ON BANACH SPACES Y FOR WHICH $B(C(\Omega), Y) = K(C(\Omega), Y)$

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Let \( \Omega \) be a compact Hausdorff space. In this paper we give some necessary conditions and some sufficient conditions on a Banach space \( Y \) in order that all continuous linear operators from \( C(\Omega) \) into \( Y \) are compact. We prove that for a nonscattered compact Hausdorff space \( \Omega \), for \( Y \) belonging to a large class of Banach spaces all operators from \( C(\Omega) \) into \( Y \) are compact if and only if all operators from \( l^2 \) into \( Y \) are compact.

Introduction. In this paper by the word “operator” we will mean a “continuous linear operator.” E. Dubinsky, A. Pelczynski, and H.P. Rosenthal [8] have given a characterization of all Banach spaces \( Y \) for which all operators from \( L_\infty \) into \( Y \) are absolutely 2-summing. Here, our aim is to characterize all Banach spaces \( Y \) for which all operators from a \( C(\Omega) \)-space into \( Y \) are compact. We noticed that such a characterization depends on whether the compact Hausdorff space \( \Omega \) is scattered (dispersed) or nonscattered (nondispersed). So we consider two cases separately.

Case 1: \( \Omega \) is an infinite scattered compact Hausdorff space. In this case, from some known results we deduce that all operators from \( C(\Omega) \) into a Banach space \( Y \) are compact if and only if all operators from a closed subspace of \( c_0 \) into \( Y \) are compact if and only if \( Y \) does not contain a copy of \( c_0 \).

Case 2: \( \Omega \) is a nonscattered compact Hausdorff space. In this case, we present a necessary condition on a Banach space \( Y \) for all operators from \( C(\Omega) \) into \( Y \) to be compact. Specifically, if each operator from \( C(\Omega) \) into \( Y \) is compact, then each operator from \( l^2 \) into \( Y \) is compact. Consequently, for a Banach space \( Y \) for which each operator from \( C(\Omega) \) into \( Y \) is absolutely 2-summing, each operator from \( C(\Omega) \) into \( Y \) is compact if and only if each operator from \( l^2 \) into \( Y \) is compact. Another necessary condition is given by
a theorem of T. Terzioglu. Namely, if each operator from $C(\Omega)$ into $Y$ is compact, then each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$. Next, we see that the above two necessary conditions together are also sufficient. Putting together: Each operator from $C(\Omega)$ into $Y$ is compact if and only if each operator from $l^2$ into $Y$ is compact and each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$.

In order to prove that another related condition is also sufficient we first generalize a theorem of N.J. Kalton. Then, employing this generalization, and a result of L. Drewnowski we prove: Each operator from $C(\Omega)$ into $Y$ is compact if and only if each operator from $l^2$ into $Y$ is compact and each operator from $C(\Omega)$ into $Y$ has a weak unconditional compact netted expansion (Definition 3.5). Consequently, for a Banach space $Y$ with an unconditional basis consisting of finite dimensional subspaces all operators from $C(\Omega)$ into $Y$ are compact if and only if all operators from $l^2$ into $Y$ are compact. The conclusion is that the class of all Banach spaces $Y$ for which all operators from $C(\Omega)$ into $Y$ are compact if and only if all operators from $l^2$ into $Y$ are compact is big (see Conclusion 3.12).

In the way we present a necessary and sufficient condition on a Banach space $Y$ for all operators from $l^p$ into $Y$ to be compact for each $p \in [1, \infty)$. We conclude this paper with some results that relate the space of all compact operators on $C(\Omega)$ with the space $\Phi_{c_0}(C(\Omega))$ for all operators factoring through $c_0$.

1. Notations. Suppose $X$ and $Y$ are Banach spaces. We will denote the space of all bounded linear operators, compact operators, and absolutely 2-summing operators from $X$ into $Y$ by $B(X, Y)$, $K(X, Y)$, and $\Pi_2(X, Y)$, respectively. By "$X \leftrightarrow Y$" we will mean "$Y$ contains a copy of $X$.

1.1. Scattered-Compact Spaces. Recall that a topological space $S$ is said to be scattered or dispersed if every nonempty closed subset of $S$ has an isolated point in its induced topology (see [22]). In this section we will assume that $S$ is a scattered compact Hausdorff space.

Proposition 1.1. Suppose $X$ is an infinite dimensional closed subspace of $c_0$ and $Y$ is a Banach space. Then, $B(X, Y) = K(X, Y)$
if and only if \( Y \) does not contain any copy of \( c_0 \).

Proof. Suppose \( Y \) does not contain any copy of \( c_0 \). Let \( T \in B(X,Y) \). Let \( \{x_n\} \) be any norm bounded sequence in \( E \). We will show that \( \{Tx_n\} \) has a norm convergent subsequence. Since \( c_0 \) does not contain any copy of \( l^1 \), the space \( E \) does not contain any copy of \( l^1 \). So by the celebrated \( l^1 \)-theorem of H.P. Rosenthal [20], a subsequence of \( \{x_n\} \) is weakly Cauchy. By passing to the subsequence we can assume that the \( \{x_n\} \) itself is weakly Cauchy. Let \( y_{m,n} = x_n - x_m \). Then the net \( \{y_{m,n}\} \) is weakly null. So is the net \( \{Ty_{m,n}\} \). We claim that \( ||Ty_{m,n}|| \to 0 \). To arrive at a contradiction suppose this is not the case. Then there exists an \( \epsilon > 0 \) and sequences \( \{m_k\} \) and \( \{n_k\} \) of natural numbers such that \( m_k > m_{k-1} \geq k -1, \ n_k > n_{k-1} \geq k -1 \), and \( ||Ty_{m_k,n_k}|| > \epsilon \). Now by a theorem of C. Bessaga and A. Pelczynski [4] a subsequence of \( Ty_{m_k,n_k} \) itself is a basic sequence. Since \( y_{m_k,n_k} \) is a weakly null sequence in \( c_0 \) such that \( \inf ||y_{m_k,n_k}|| > 0 \), a subsequence of this sequence is a basic sequence and a subsequence of the basic sequence is equivalent to a block basis of the standard basis of \( c_0 \). Since every normalized block basis of the standard basis is equivalent to the standard basis, it follows that a subsequence of \( \{y_{m_k,n_k}\} \) is equivalent to the standard basis of \( c_0 \). By passing to the subsequence we can assume that \( \{y_{m_k,n_k}\} \) itself is such a sequence. That is, \( \{y_{m_k,n_k}\} \) is equivalent to the standard basis of \( c_0 \). Now it is easy to verify that \( \sum a_k y_{m_k,n_k} \) converges if and only if \( \sum a_k Ty_{m_k,n_k} \) does. So, the subspace \( [Ty_{m_k,n_k}] \) of \( Y \) is isomorphic to \( c_0 \). This contradicts the hypothesis. The converse is obvious.  

The next result is a corollary of some known results and Proposition 1.1.

**Corollary 1.2.** For a Banach space \( Y \) the following are equivalent

(a) For all infinite scattered compact Hausdorff spaces \( S \), we have \( B(C(S),Y) = K(C(S),Y) \).

(b) For some infinite scattered compact Hausdorff space \( S \), we have \( B(C(S),Y) = K(C(S),Y) \).

(c) \( Y \) does not contain a copy of \( c_0 \).

(d) For all infinite dimensional subspaces \( X \) of \( c_0 \), we have
\[ B(X, Y) = K(X, Y). \]

(e) For some infinite dimensional subspace \( X \) of \( c_0 \), we have \( B(X, Y) = K(X, Y) \).

**Proof.** (a) \( \Rightarrow \) (b) This is obvious.

(b) \( \Rightarrow \) (c) By way of contradiction, suppose that \( Y \) contains a copy of \( c_0 \). Since \( S \) is an infinite scattered space, there exists a complemented subspace \( M \) of \( C(S) \) isomorphic to \( c_0 \) see [19, p. 201]). Let \( P \) be the projection of \( C(S) \) onto \( M \) and \( T \) be an isomorphism of \( M \) onto an isomorphic copy of \( c_0 \) in \( Y \). Then \( TP \in B(C(S), Y) \) is a noncompact operator. This contradiction proves (c).

(c) \( \Rightarrow \) (a) Let \( S \) be an arbitrary infinite scattered compact Hausdorff space. Let \( T \in B(C(S), Y) \) be arbitrary. Since \( Y \) does not contain any copy of \( c_0 \), by a result of A. Pelczynski [17], the operator \( T \) is weakly compact. So, its adjoint \( T^* : Y^* \rightarrow C(S)^* \) is weakly compact. By a well known theorem of W. Rudin [21], or (see [22, Corollary 19.7.7]), we have \( C(S)^* \cong l^1(S) \). By a theorem of Schur (see [22, p. 338]), the space \( l^1(S) \) has the Schur property. So, \( T^* \) is compact. Hence, \( T \) is compact.

(c) \( \Leftrightarrow \) (d) \( \Leftrightarrow \) (e) This is Proposition 1.1. \( \square \)

**Corollary 1.3 (Pitt).** For \( 1 \leq p < \infty \), we have \( B(c_0, l^p) = K(c_0, l^p) \).

**Proof.** We know that \( c_0 \cong C(S) \) for the infinite scattered compact Hausdorff space \( S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \). We also know that \( l^p \) does not contain any copy of \( c_0 \). So, by Corollary 1.2, we have \( B(c_0, l^p) = K(c_0, l^p) \). \( \square \)

### 1.2. \( l^p_w \)-Sequences.

This section gives a complete characterization of all Banach spaces \( Y \) (in terms of \( l^p_w \)-sequences) for which \( B(X, Y) = K(X, Y) \) for \( X = c_0 \) or \( l^p \) \((1 \leq p < \infty)\). The results for \( X = c_0 \) and \( l^2 \) are already known. We fill in the gap by giving the characterization in the case \( X = l^p \) for \( 1 \leq p < \infty \). This ties the results for \( c_0 \), \( l^2 \), and \( l^p \) \((p \neq 2)\) together.

Recall that a sequence \( \{y_n\} \) of elements in a Banach space \( Y \) is said to be a weak \( l^p \)-sequence, or in short an \( l^p_w \)-sequence in \( Y \), where \( p \in [1, \infty) \), if for every \( f \in Y^* \) we have \( \sum_{n=1}^{\infty} |f(y_n)|^p < \infty \). The set of all \( l^p_w \)-sequences of a Banach space \( Y \) is denoted by \( l^p_w(Y) \).
For any real number $p > 1$, we denote the number $p/(p-1)$ by $q$. Note that $1/p + 1/q = 1$.

**Remark.**
(a) If $\{y_n\} \in l^p_w(Y)$, $p \geq 1$, then $\{y_n\} \in l^r_w(Y)$ for any $r \geq p$.
(b) If $\{e_n\}$ is the standard unit vector basis of $l^p$, $1 < p < \infty$, then $\{e_n\} \in l^q_w(l^p)$.
(c) If $\{e_n\}$ is the standard unit vector basis of $c_0$, then $\{e_n\} \in l^1_w(c_0)$.

The next proposition is motivated by [3] and [4].

**Proposition 2.1.** If $\{y_n\}$ is a sequence in a Banach space $Y$ and $1 < p < \infty$, then the following three conditions are equivalent:

(a) The sequence $\{y_n\} \in l^p(Y)$.
(b) The series $\sum_{n=1}^{\infty} a_n y_n$ converges unconditionally for all $\{a_n\} \in l^q$.
(c) There exists an operator $T \in B(l^q, Y)$ such that $T e_n = y_n$, where $\{e_n\}$ is the standard unit vector basis of $l^q$.

**Proof.** (a) $\Rightarrow$ (b) We suppose that $\{y_n\} \in l^p_w(Y)$, that is, $\{f(y_n)\} \in l^p$ for each $f \in Y^*$. First define a linear operator $S : Y^* \rightarrow l^p$ by $Sf = \{f(y_n)\}$ for $f \in Y^*$. We will use the closed graph theorem to prove continuity of $S$. So suppose $\{f_n \oplus S f_n\}$ is a Cauchy sequence in the product space $Y^* \oplus l^p$. Then both $\{f_n\}$ and $\{S f_n\}$ are Cauchy sequences in $Y^*$ and $l^p$, respectively. Let $f_n \rightarrow f \in Y^*$. We will show that $S f_n \rightarrow S f$. For every $\varepsilon > 0$ there exists a natural number $n_0$ such that $\|S f_i - S f_j\| < \varepsilon$ for all $i, j > n_0$. That is, $\sum_{i=1}^{\infty} |f_i(y_n) - f_j(y_n)|^p < \varepsilon^p$ for all $i, j > n_0$. In particular, $\sum_{n=1}^{N} |f_i(y_n) - f_j(y_n)|^p < \varepsilon^p$, for all natural numbers $N$ and all natural numbers $i, j > n_0$. By letting $j \rightarrow \infty$ we get $\sum_{n=1}^{N} |f_i(y_n) - f(y_n)|^p \leq \varepsilon^p$. Since this holds for all natural numbers $N$ we get

$$\|S f_i - S f\|^p = \sum_{n=1}^{\infty} |f_i(y_n) - f(y_n)|^p \leq \varepsilon^p$$

for all $i > n_0$. So, $S f_n \rightarrow S f$ in norm. Hence, $S$ is continuous.

Now let $\{a_n\} \in l^q$ be arbitrary, $f \in Y^*$ be such that $\|f\| = 1$, and
\(i, j\) be any natural numbers. Then
\[
\left\| f \left( \sum_{n=1}^{j} a_n y_n \right) \right\| = \left\| \sum_{n=1}^{j} a_n f(y_n) \right\|
\]
\[
= \left| \{0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots\} S(f) \right|
\]
\[
\leq \left( \sum_{n=1}^{j} |a_n|^q \right)^{\frac{1}{q}} \|S\|,
\]
where \((0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots)\) is treated as an element of \((l^p)^*\).
So,
\[
\sup_{\|f\| \leq 1} \left| f \left( \sum_{n=i}^{j} a_n y_n \right) \right| \leq \left( \sum_{n=i}^{j} |a_n|^q \right)^{\frac{1}{q}} \|S\|.
\]
Since
\[
\sup_{\|f\| \leq 1} \left| f \left( \sum_{n=i}^{j} a_n y_n \right) \right| = \left\| \sum_{n=1}^{j} a_n y_n \right\|,
\]
we obtain
\[
(1) \quad \left\| \sum_{n=i}^{j} a_n y_n \right\| \leq \left( \sum_{n=i}^{j} |a_n|^q \right)^{\frac{1}{q}} \|S\|,
\]
for all natural numbers \(i, j\). Since \(\{a_n\} \in l^q\), \(\left( \sum_{n=i}^{j} |a_n|^q \right)^{\frac{1}{q}} \to 0\)
as \(n \to \infty\). So, \(\| \sum_{n=i}^{j} a_n y_n \| \to 0\) as \(n \to \infty\). Hence, the series \(\sum_{n=1}^{\infty} a_n y_n\) converges. Since \(\{a_n\} \in l^q\) implies \(\{\epsilon_n a_n\} \in l^q\), for any sequence \(\{\epsilon_n\}\) of numbers +1 and −1, we certainly have that the series \(\sum_{n=1}^{\infty} \epsilon_n a_n y_n\) converges. That is, the series \(\sum_{n=1}^{\infty} a_n y_n\) converges unconditionally in \(Y\).

(b) \(\Rightarrow\) (c) Define the operator \(T : l^q \to Y\) by \(T(\{a_n\}) = \sum_{n=1}^{\infty} a_n y_n\). Clearly, \(T\) is linear and \(T(\epsilon_n) = y_n\). We will prove that \(T\) is bounded. Let \(S\) be the bounded linear operator defined above. By letting \(i = 1\) and \(j \to \infty\) in (1), we obtain \(\| \sum_{n=1}^{\infty} a_n y_n \| \leq \|\{a_n\}\| \|S\|\). So, \(\|T\| \leq \|S\|\).

(c) \(\Rightarrow\) (a) Suppose \(T \in B(l^q, Y)\) and \(T(\epsilon_n) = y_n\), for \(n = 1, 2, \ldots\). We need to prove that \(\{y_n\} \in l^p_w(Y)\). Let \(f \in Y^*\) be arbitrary. Then \(\sum_{n=1}^{\infty} |f(y_n)|^p = \sum_{n=1}^{\infty} |f \circ T(\epsilon_n)|^p < \infty\), because \(f \circ T \in (l^q)^*\) and \(\{\epsilon_n\} \in l^p_w(l^q)\). \(\Box\)
REMARK. On replacing \( l^p \) by \( l^1 \) and \( l^q \) by \( c_0 \) in the statement of Proposition 2.1, we obtain a result of C. Bessaga and A. Pelczynski [4], whereas on replacing \( l^p \) by \( l^2 \) and \( l^q \) by \( l^2 \) we get a result given in the paper of R. Anantharaman and J. Diestel [3].

The next proposition is motivated by a paper of L. Drewnowski [7]. Part (c) of the proposition is well known and is included here for the sake of completeness.

**Proposition 2.2.** For a Banach space \( Y \) and an arbitrary \( 1 < p < \infty \), the following statements are true.

(a) The equality \( B(l^p, Y) = K(l^p, Y) \) holds if and only if every \( l^p \)-sequence in \( Y \) is a norm null sequence.

(b) The equality \( B(c_0, Y) = K(c_0, Y) \) holds if and only if every \( l^1 \)-sequence in \( Y \) is a norm null sequence.

(c) The equality \( B(l^1, Y) = K(l^1, Y) \) holds if and only if \( Y \) is of finite dimension.

**Proof.** (a) Suppose \( B(l^p, Y) = K(l^p, Y) \). Let \( \{y_n\} \) be an arbitrary \( l^p \)-sequence in \( Y \). By Proposition 2.1, there is an operator \( T \in B(l^p, Y) \) such that \( T(e_n) = y_n \) for all \( n = 1, 2, \ldots \), where \( \{e_n\} \) is the standard unit vector basis of \( l^p \). By way of contradiction, suppose that \( \{y_n\} \) is not norm null. So, there exists a subsequence, say \( \{y_{n_k}\} \), such that \( \|y_{n_k}\| > \epsilon \) for some \( \epsilon > 0 \) and for all \( k = 1, 2, \ldots \). Since \( \{e_{n_k}\} \) is a norm bounded sequence, and \( T \) is a compact operator, the sequence \( \{Te_{n_k}\} \), (i.e., \( \{y_{n_k}\} \)) has a norm convergent subsequence, say \( \{y_{n_{k_l}}\} \). Suppose \( y_{n_{k_l}} \overset{\|}{\rightharpoonup} y \in Y \). Then \( y_{n_{k_l}} \overset{w}{\rightarrow} y \) in \( Y \). Since \( \{y_n\} \) is an \( l^p \)-sequence, it is a weakly null sequence. So, \( y_{n_{k_l}} \overset{w}{\rightarrow} 0 \). Thus, \( y = 0 \). Hence, \( \|y_{n_{k_l}}\| \overset{\|}{\rightharpoonup} 0 \), a contradiction.

For the converse, suppose that every \( l^p \)-sequence of \( Y \) is a norm null sequence and take an arbitrary \( T \in B(l^p, Y) \). Let \( \{x_n\} \) be any norm bounded sequence in \( l^p \). We will show that \( \{T(x_n)\} \) has a norm convergent subsequence. Since \( l^p \) is reflexive, the sequence \( \{x_n\} \) has a weakly convergent subsequence. Without loss of generality we can assume that \( \{x_n\} \) itself is weakly convergent. Suppose \( x_n \overset{w}{\rightharpoonup} x \in l^p \). If \( \liminf \|x_n - x\| = 0 \), then \( \{x_n\} \) has a norm convergent subsequence, and consequently, \( \{T(x_n)\} \) has a norm convergent subsequence. So suppose that \( \lim \|x_n - x\| > 0 \). By the Bessaga-
Pelczynski theorem (see [6]), there exists a subsequence of \( \{x_n - x\} \) which is a basic sequence. Since \( \{x_n - x\} \) is a basic sequence in \( l^p \) and \( \liminf \|x_n - x\| > 0 \), by a theorem of A. Pelczynski [16, p. 7], there is a subsequence of \( \{x_n - x\} \), which is equivalent to a block basis of the standard basis of \( l^p \). Again by passing to a subsequence, we can assume that \( \{x_n - x\} \) itself is equivalent to a block basis of the standard basis. Since every block basis of the standard basis of \( l^p \) is equivalent to the standard basis (see [16]), \( \{x_n - x\} \) is equivalent to the standard basis. Since the standard basis is an \( l^p\)-sequence, \( \{x_n - x\} \) is an \( l^p\)-sequence. And so \( \{T(x_n - x)\} \) is an \( l^p\)-sequence. Consequently, by the hypothesis, \( \{T(x_n - x)\} \) is a norm null sequence. That is, \( Tx_n \rightarrow Tx \) in norm. In other words, for every norm bounded sequence \( \{x_n\} \) the sequence \( \{Tx_n\} \) has a norm convergent subsequence.

(b) Suppose \( B(c_0, Y) = K(c_0, Y) \). Let \( \{y_n\} \in l^1(Y) \) be arbitrary. By Proposition 2.1 there is an operator \( T \in B(c_0, Y) \) such that \( T(e_n) = y_n \). Note that \( \{y_n\} \) converges weakly to zero. So, every subsequence of it converges weakly to zero. Since \( T \) is compact, every subsequence of \( \{Te_n\} \) (i.e., of \( \{y_n\} \)) has a subsequence which converges to zero in norm. So, \( \{y_n\} \) itself converges to zero in norm.

For the converse, suppose that every \( l^p\)-sequence of \( Y \) converges in norm to zero. Notice that the standard unit vector basis \( \{e_n\} \) of \( c_0 \) is an \( l^p\)-sequence, which does not converge to zero in norm. So, \( Y \) does not contain any copy of \( c_0 \). Since \( c_0 \cong C(S) \), for some infinite scattered compact Hausdorff space \( S \), Corollary 1.2 implies that all operators from \( c_0 \) into \( Y \) are compact.

(c) This follows from the well known fact that every separable Banach space is a quotient of \( l^1 \).

NOTE 2.3. For the comparison we mention now the following result that follows from Corollary 3.11. If a Banach space \( Y \) has an unconditional basis of finite dimensional subspaces (or more generally, a weak unconditional compact netted expansion of identity), then \( B(l_\infty, Y) = K(l_\infty, Y) \) if and only if every \( l^2_w\)-sequence in \( Y \) is a norm null sequence.

**Corollary 2.4.** Suppose \( Y \) is a Banach space and suppose \( p \in [1, \infty) \). If \( B(l^p, Y) = K(l^p, Y) \), then

(a) \( B(l^r, Y) = K(l^r, Y) \) for all \( r \in [p, \infty) \) and
(b) \( B(c_0, Y) = K(c_0, Y) \).

Proof. (a) For \( p = 1 \) the result follows from Proposition 2.2(c). Suppose now that \( 1 < p \leq r < \infty \) and \( B(l^p, Y) = K(l^p, Y) \). Then by Proposition 2.2(a) every \( l^q_w \)-sequence of elements in \( Y \) converges to zero in norm. Since \( p \leq r \) implies that the conjugate number \( r' \) satisfies \( r' \leq q \), we see that every \( l^r'_w \)-sequence of elements in \( Y \) is an \( l^q_w \)-sequence. So, every \( l^r'_w \)-sequence of elements in \( Y \) converges to zero in norm. By Proposition 2.2(a), we get \( B(l^r, Y) = K(l^r, Y) \).

(b) Since \( B(l^p, Y) = K(l^p, Y) \) for some \( 1 \leq p < \infty \), the space \( Y \) does not contain any copy of \( c_0 \). Since \( c_0 \cong C(s) \), for some infinite compact scattered Hausdorff space, by Corollary 1.2 we get \( B(c_0, Y) = K(c_0, Y) \).

We conclude this section with the following remark.

Remark 2.5. For a Banach space \( Y \) the following are equivalent.
(a) For all infinite dimensional Hilbert spaces \( H \) we have \( B(H, Y) = K(H, Y) \).
(b) For some infinite dimensional Hilbert space \( H \) we have \( B(H, Y) = K(H, Y) \).
(c) We have \( B(l^2, Y) = K(l^2, Y) \).
(d) Every \( l^2_w \)-sequence in \( Y \) is a norm null sequence.

3. Nonscattered-Compact Spaces. Recall that a topological space \( \Omega \) is said to be nonscattered or nondispersed if \( \Omega \) contains a nonempty closed set which has no isolated point in its induced topology. In this section we assume that \( \Omega \) is a nonscattered compact Hausdorff space. We begin with a note whose proof is left to the readers.

Note 3.1. If \( Y \) is a Banach space with the Schur property, then \( B(C(\Omega), Y) = K(C(\Omega), Y) \).

Theorem 3.2. Let \( \Omega \) be a nonscattered compact Hausdorff space, \( Y \) be a Banach space. If \( B(C(\Omega), Y) = K(C(\Omega), Y) \), then \( B(l^2, Y) = K(l^2, Y) \). Furthermore, \( Y \) is a Banach space. If \( B(C(\Omega), Y) = K(C(\Omega), Y) \), then \( B(l^p, Y) = K(l^p, Y) \) for \( p \geq 2 \).

Proof. By Corollary 2.4 only the case \( p = 2 \) needs a proof. We proceed by contradiction and assume that \( B(l^2, Y) \neq K(l^2, Y) \). Then
there is a noncompact operator \( T \) in \( B(\ell^2, Y) \). From the proof of Proposition 2.2 it follows that there is a basic sequence \( \{u_n\} \) in \( \ell^2 \) equivalent to a block basis of the standard basis of \( \ell^2 \) such that \( \{Tu_n\} \) is an \( \ell^2_w \)-sequence with no norm convergent subsequence.

Now we will define a bounded linear operator \( \Psi(T) : C(\Omega) \to Y \) which is not compact. Since \( \Omega \) is a nonscattered compact Hausdorff space, by a theorem of A. Pelczynski, W. Rudin, and Z. Semedeni (see [22, Theorem 19.7.6]) there exists a purely nonatomic Borel probability measure \( \mu \) on \( \Omega \). Let \( \{r_n\} \) be a sequence of Rademacher-like functions in \( L^2(\mu) \). Then the sequence \( \{r_n\} \) is a basic sequence of orthonormal functions. Observe that since \( \mu \) is a regular Borel measure, for each function \( r_n \) and for each natural number \( k \) there exists an \( f_{nk} \in C(\Omega) \) such that \( \|f_{nk}\| = \sup \{|f_{nk}(w)| : w \in \Omega\} = 1 \) and \( \|f_{nk} - r_n\|_2 < \frac{1}{k} \). Let \( M \) be the closed subspace of \( L^2(\mu) \) spanned by the sequence \( \{r_n\} \) and the sequences \( \{f_{nk}\} \) for \( n = 1, 2, \ldots \). Let \( M_1 \) be the closed subspace of \( M \) spanned by the sequence \( \{r_n\} \) and \( M_0 \) be the orthogonal complement of \( M_1 \) in \( M \). Then \( M \) is the internal direct sum of \( M_1 \) and \( M_0 \) (i.e., \( M = \{x_1 + x_2 : x_1 \in M_1, x_2 \in M_2\} \) and \( \|x_1 + x_2\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}} \)). Let \( N \) be the closed linear subspace spanned by \( \{u_n\} \). We have

\[
C(\Omega) \xrightarrow{\Lambda} L^2(\mu) \xrightarrow{P} M \xrightarrow{I} M_1 \oplus M_0 \xrightarrow{J} N \xrightarrow{T|_N} Y,
\]

where \( \Lambda(f) = f = \) the equivalence class of \( f \) in \( L^2(\mu) \); the operator \( P \) is the orthogonal projection from \( L^2(\mu) \) onto \( M \); \( I \) is the identity map from \( M \) onto \( M_1 \oplus M_0 \); and \( J : M_1 \oplus M_0 \to N \) is the operator defined by \( J(r_n) = u_n \) for \( n = 1, 2, \ldots \) and \( J(x) = 0 \) for each \( x \in M_0 \). (Since \( \{u_n\} \) is a basic sequence in \( \ell^2 \), \( J \) is an isomorphism from \( M_1 \) onto \( N \).) Let \( \Psi(T) = T|_N JI \Lambda \). Clearly, \( \Psi(T) \) maps \( C(\Omega) \) into \( Y \). We claim that \( \Psi(T) \) is not compact. For this it is enough to show that \( \{Tu_n\} \subseteq \{\Psi(T)(f) : f \in C(\Omega) \text{ and } \|f\| = 1\} \). To this end, note that

\[
\|Tu_n - \Psi(T)f_{nk}\| = \|TJP_{n} - TJJ\Lambda f_{nk}\|
\leq \|T\| \|J_{P_{n}} - JP_{f_{nk}}\|
\leq \|T\| \|J\| \|P\| \frac{1}{k} \to 0 \quad \text{as } k \to \infty.
\]

\[\square\]
COROLLARY 3.3. If $Y$ is a Banach space such that $B(C(\Omega), Y) = \Pi_2(C(\Omega), Y)$, then $B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(l^2, Y) = K(l^2, Y)$.

Proof. In view of Theorem 3.2 we need only to prove that if $B(l^2, Y) = K(l^2, Y)$, then $B(C(\Omega), Y) = K(C(\Omega), Y)$. This follows from Remark 2.5 and the factorization theorem of A. Pietsch [18], which states that every absolutely 2-summing operator factors through a Hilbert space. □

COROLLARY 3.4. For any compact nonscattered Hausdorff space $\Omega$ and any Banach space $Y$, the following are equivalent.

(a) $B(C(\Omega), Y) = K(C(\Omega), Y)$.
(b) $B(l^2, Y) = K(l^2, Y)$ and each $T \in B(C(\Omega), Y)$ factors through a closed subspace of $c_0$.

Proof. (a)$\implies$ (b) This follows from a theorem of T. Terzioglu [24] (or see [1, Theorem 16.5]) and Theorem 3.2.
(b)$\implies$(a) Since $B(l^2, Y) = K(l^2, Y)$, $Y$ does not contain any copy of $c_0$. So, every operator from $c_0$ into $Y$ is compact. Now (a) is clear. □

To present Theorem 3.9 we need some discussion on the spaces of compact operators. Recall [11] that an operator $T \in B(X, Y)$ is said to have a unconditional compact expansion if there is a sequence $\{T_n\}$ of compact operators from $X$ into $Y$ such that for each $x \in X$ we have $Tx = \sum_{n=1}^{\infty} T_n x$, where the series converges unconditionally in $Y$. Recall also that $T$ is said to have a finite dimensional expansion if the operators $T_n$ are of finite rank. We shall now formulate the following definitions.

DEFINITION 3.5. An operator $T \in B(X, Y)$ is said to have a weak unconditional compact netted expansion if there is a net $\{T_\mu\}$ of compact operators from $X$ into $Y$ such that for each $x \in X$

$$Tx = \sum_{\mu} T_\mu x,$$

where the series converges weakly unconditionally in $Y$.

DEFINITION 3.6. A Banach space $B$ is said to have a weak unconditional compact netted expansion of identity if the
identity operator $I_B$ on $B$ has a weak unconditional compact netted expansion.

Recall that if $I_B$ in the above definition has an unconditional finite dimensional expansion, then $B$ is said to have an **unconditional finite dimensional expansion of identity**.

**REMARKS.** Suppose $T$ in $B(X, Y)$ factors through a Banach space $E$.

(a) If $E$ has a weak unconditional compact netted expansion of identity, then $T$ has a weak unconditional compact netted expansion.

(b) If $E$ has an unconditional finite dimensional expansion of identity, then $T$ has an unconditional finite dimensional expansion.

The part (a) of the next proposition is motivated by a result of N.J. Kalton [13] and is slightly more general than other known generalizations of the same result.

**Proposition 3.7.** Suppose $c_0$ does not embed in $K(X, Y)$ and $T \in B(X, Y)$.

(a) If $T$ has a weak unconditional compact netted expansion, then $T$ is compact.

(b) If $T$ has a weak unconditional compact netted expansion, then $T$ factors through a closed subspace of $c_0$.

**Proof.** (a) Let $\{T_\mu\}$ be a weak unconditional compact netted expansion of $T$. We claim that $\{T_\mu\}$ is an unconditional compact netted expansion of $T$. By way of contradiction suppose that for some $x \in B$ the series $\sum_\mu T_\mu x$ does not converge unconditionally. Then there exists an $\epsilon > 0$ and sequences $(F_n), (F'_n)$ of finite subsets of the index set such that for all $m$ and $n$ the sets $F_n$ and $F'_m$ are disjoint and

$$\left\| \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_m} \epsilon_\eta T_\eta x \right\| > \epsilon.$$  

for some choices of signs $\epsilon_\eta$. Set $y_n = \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_m} \epsilon_\eta T_\eta x$. Then, the series $\sum_n y_n$ converges weakly unconditionally Cauchy in $Y$ and $\inf \|y_n\| \geq \epsilon$. So, by a theorem of Bessaga and Pelczynski [4] the space $Y$ contains a copy of $c_0$. This contradicts the hypothesis.
Since the series \( \sum_{\mu} T_{\mu}x \) converges unconditionally for every \( x \in B \), by the uniform boundedness principle

\[
\sup \left\| \sum_{\mu \in F} T_{\mu} \right\| < \infty,
\]

where the supremum is taken over all finite subsets \( F \) of the index set \( M \). Equivalently, the series \( \sum_{\mu} T_{\mu} \) is weakly unconditionally Cauchy in \( K(X, Y) \). Since \( K(X, Y) \) does not contain any copy of \( c_0 \) by a theorem of Bessaga and Pelczynski [4], the series converges in norm. Clearly, it converges to \( T \).

(b) This is immediate from (a) and a theorem of T. Terzioglu [24]. \( \square \)

This completes the necessary discussion on the spaces of compact operators. The following theorem due to L. Drewnowski [7] will also be useful in the proof of Theorem 3.9. Here, the Banach space of all countably additive vector measures from the \( \sigma \)-algebra \( \Sigma \) into the Banach space \( Y \) is denoted by \( ca(\Sigma, Y) \).

**Theorem 3.8 (Drewnowski).** If a \( \sigma \)-algebra \( \Sigma \) admits an atomless probability measure, then for any Banach space \( Y \) the following statements are equivalent.

(a) \( l_\infty \hookrightarrow ca(\Sigma, Y) \).
(b) \( c_0 \hookrightarrow ca(\Sigma, Y) \).
(c) \( B(l^2, Y) \neq K(l^2, Y) \).

The following theorem gives another necessary and sufficient condition on a Banach space \( Y \) for all operators from \( C(\Omega) \) into \( Y \) to be compact.

**Theorem 3.9.** For any compact nonscattered Hausdorff space \( \Omega \) and any Banach space \( Y \) the following are equivalent.

(a) \( B(C(\Omega), Y) = K(C(\Omega), Y) \).
(b) \( B(l^2, Y) = K(l^2, Y) \) and each \( T \in B(C(\Omega), Y) \) has a weak unconditional compact netted expansion.

**Proof.** (a) \( \implies \) (b) We get the equality \( B(l^2, Y) = K(l^2, Y) \) from Theorem 3.2 and that each \( T \in B(C(\Omega), Y) \) admits a weak unconditional compact netted expansion is obvious.
(b) \implies (a) Since \( \Omega \) is nonscattered, by a theorem of A. Pelczynski, W. Rudin, and Z. Semadeni (see [22, p. 338]), it admits an atomless regular Borel probability measure. Since \( B(l^2, Y) = K(l^2, Y) \), by Theorem 3.8, it follows that \( c_0 \not\subset ca(\Sigma, Y) \), where \( \Sigma \) denotes the \( \sigma \)-algebra of all Borel subsets of \( \Omega \). Since \( K(C(\Omega), Y) \) is isometrically embeddable in \( ca(\Sigma, Y) \) (see [5, pp. 152–154]), \( c_0 \not\subset K(C(\Omega), Y) \). Now the conclusion follows from Proposition 3.7.

\textbf{Corollary 3.10.} If for some \( p \) with \( 1 \leq p \leq 2 \), \( B(l^p, Y) = K(l^p, Y) \) and each operator in \( B(C(\Omega), Y) \) has a weak unconditional compact netted expansion, then \( B(C(\Omega), Y) = K(C(\Omega), Y) \).

\textit{Proof.} This follows from Corollary 2.4 and Theorem 3.9.

Recall that a Banach space is said to be \textit{separably universal} if it contains an isometric copy of every separable Banach space. Recall also that for a compact Hausdorff space \( \Omega \) the space \( C(\Omega) \) is separably universal if and only if \( \Omega \) is nonscattered (see [14]). Note that if \( \mu \) is a regular Borel measure whose support is an infinite compact Hausdorff space, then there exists a nonscattered compact Hausdorff space \( \Omega' \) such that \( L^\infty(\mu) \cong C(\Omega') \). In particular, \( l^\infty \cong C(\Omega') \) for some nonscattered compact Hausdorff space \( \Omega' \).

\textbf{Corollary 3.11.} For any nonscattered compact Hausdorff space \( \Omega \), any Banach space \( Y \) with a weak unconditional compact netted expansion of identity, and any regular Borel measure \( \mu \) on a compact Hausdorff space the following statements hold.

(a) \( B(C(\Omega), Y) = K(C(\Omega), Y) \) if and only if \( B(l^2, Y) = K(l^2, Y) \).

(b) For any nonscattered compact Hausdorff space \( \Omega' \) we have \( B(C(\Omega), Y) = K(C(\Omega), Y) \) if and only if \( B(C(\Omega'), Y) = K(C(\Omega'), Y) \).

(c) \( B(C(\Omega), l^p) = K(C(\Omega), l^p) \) for \( 1 \leq p < 2 \).

(d) \( B(C(\Omega), l^p) \neq K(C(\Omega), l^p) \) for \( 2 \leq p < \infty \).

(e) \( B(L^\infty(\mu), l^p) = K(L^\infty(\mu), l^p) \) for \( 1 \leq p < 2 \).

(f) \( B(L^\infty(\mu), l^p) \neq K(L^\infty(\mu), l^p) \) for \( 2 \leq p < \infty \).

\textit{Proof.} (a) This follows from Theorem 3.2 and Theorem 3.9.

(b) This follows from (a).
(c) Since $1 \leq p < 2$, by a result of H.R. Pitt [16], we have $B(l^2, l^p) = K(l^2, l^p)$. We know that $l^p$ has a weak unconditional compact netted expansion of identity, so by (a) we get $B(C(\Omega), l^p) = K(C(\Omega), l^p)$.

(d) Since $2 \leq p < \infty$, we obviously have $B(l^2, l^p) \neq K(l^2, l^p)$. Now the conclusion follows from Theorem 3.2.

(e) follows from (c) and (f) follows from (d).

Parts (e) and (f) of Corollary 3.11 follow also from [19, Remark 2].

The following conclusion is clear from what we have proved so far.

**Conclusion 3.12.** Let $\Sigma(\Omega)$ denote the class of all Banach spaces $Y$ for which all operators from $C(\Omega)$ into $Y$ are compact iff all operators from $l^2$ into $Y$ are compact. Then, for a Banach space $Y$ the following statements hold.

(a) If $Y$ has an unconditional basis, then $Y \in \Sigma(\Omega)$.

(b) If $Y$ has an unconditional basis consisting of finite dimensional subspaces, then $Y \in \Sigma(\Omega)$.

(c) If $Y$ has a weak conditional compact netted expansion of identity, then $Y \in \Sigma(\Omega)$.

(d) If each operator from $C(\Omega)$ into $Y$ admits a weak unconditional compact netted expansion, then $Y \in \Sigma(\Omega)$.

(e) If each operator from $C(\Omega)$ into $Y$ factors through a closed subspace of $c_0$, then $Y \in \Sigma(\Omega)$.

(f) If each operator from $C(\Omega)$ into $Y$ is absolutely 2-summing, then $Y \in \Sigma(\Omega)$.

(g) If $Y$ has the Schur property, then $Y \in \Sigma(\Omega)$.

We conclude this section with a remark, whose proof is left to the reader.

**Remark.** In Theorem 3.8 the space $l^2$ can not be replaced by an $l^p$-space with $p \neq 2$.

**4. Factorization.** In this section $\Omega$ is any (scattered or nonscattered) compact Hausdorff space. Now we will use some of our earlier theorems to prove some results regarding the space $\Phi_{c_0}(C(\Omega))$ of all operators on $C(\Omega)$ factoring through $c_0$. 
Proposition 4.1. For an infinite compact Hausdorff space \( \Omega \), and for a closed subspace \( X \) of \( c_0 \) the following inclusions hold.

(a) \( \Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega)) \).

(b) \( K(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega)) \), but \( K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega)) \).

Proof. (a) Let \( T \in \Phi_X(C(\Omega)) \) be arbitrary and \( T = T_2T_1 \) be a factorization of \( T \) through \( X \). Since \( X \) is a closed subspace of \( c_0 \), by a theorem of J. Lindenstrauss and A. Pelczynski [15, Theorem 3.1], \( T_2 \) extends to a bounded linear operator \( \tilde{T}_2 \) from \( c_0 \) into \( C(\Omega) \). Clearly, \( T = \tilde{T}_2T_1 \in \Phi_{c_0}(C(\Omega)) \).

(b) Let \( T \in K(C(\Omega)) \) be arbitrary. Then by the theorem of T. Terzioglu [24], \( T \) factors through a closed subspace of \( c_0 \). Hence, by (a) \( T \in \Phi_{c_0}(C(\Omega)) \), (i.e., \( K(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega)) \)). To prove that \( K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega)) \) let us first suppose \( \Omega \) is scattered. Since \( \Omega \) is an infinite set, the space \( C(\Omega) \) contains a complemented subspace \( M \) isomorphic to \( c_0 \) (see [19, p. 201]). Let \( P : C(\Omega) \to M \) be a continuous projection onto \( M \), let \( M \to C(\Omega) \) be the inclusion map. Clearly, \( JP \) factors through \( c_0 \) and is noncompact. Now suppose \( \Omega \) is nonscattered. First note that there is a noncompact operator \( T \) in \( B(C(\Omega)) \). (For, otherwise our Theorem 3.2 would imply that \( B(l^2, c_0) = K(l^2, c_0) \). On the other hand, the formal identity map from \( l^2 \) to \( c_0 \) is not compact.) Now note that since \( \Omega \) is nonscattered there exists an isometry \( J \) in \( B(c_0, C(\Omega)) \). Clearly, \( JT \in \Phi_{c_0}(C(\Omega)) \) and \( JT \) is noncompact. \( \square \)

Theorem 4.2. For a compact Hausdorff space \( \Omega \) and for a separable Banach space \( X \) the following are equivalent.

(a) \( \Phi_X(C(\Omega)) \subseteq K(C(\Omega)) \).

(b) \( \Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega)) \), but \( \Phi_X(C(\Omega)) \neq \Phi_{c_0}(C(\Omega)) \).

Proof. (a) \( \implies \) (b) This is immediate from Proposition 4.1.

(b) \( \implies \) (a) First observe that \( c_0 \not\subseteq X \). For, otherwise since \( X \) is separable, a theorem of Sobczyk [23], would imply that an isomorphic copy of \( c_0 \) is complemented in \( X \). So, we would get \( \Phi_{c_0}(C(\Omega)) \subseteq \Phi_X(C(\Omega)) \), contrary to our assumption. To prove that \( \Phi_X(C(\Omega)) \subseteq K(C(\Omega)) \), it suffices to prove that \( B(C(\Omega), X) = K(C(\Omega), X) \). If \( \Omega \) is scattered, then \( B(C(\Omega), X) = K(C(\Omega), X) \) by Corollary 1.2. If \( \Omega \) is nonscattered, then \( C(\Omega) \) is separably universal. So, there is an isometry \( J : X \to C(\Omega) \). If \( T \in B(C(\Omega), X) \), then by our
hypothesis $JT \in \Phi_{c_0}(C(\Omega))$. So, suppose $JT = T_2T_1$ is a factorization through $c_0$. Note that $T_2 \in B(c_0, J(X))$ and $c_0 \cong C(S)$ for some scattered compact Hausdorff space $S$. Since $c_0 \not\rightarrow J(X)$, by Corollary 1.2 the operator $T_2$ is compact.

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Added in proof. After this paper was accepted for publication we learned that Corollary 1.2 (a)$\iff$(b)$\iff$(c) was already known. See Proposition 2 of the following paper.


References


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