^{\otimes f}(A) \pi_b(d) \right) = \frac{1}{n^{k-m_0}} \text{Tr} \left( \rho^{\otimes f}(A) \pi_b(e^{\otimes k} \otimes \gamma_\mu) \right) = \frac{1}{n^{k-m_0}} n^k p_\mu(x_1, \ldots, x_n).
\]

(c) The proof is exactly as in the orthogonal case except that one has that for $A \in \text{Sp}(2m)$ and $e = e_1 \in D_2(-2m)$,
\[
\text{Tr} \left( \rho^{\otimes 2}(A) \pi_c(e) \right) = \sum_{1 \leq i, j \leq 2m} A v_i v_j e |v_i v_j = \sum_{i, j} -\epsilon_{ij} \sum_{k=1}^m v_k v_k' - v_k v_k' |v_i v_j = -2m,
\]
by (4.6), and that, by part (a),
\[ \text{Tr} \left( \rho^{\otimes r}(A) \pi_c(\gamma_r) \right) = (-1)^{r-1} p_r(x_1, \ldots, x_{2m}). \]

5. The irreducible characters of $D_f(x)$. Our developments in the previous section give us all necessary tools to derive a formula for the irreducible characters of $D_f(x)$ in terms of the characters of the symmetric group $S_f$.

**Theorem 5.1.** Let $\lambda$ be a partition of $f - 2k$ and let $d$ be a diagram on $f$ dots of type $\mu \vdash f - 2h$. Then the irreducible character of $D_f(x)$ corresponding to $\lambda$ is given by

\[ \chi_{(f,x)}^\lambda(d) = x^h \sum_{\nu \vdash f - 2h} \left( \sum_{\beta \text{ even}} c_{\lambda \beta}^\nu \right) \chi_{S_{f-2h}}^\nu(\gamma_\mu). \]

**Proof.** Let $n = 2m + 1$, $n \geq 2j$, and let $d = e^{\otimes h} \otimes \gamma_\mu$. Combining Theorem (4.6)(b), Theorem (4.5)(b), and Theorem (1.2)(b), we have

(1) $n^h p_\mu(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}, 1)$

\[ = \sum_{\lambda \vdash f - 2k} \chi_{(f,n)}^\lambda(d) s_{b_\lambda}(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}), \]

and similarly by Theorem (4.6)(a), Theorem (4.5)(a), and Theorem (1.2)(a),

(2) $n^h p_\mu(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}, 1)$

\[ = n^h \sum_{\lambda \vdash f - 2h} \chi_{S_{f-2h}}^\lambda(\gamma_\mu) s_{a_\nu}(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}, 1). \]

The branching rule (1.1) gives

(3) $s_{a_\nu}(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}, 1) = \sum_{\lambda \subseteq \nu} \left( \sum_{\beta \text{ even}} c_{\lambda \beta}^\nu \right) s_{b_\lambda}(x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}).$

Setting (1) and (2) equal and using (3) to expand the $s_{a_\lambda}$ in terms of the $s_{b_\lambda}$,

\[ n^h p_\mu = \sum_{\lambda} \chi_{(f,n)}^\lambda(d) s_{b_\lambda} \]

\[ = n^h \sum_{\nu \vdash f - 2h} \chi_{S_{f-2h}}^\nu(\gamma_\mu) \sum_{\lambda \subseteq \nu} \left( \sum_{\beta \text{ even}} c_{\lambda \beta}^\nu \right) s_{b_\lambda}. \]
Since the $sb_\lambda$ are algebraically independent ($n > 2f$) we can equate coefficients of $sb_\lambda$ to get that

$$\chi_{(f,n)}^\lambda(d) = n^h \sum_{\nu \geq \lambda } \left( \sum_{\beta \text{ even}} c_{\nu,\beta}^\lambda \right) \chi_{S_{f-2h}}^\nu(\gamma_\mu).$$

This identifies the irreducible character $\chi_{(f,n)}^\lambda$ of $D_f(n)$ for all odd $n > 2f$.

Let

$$ch(x) = x^h \sum_{\nu \geq \lambda \geq \nu + f - 2h} \left( \sum_{\beta \text{ even}} c_{\nu,\beta}^\lambda \right) \chi_{S_{f-2h}}^\nu(\gamma_\mu).$$

Then,

$$ch(n) = \chi_{(f,n)}^\lambda(d) = \chi_{(f,x)}^\lambda(d)|_{x=n}$$

for infinite number of $n \in \mathbb{Z}$, where the first equality follows from (4) and the second from Theorem (2.4). Since both $ch(x)$ and $\chi_{(f,x)}^\lambda(d)$ are rational functions in $x$ and they are equal at an infinite number of points they must be equal everywhere. \hfill \Box

**Corollary 5.2.** If $|\lambda| < f$ and $d$ is a diagram on $f$ dots of the form $e^{\otimes h} \otimes \gamma_\mu$ with $|\mu| < f$ then

$$\chi_{(f,x)}^\lambda(d) = x \chi_{(f-2,x)}^\lambda(d'),$$

where $d' = e^{\otimes h-1} \otimes \gamma_\mu$.

**Proof.** Let $|\mu| = k$. Then

$$\chi_{(f,x)}^\lambda(d) = x^h \sum_{\nu \geq \lambda \geq \nu + k \geq \nu + f - 2h} \left( \sum_{\beta \text{ even}} c_{\nu,\beta}^\lambda \right) \chi_{S_k}^\nu(\gamma_\mu),$$

$$\quad = x \left( x^{h-1} \sum_{\nu \geq \nu + k \geq \nu \geq \nu + f - 2h} \left( \sum_{\beta \text{ even}} c_{\nu,\beta}^\lambda \right) \chi_{S_k}^\nu(\gamma_\mu) \right),$$

$$\quad = x \chi_{(f-2,x)}^\lambda(d').$$

\hfill \Box
COROLLARY 5.3. If \( \lambda \vdash f \) and \( d \) is a diagram on \( f \) dots of the form \( e^{\otimes h} \otimes \gamma_\mu \) with \( |\mu| < f \) then

\[ \chi^\lambda_{(f,x)}(d) = 0. \]

Proof. Since \( \lambda \) is a partition of \( f \) and \( |\mu| < f \), we have \( c^\mu_{\lambda\beta} = 0 \) for all \( \beta \).

COROLLARY 5.4. If \( \lambda \vdash f \) and if \( d \) is a diagram on \( f \) dots of the form \( \gamma_\mu \) with \( \mu \vdash f \), then

\[ \chi^\lambda_{(f,x)}(d) = \chi^\lambda_{S_f}(\gamma_\mu). \]

Proof.

\[ c^\nu_{\lambda\beta} = \begin{cases} 1, & \text{if } \lambda = \nu \text{ and } \beta = \emptyset, \\ 0, & \text{otherwise,} \end{cases} \]

giving that

\[ \chi^\lambda_{(f,x)} = x^0 \sum_{\nu \vdash f - 2h, \nu \subseteq \lambda} \left( \sum_{\beta \text{ even}} c^\nu_{\lambda\beta} \right) \chi^\nu_{S_f}(\gamma_\mu) = \chi^\lambda_{S_f}(\gamma_\mu). \]

Let \( \Xi_f \) denote the character table of \( D_f(x) \), i.e. \( \Xi_f \) is a matrix with rows and columns indexed by partitions of \( f - 2k, 0 \leq k \leq \lfloor f/2 \rfloor \), and the entry in the \( \lambda \)th row and the \( \mu \)th column of \( \Xi_f \) is \( \chi^\lambda_{(f,x)}(e^{\otimes h} \otimes \gamma_\mu) \) where \( h = (f - |\mu|)/2 \). We can summarize the results of Corollaries (5.2)-(5.4) by observing that the character table \( \Xi_f \) of \( D_f(x) \) can be given in the form

\[ \Xi_f = \begin{pmatrix} x^{\Xi_{f-2}} & \ast \\ 0 & \Xi_{S_f} \end{pmatrix}, \]

where \( \Xi_{f-2} \) is the character table of \( D_{f-2}(x) \) and \( \Xi_{S_f} \) is the character table of the symmetric group \( S_f \). More specifically, the character table of \( D_f(x) \) can be given in block upper triangular form where the diagonal blocks are of the form \( x^k \Xi_{S_{f-2k}}, 0 \leq k \leq \lfloor f/2 \rfloor \).
6. A combinatorial rule for computing the characters of $D_f(x)$. In this section we analyze the combinatorics of formula (1) in the proof of Theorem (5.1). We begin by reviewing the basic properties of alternating and symmetric functions for the hyperoctahedral group. Note that these basic results hold for any Weyl group, see [Bou]. We shall treat only the special case of the hyperoctahedral group in the following.

Define the hyperoctahedral group $B_n$ as the group of $n \times n$ matrices $w = (w_{ij})$ such that

1. $w_{ij} \in \{0, 1, -1\}$, for each $1 \leq i, j \leq n$, and
2. the matrix $(|w_{ij}|)$ is a permutation matrix.

The symmetric group $S_n$ is a subgroup of $B_n$. For each $w \in B_n$ define the sign of $w$ by $\epsilon(w) = \det(w)$. As $B_n \subset M_n(\mathbb{C})$, there is a natural action of elements of $B_n$ on elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$. For each $\alpha \in \mathbb{Z}^n$ let $x^\alpha$ denote the monomial $x^{\alpha} = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$. Define an action of elements of $B_n$ on monomials in the variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ by defining $wx^{\alpha} = x^{w\alpha}$.

Let $1/2 + \mathbb{Z}$ denote the set $\{1/2 + p, p \in \mathbb{Z}\}$ of half-integers and set

$$\delta = (n - 1/2, n - 3/2, \ldots, 3/2, 1/2).$$

For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in (1/2 + \mathbb{Z})^n$ define

$$b_\alpha = \sum_{w \in B_n} \epsilon(w)wx^{\alpha}.$$

Then, if $\lambda$ is a partition $\ell(\lambda) \leq n$,

$$sb_\lambda = \frac{b_{\lambda + \delta}}{b_\delta}. \tag{6.1}$$

The polynomial $b_\alpha$ is skew symmetric under the action of $B_n$, i.e.,

$$w(b_\alpha) = \epsilon(w)b_\alpha = b_{w\alpha}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (1/2 + \mathbb{Z})^n$. Let $m$ be the number of $\alpha_i < 0$ in $\alpha$ and let $|\alpha|$ denote the vector $(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|)$. Let $\text{Re}(|\alpha|)$ denote the sequence given by rearranging the parts of $|\alpha|$ in decreasing order and let $\pi$ denote the permutation such that $\pi(|\alpha|) = \text{Re}(|\alpha|)$. Let $\lambda = \text{Re}(|\alpha|) - \delta$. Since $b_\alpha$ is skew symmetric under the action
of $B_n$, $b_{\text{Re}(|\alpha|)} = 0$ unless $|\alpha_1|, |\alpha_2|, ..., |\alpha_n|$ are different. If the $|\alpha_i|$ are all positive and distinct then $\lambda$ is a partition. We have that

$$b_\alpha = (-1)^m b_{|\alpha|} = (-1)^m \epsilon(\pi) b_{\text{Re}(|\alpha|)}$$

$$= \begin{cases} 
(-1)^m \epsilon(\pi) b_{\lambda+\delta}, & \text{if } \lambda \text{ is a partition}, \\
0, & \text{otherwise}.
\end{cases}$$

In the standard fashion (see [Mac]) we shall associate to each sequence of positive integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ a diagram consisting of $n$ rows of boxes such that row $i$ contains $\alpha_i$ boxes. Let $e_i$ denote the vector $(0, ..., 0, 1, 0, ..., 0)$ where the 1 appears in the $i$th entry. Let $\mu = (\mu_1, \mu_2, ..., \mu_n)$ be a partition. Let

$$\lambda = \text{Re}(|\mu + \delta + re_i|) - \delta.$$ 

We say that the sequence $\lambda$ is given by adding $r$ boxes along the boundary of the diagram of $\mu$, beginning in row $i$ and continuing in rows $< i$.

Let

$$\nu = \text{Re}(|\mu + \delta - re_i|) - \delta.$$ 

We say that the sequence $\nu$ is given by removing a slinky of length $r$ from $\mu$ beginning at the row $i$. Pictorially, the diagram of $\nu$ is given by removing $r$ boxes from the boundary of the diagram of $\mu$, beginning with the last box in row $i$ and continuing in rows $> i$ as in diagram (6.4)(a). It may happen that $r$ is large enough that not all $r$ boxes are removed before reaching the $\ell(\mu)$th row. In this case one proceeds by continuing to remove boxes from an imaginary wall of height $n$ adjacent to $\lambda$. In this case $\nu$ will be of the form $(*, *, ..., *, -1, -1, ..., -1, 0, ..., 0)$ where the * entries are positive integers. Pictorially $\nu$ is given as in diagram (6.4)(b). If, in the process of removing boxes, the height $n - 1$ in the imaginary wall is reached and still $r - 1$ boxes have not been removed, then
one begins placing boxes, first in the holes in the wall, then along the boundary of the shape, until a total of \( r - 1 \) boxes have either removed or placed, diagram (6.4)(c).

(6.4)

\[ \text{(a)} \quad \text{(b)} \quad \text{(c)} \]

It will be clear that the sequence \( \nu \) given by removing a slinky of length \( r \) at row \( i \) is given by this diagram from the proof of Lemma (6.6).

Represent the monomial \( x^{\mu+\delta} \) by the diagram

(6.5)

Numbering the rows from the bottom to top, row \( i \) contains \( \delta_i = n - i + 1/2 \) boxes to the left of the vertical bar and \( \mu_i \) boxes to the right of the vertical bar. One can view the action of \( B_n \) on monomials as an action of \( B_n \) on these diagrams. Let \( s_i \) denote the transposition \((i, i + 1)\) and let \( s_0 \) denote the element of \( B_n \) given by the matrix \((w_{ij})\), where \( w_{ij} = 0 \) for \( 1 \leq i \leq n - 1 \) and \( w_{nn} = -1 \).
LEMMA 6.6. Let μ be a partition and r > 0.

(a) Let λ be the sequence given by adding a slinky of length r to μ at row i. Let k be the number of rows spanned by the slinky. Then

$$
\sum_{w \in B_n} \epsilon(w)wx_i^r x_i^\mu + \delta = \begin{cases} (-1)^{k-1} b_{\lambda+\delta}, & \text{if } \lambda \text{ is a partition,} \\ 0, & \text{otherwise.} \end{cases}
$$

(b) Let ν be the sequence given by removing a slinky of length r from μ beginning at row i. Let k be the number of rows in the slinky. Then

$$
\sum_{w \in B_n} \epsilon(w)wx_i^{-r} x_i^\mu + \delta = \begin{cases} (-1)^{k-1} b_{\nu+\delta}, & \text{if } \nu \text{ is a partition,} \\ 0, & \text{otherwise.} \end{cases}
$$

Proof. (a) The diagram representing the monomials $x_i^r x_i^\mu + \delta$ and $s_i^{-1} x_i^r x_i^\mu + \delta$ looks like

In some sense the factors from $x_i^r$ have "slinkyed" one row down the shape of ν. Let 1 ≤ j ≤ i be the greatest j ≤ i such that in the diagram of $s_j s_{j+1} \cdots s_{i-2} s_{i-1} x_i^r x_i^\mu + \delta$ the number of boxes in the j row is less than or equal to the number of boxes in the j + 1st row. Pictorally j is such that the diagram of $s_j s_{j+1} \cdots s_{i-2} s_{i-1} x_i^r x_i^\mu + \delta$ looks like that in (6.3) and the factors from $x_i^r$ have slinkyed down the shape of μ as far as possible.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be given by adding a slinky of length r to μ beginning at row i, as in (6.3). Setting $\pi = s_j \cdots s_{i-2} s_{i-1}$, we have that $\epsilon(\pi) = i - j = k - 1$, where k is the number of rows in the
slinky, and that \( x^{\lambda+\delta} = \pi x_i^r x^{\mu+\delta} \). Then

\[
\sum_{w \in B_n} \epsilon(w) wx_i^r x^{\mu+\delta} = \sum_{w \in B_n} \epsilon(w) wx^{\mu+\delta+r \epsilon_i} = \epsilon(\pi) \sum_{w \in B_n} \epsilon(w) wx^{\lambda+\delta} = \begin{cases} (-1)^{k-1} b_{\lambda+\delta} & \text{if } \lambda \text{ is a partition,} \\ 0 & \text{otherwise.} \end{cases}
\]

(b) \( x_i^{-r} x^{\mu+\delta} = x^{\mu+\delta-r \epsilon_i} \) and \( s_i x_i^{-r} x^{\mu+\delta} \) have diagrams of the form

\[
\[
\]

In this case rather than adding a slinky of length \( r \) we are removing a slinky of length \( r \) beginning in row \( i \). Let \( \lambda \) be the sequence given by removing a slinky of length \( r \) from \( \mu \) beginning at row \( i \). If \( \lambda \) is as in (6.4)(a) or (6.4)(b) let \( \pi = s_j s_{j-1} \cdots s_i \). If \( \lambda \) is as in (6.4)(c) then let \( \pi = s_j s_{j+1} \cdots s_{n-1} s_0 s_{n-1} \cdots s_i \). Then, in each case, we shall have that \( x^{\lambda+\delta} = \pi x_i^{-r} x^{\mu+\delta} \).

Let \( k = |j - i| + 1 \), so that \( k \) is the number of rows spanned by the slinky. The result follows as in (6.7).

**Remark.** Notice that in the case of Lemma (6.6)(b), if \( n \) is large \((n > r + \ell(\mu))\), then one has that \( \sum_{w \in B_n} \epsilon(w) wx_i^{-r} x^{\mu+\delta} = 0 \) unless the sequence \( \nu \) given by removing a slinky of \( r \) boxes from \( \mu \) is either

1. as in figure (6.4)(a), or
2. \( r \) is odd and \( n - 1 - i + 1 = (r - 1)/2 \), in which case \( \nu = \mu \).

These are the only cases for which \( \nu \) will be a partition.
Fix $n$. Let $\lambda$ and $\mu$ be partitions. Then we say that $\lambda$ differs from $\mu$ by an $r$-slinky if $\lambda$ is given by either adding or removing a slinky of length $r$ from $\mu$.

**Theorem 6.8.** Let $\mu$ be a partition. Then

$$p_r s\beta_\mu = s\beta_\mu + \sum_{\lambda} (-1)^{k(\lambda)-1} s\beta_\lambda,$$

where the sum is over all partitions $\lambda$ such that $\lambda$ differs from $\mu$ by an $r$-slinky and $k(\lambda)$ is a number of rows in this slinky. This expansion is independent of $n$ for $n > r + \ell(\mu)$.

**Proof.**

$$p_r(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1) b_{\mu+\delta}$$

$$= \left(1 + \sum_{i=1}^{n} x_i^r + x_i^{-r}\right) \left(\sum_{w \in B_n} \epsilon(w) x \mu+\delta\right)$$

$$= b_{\mu+\delta} + \sum_{i=1}^{n} \left(\sum_{w \in B_n} \epsilon(w) x_i^r x \mu+\delta + \sum_{w \in B_n} \epsilon(w) x_i^{-r} x \mu+\delta\right).$$

Using Lemma (6.6) we have that

$$p_r b_{\mu+\delta} = b_{\mu+\delta} + \sum_{\lambda} (-1)^{k(\lambda)-1} b_{\lambda+\delta},$$

where the sum is over all $\lambda$ that differ from $\mu$ by a slinky of length $r$ and $k(\lambda)$ is a number of rows in this slinky. The result follows by dividing by $b_\delta$.

The fact that this expansion is independent of $n$ for large $n$ follows from the remark following Lemma (6.6).

**Remark.** There is a slight subtlety in the definition of when $\lambda$ differs from $\mu$ by an $r$-slinky. Let us restate this in the language of border strips (connected skew diagrams with no $2 \times 2$ blocks of boxes), see [Mac, §3, Ex. 11]. Assume $n$ is large $n > r + \ell(\mu)$. Then $\lambda$ differs from $\mu$ by an $r$-slinky if either

1. $\mu \subseteq \lambda$ and $\lambda/\mu$ is a border strip of length $r$,
2. $\lambda \subseteq \mu$ and $\mu/\lambda$ is a border strip of length $r$, or
(3) $r$ is odd and $\lambda = \mu$.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a partition. Define a $\mu$-slinky tableau of shape $\lambda$ to be a sequence of partitions

$$T = (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(k)} = \lambda)$$

such that for each $i$ either $\lambda^{(i)} = \lambda^{(i+1)}$ or $\lambda^{(i+1)}$ differs from $\lambda^{(i)}$ by a $\mu_i$ slinky.

**Theorem 6.9.** Let $d$ be a diagram on $f$ dots of the form $e^{\otimes h} \otimes \gamma_\mu$. Then

$$\chi_{(f,x)}^\lambda(d) = x^h \sum_T \text{wt}(T),$$

where the sum is over all $\mu$-slinky tableaux $T$ of shape $\lambda$ and

$$\text{wt}(T) = \prod_{\text{slinkies in } T} (-1)^{\# \text{ of rows in slinky}-1},$$

where the product is over all slinkys in $T$.

**Proof.** By Corollary (5.2)

$$\chi_{(f,x)}^\lambda(e^{\otimes h} \otimes \gamma_\mu) = \chi_{(f-2h,x)}^\lambda(\gamma_\mu).$$

thus it is sufficient to prove the theorem for $h = 0$.

Let $h = 0$ and let $m = 2n + 1 > 2f + 1$ be odd. Then from the proof of Theorem (5.1) one knows that

$$\chi_{(f,x)}^\lambda = \chi_{(f,m)}^\lambda,$$

and further that

$$m^hp_\mu(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1) = \sum_{\lambda \vdash f-2k} \chi_{(f,m)}^\lambda(\gamma_\mu) sb_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}).$$

Thus, $\chi_{(f,x)}^\lambda$ is given by the coefficient of $sb_\lambda$ in the expansion of $\neg p_\mu(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1)$. This coefficient is given by repeated application of Theorem (6.8). In view of the fact that $n$ is large ($n > f \geq |\mu|$) this expansion is independent of $n$. The theorem follows. \qed
REMARK. J. Stembridge [Ste] has given a combinatorial rule for computing the characters of the hyperoctahedral group $B_n$ which involves placing and removing slinkys in much the same fashion as for the Brauer algebra. Although this may seem to be merely coincidence it seems that there is a deeper connection between the hyperoctahedral group and the Brauer algebra which is also reflected in the work of Hanlon and Wales [HW1-2].

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