FINITE GROUPS WITH A SPECIAL 2-GENERATOR PROPERTY

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This paper deals with finite groups. J. L. Brenner and James Wiegold defined a finite group $G$ as lying in $\Gamma_1^{(2)}$ if $G$ is nonabelian and for every $1 \neq x \in G$, either $x$ is an involution and $G = \langle x, y \rangle$ for some $y \in G$ or $x$ is not an involution and there is an involution $z \in G$ with $G = \langle x, z \rangle$. In this paper we expand the work of J. L. Brenner and James Wiegold, and that of Martin J. Evans in the investigation of which finite groups lie in $\Gamma_1^{(2)}$.

1. Introduction.

**Definition 1.1.** An element $x$ in a group is called a mate of an element $y \in G$ if $G = \langle x, y \rangle$.

**Definition 1.2.** We say that a finite group $G$ lies in $\Gamma_1^{(2)}$ if $G$ is nonabelian and for every $1 \neq x \in G$, either $x$ is an involution and $x$ has a mate $y \in G$ or $x$ is not an involution and $x$ has an involution $z \in G$ as a mate (see [3]).

Brenner and Wiegold [3] proved that $PSL(2, q)$ lies in $\Gamma_1^{(2)}$ except when $q = 9$, and that $PSL(n, q)$ does not lie in $\Gamma_1^{(2)}$ for $n \geq 3$ unless $n = 3$ and $q = 2$, or 4.

Evans [6] proved that if $G = Sz(2^{2n+1})$ is a Suzuki group, then $G$ lies in $\Gamma_1^{(2)}$; moreover if $G$ is a simple Chevalley group over a finite field $K$ of odd characteristic and $G$ lies in $\Gamma_1^{(2)}$, then $G \cong PSL(2, K)$.

Using the fact that each of the groups $S_p(2n, K)$, $P\Omega_2^+(K)$, and $P\Omega_2^-(K)$ act irreducibly on $V = K^{2n}$, and $PSU_m(K)$ acts irreducibly on $V = K^m$, we find an element $x$ of order greater than 2 that acts trivially on a “large enough” subspace, so that two conjugates of $x$ can not act irreducibly on $V$, if dimension of $V$ is large. We get in [7] that over a field $K$ of characteristic 2, $P\Omega_2^-(K)$, for $n \geq 4$ or $|K| > 2$, and $PSU_m(K)$ for $n > 3$, $S_p(2n, K)$ for $n \geq 3$, and $P\Omega_2^+(K)$ for $n \geq 5$, or $|K| > 2$, do not lie in $\Gamma_1^{(2)}$. Similarly we show in [7] that $\Sigma_n$ for $n \geq 5$ and $A_n$ for $n > 5$ do not lie in $\Gamma_1^{(2)}$.
In this paper we classify all those solvable groups that lie in $\Gamma_1^{(2)}$, and we show that a finite non-simple non-solvable group lies in $\Gamma_1^{(2)}$ if it is isomorphic to the semi-direct product of $N$ and $\langle x \rangle$ where $x$ is an involution and $N$ is a simple nonabelian group. Many simple groups are excluded from being candidates for the $N$ above. We also continue in the investigation of which simple groups lie in $\Gamma_1^{(2)}$.

2. A Preliminary look at $\Gamma_1^{(2)}$.

This section deals with general facts about groups which lie in $\Gamma_1^{(2)}$. We state Lemmas 2.1-2.6 but since they are obvious we omit the proofs.

Lemma 2.1. If $G$ is a group and $G = \langle x, y \rangle$ where $y$ is an involution and $x \neq 1$, then $H = \langle x, x^y \rangle$ is a normal subgroup of $G$.

Lemma 2.2. If $G$ is a nonabelian simple group and $G < M < \text{Aut}(G)$, and $M = \langle x, y \rangle$ where $1 \neq x \in G$, $y \in M$ and is an involution, then $G = \langle x, x^y \rangle$.

Corollary 2.3. If $G$ is simple and $G$ lies in $\Gamma_1^{(2)}$, then every conjugacy class of $G$ other than the classes of elements of order 1 or 2 contains a pair of conjugate elements which generate $G$.

Lemma 2.4. If $x$ and $y$ are conjugate in $G$ and $y$ has a mate, then $x$ has a mate.

Lemma 2.5. If $G$ lies in $\Gamma_1^{(2)}$, then $Z(G) = 1$.

Lemma 2.6. If $G$ lies in $\Gamma_1^{(2)}$ and $N$ is a nontrivial normal subgroup of $G$, then $G^* = G/N$ is cyclic.

Definition 2.7. A finite nonabelian group is called a $\Gamma$-group if $G = NP$, where $N$ is an elementary abelian normal 2-subgroup, and $P$ is a cyclic group of prime order acting irreducibly on $N$.

The last condition says that no proper subgroup of $N$ is $P$-invariant. Since $C_N(P)$ is $P$-invariant, $C_N(P) = 1$ or $C_N(P) = N$. Since $G$ is nonabelian $C_N(P) = 1$.

Remark 2.8. If $G$ is a $\Gamma$-group, then $p = |P|$ is an odd prime.

Remark 2.9. Given an odd prime $p$, there is a $\Gamma$-group $G$ of order divisible by $p$. It is the subgroup of $\text{Aff}(1,2^n)$, where $n$ is the unique positive integer such that $n|(p-1)$, and $p|(2^n-1)$ but $p$ does not divide $2^m-1$ for $0 < m < n$.

Lemma 2.10. A $\Gamma$-group $G = NP$ ($N$ and $P$ as in Definition 2.7) has the following properties:
1) \( G \) lies in \( \Gamma_1^{(2)} \).
2) \( N \) is the commutator subgroup of \( G \).

**Proof.** Let \( y \in G - \{1\} \) and \( P = \langle x \rangle \).

If \( y \in N - \{1\} \), then \( \langle y \rangle \langle x \rangle \) is a non-trivial \( P \)-invariant subgroup of \( N \) and \( \langle y \rangle \langle x \rangle = N \). Therefore, \( G = \langle x, y \rangle \). If \( y \in G - N \), then \( y \) is \( N \)-conjugate to a power of \( x \), so that any \( \langle y \rangle \)-invariant subgroup of \( N \) is also \( P \)-invariant. Thus, \( G = \langle y, n \rangle \) for any \( n \in N - \{1\} \), and \( G \) lies in \( \Gamma_1^{(2)} \). Since \( P \) is an abelian group, \( G' \leq N \) and \( G' = N \) because \( G' \) is \( P \)-invariant and nontrivial. \( \square \)

**Notation.** If \( y \in G \), then \( O(y) \) is the order of \( y \).

**Theorem 2.11.** If \( G \) lies in \( \Gamma_1^{(2)} \), then \( G \) has a proper normal subgroup of odd index if and only if it is a \( \Gamma \)-group.

**Proof.** By Lemma 2.10 any \( \Gamma \)-group lies in \( \Gamma_1^{(2)} \), and has a proper normal subgroup of odd index.

Assume that \( G \) has a proper normal subgroup \( K \) of odd index. Since \( |G/K| \) is odd and \( |G| \) is even, \( K \) is nontrivial; therefore, Lemma 2.6 gives that \( G/K \) is cyclic. Let \( M^* \) be a subgroup of \( G/K \) of odd prime index \( p \). If \( M \leq G \) such that \( M/K = M^* \), then \( M \) is normal in \( G \) and \( |G/M| = p \) is an odd prime.

If \( x \) is an involution in \( G \), then \( x \) is in \( M \) because \( O(xM) = 1 \). If \( y \) is an element of \( M - \{1\} \) with \( O(y) \neq 2 \), then \( y \) has a mate \( t \) of order 2. Thus, \( t \) is in \( M \). But \( G = \langle y, t \rangle \) gives that \( G = M \), a contradiction. Therefore, \( |M| = 2^n \) and \( M \) is an elementary abelian 2-group. Since, \( |G/K| \) is odd and \( K \leq M, M = K \).

Let \( x \) be an element of order \( p \) in \( G \), and set \( P = \langle x \rangle \). Then \( G \) is the semidirect product of \( M \) by \( P \), because \( M \cap P = \{1\} \) and \( G = MP \). If \( z \in G - M \), then \( O(zM) \neq 1 \) and so \( O(zM) = p \). Thus, \( p|O(z) \) so we see that \( O(z) = 2^kp \) for some \( k \). If \( t = z^p \), then \( t \in M \cap C_G(z) \); therefore \( t \) is in \( Z(G) \) and by Lemma 2.5, \( t = 1 \). Therefore, \( O(z) = p \).

Let \( N \) be a minimal \( P \)-invariant subgroup of \( M \), and \( d \) an element of \( N - \{1\} \). Now, \( d \) has a mate \( y \). Since \( G = \langle d, y \rangle \), \( y \) is not an element of \( M \). By the preceding paragraph \( O(y) = p \). Hence, \( \langle y \rangle \) is \( M \)-conjugate to \( P \), by Sylow's Theorem. Thus, \( \langle y \rangle^m = P \) for some \( m \in M \), and because \( M \) is abelian \( d^m = d \). It follows that

\[ G = \langle d, y \rangle = \langle d^m, y^m \rangle = \langle d, P \rangle = \langle N, P \rangle \text{ (because } d \in N \text{).} \]

Hence, \( G = NP \) and \( |G| = |N| \cdot |P| \). This implies that \( N = M \). Therefore \( P \) acts irreducible on \( M \) and \( G \) is a \( \Gamma \)-group. \( \square \)
Theorem 2.12. If $G$ lies in $\Gamma_1^{(2)}$ and $G \neq G'$, then either
1) $G$ is a $\Gamma$-group,
or
2) $G$ is isomorphic to a semi-direct product of $G'$ by $\langle y \rangle$, where $y$ is an involution. If, in this case, $G'$ is abelian, then $G$ is isomorphic to $D_{2p}$, where $p$ is an odd prime.

Proof. If $G^* = G/G'$, then $G^*$ is cyclic by Lemma 2.6.

If $G^*$ is not a 2-group, then there exists a normal subgroup $N$ of $G$ with $G' \leq N < G$ and $|G/N|$ is odd. Since $|G/N| > 1$ is odd, $G$ is a $\Gamma$-group and $G' = N$, by Lemma 2.10.

If $G^*$ is a 2-group, note that $G$ is not a 2-group, for $Z(G) = 1$. Therefore there exists an $x \in G'$ of odd prime order $p$. The element $x$ has a mate $y$ of order 2. Since $G = \langle x, y \rangle$, $G^* = \langle xG', yG' \rangle = \langle yG' \rangle$; therefore $|G^*| = 2$, $G' \cap \langle y \rangle = 1$, and $G$ is isomorphic to a semi-direct product of $G'$ and $\langle y \rangle$.

If $G'$ is abelian, then $y^2 = z$ for any $z \in G'$ if and only if $z = 1$, because $Z(G) = 1$. Hence, $y$ acts fixed point free on $G'$ and, since $y$ is an involution, $G'$ is of odd order and for all $z \in G' - \{1\}$ $z^y = z^{-1}$. Since $G = \langle x, y \rangle$, $G' = \langle x \rangle$ and $|G'| = p$. Therefore $G \cong D_{2p}$, for $|G| = 2p$ and $G = \langle x, y \rangle$, where $x^p = y^2 = 1$, and $x^y = x^{-1}$.

Lemma 2.13. Let $G$ lie in $\Gamma_1^{(2)}$. If $G$ is isomorphic to a semi-direct product of $N$ by $\langle x \rangle$, where $x$ is an involution, then $x$ acts on $N$ as an outer automorphism of $N$.

Proof. Assume $x$ acts on $N$ as conjugation by an element $y \in N$, then $xy^{-1}$ acts trivially on $N$. Therefore, $1 \neq xy^{-1}$ is in the center of $G$, contradicting Lemma 2.5.

Definition 2.14. A group is a proper semi-direct product of $N$ by $P$ if $G$ is a semi-direct product of $N$ by $P$ and $NC_G(N) \neq G$.

Note. The situation in Lemma 2.13 gives us an example of a proper semi-direct product, since $NC_G(N) = N \neq G$.

In [6] Evans shows that if a simple Chevalley group over a field of odd characteristic lies in $\Gamma_1^{(2)}$, then it is isomorphic to $PSL(2, K)$. Below we will present a generalization of this result.

Definition 2.15. 1) $\Delta$ denotes a root system of a Lie algebra.
2) $\Delta^+$ denotes the set of positive roots in $\Delta$.
3) $\Delta^-$ denotes the set of negative roots in $\Delta$. 

4) Π denotes the fundamental system of roots of Δ.
5) \(x_r(k)\) denotes the generator \(\exp(fade_r)\) of a Chevalley group.
6) \(X_s = \langle x_s(k) \mid k \in K \rangle\).
7) \(U = \langle x_r(k) \mid r \in \Delta^+, k \in K \rangle\).

Recall that each root \(r \in \Delta\) can be written as \(r = \sum k_\alpha \alpha (\alpha \in \Pi)\) with integral coefficients \(k_\alpha\) all nonnegative or all nonpositive [10, 10.1].

**Definition 2.16.** The height of \(r \in \Delta\) (relative to \(\Pi\)) is \(ht(r) = \sum k_\alpha\).

From the Steinberg decomposition of automorphisms of a finite simple Chevalley group \(M\) [4, Theorem 12.5.1], we get that if \(\tau \in \text{Aut}(M)\), then \(\tau = idgf\), where \(i\) is an inner automorphism, \(d\) is a diagonal automorphism, \(g\) is a graph automorphism, and \(f\) a field automorphism. From the Bruhat decomposition [4, Chapter 8] we get that if \(i \in \text{inn}(M) \cong M\) since \(M\) is a simple group, then \(i = u_1 h_1 n u_2 h_2\), where \(u_1, u_2 \in U, h_1, h_2 \in H\) and \(n_i \in N\). \(H\) and \(N\) denote the diagonal and monomial subgroups of \(M\) respectively.

So by combining the above remarks we get that every \(\tau \in \text{Aut}(M)\) can be written as

\[\tau = u_1 h_1 n u_2 h_2 d g f.\]

**Theorem 2.17.** Let \(M\) be a non-nilpotent subgroup of a finite simple Chevalley group \(G\) over a finite field \(K\), and \(M = \langle X_s, X_s' \rangle\), where \(y \in \text{Aut}(G)\) and \(s\) is a root of greatest height in \(\Delta^+\), then there is a homomorphism from \(\text{SL}(2, K)\) onto \(M\).

**Proof.** Let \(g\) be a graph automorphism, then since \(g(r) \in \Delta^+\) for all \(r \in \Delta^+\), \(g|_U\) and \(g|_{Z(U)}\) are automorphisms.

Since \(y \in \text{Aut}(M)\), it follows from the remarks above that

\[y = u_1 h_1 n u_2 h_2 d g f.\]

Let \(s' = g^{-1}(s)\), then by the choice of \(s\) we get that \(X_s\) and \(X_{s'}\) are central in \(U\) [4, Theorem 5.3.3], and that \(H\) [4, page 100], all diagonal automorphisms, and all field automorphisms normalize \(X_s\) and \(X_{s'}\). So we get that

\[M = \langle X_s, (X_s)^{u_1 h_1 n u_2 h_2 d g f} \rangle = \langle X_s, (X_s)^{n u_2 h_2 d g f} \rangle \cong \langle (X_s)^{f g^{-1} d^{-1} h_2^{-1} u_2^{-1}} \rangle = \langle (X_s)^{f^{-1} g^{-1} d^{-1} h_2^{-1} u_2^{-1}}, (X_s)^{n_i} \rangle = \langle X_{s'}, (X_s)^{n_i} \rangle.\]
Let \( w_t \) denote the image of \( n_t \) in the Weyl group under the natural homomorphism from \( N \) to \( N/H \cong W \). Since \( X_{s'} = X_{w_t(s)} \), \( M \cong \langle X_{s'}, X_{w_t(s)} \rangle \).

For all \( r, s \in \Delta \) which are linearly independent there exists a \( w \in W \) such that \( w(r), w(s) \in \Delta^+ \). Therefore \( \langle X_r, X_s \rangle \) is isomorphic to a subgroup of \( U \) which is nilpotent. Since \( M \) is not nilpotent, we get that \( w_t(s) = -s' \). So we see that \( M \cong \langle X_{s'}, X_{-s'} \rangle \) and by [4, Theorem 6.3.1] there is a homomorphism from \( SL(2, K) \) onto \( M \). \( \square \)

**Theorem 2.18.** Let \( M \) be a finite simple Chevalley group over a finite field \( K \) of odd characteristic \( p \), and \( M \leq G \leq \text{Aut}(M) \). If \( G \) lies in \( \Gamma_1^{(2)} \), then \( M \cong PSL(2, K) \).

**Proof.** Since \( G \) lies in \( \Gamma_1^{(2)} \) and \( O(x_s(1)) = p \neq 2 \) where \( s \) is a root of greatest height in \( \Delta^+ \), \( x_s(1) \) has a mate \( y \) of order 2. By Lemma 2.2 \( M = \langle x_s(1), x_s(1)^y \rangle \), therefore \( M = \langle X_s, X_y \rangle \). So by Lemma 2.16 we see that there is a homomorphism from \( SL(2, K) \) onto \( M \), and since \( M \) is simple \( M \cong PSL(2, K) \). \( \square \)

### 3. Non-Simple Groups that lie in \( \Gamma_1^{(2)} \).

An easy consequence of Lemma 2.6 and Theorem 2.12 is:

**Corollary 3.1.** A solvable group \( G \) lies in \( \Gamma_1^{(2)} \) if and only if \( G \cong D_{2p} \), where \( p \) is an odd prime, or \( G \) is a \( \Gamma \)-group.

**Theorem 3.2.** If \( G \) is a non-nilpotent group lying in \( \Gamma_1^{(2)} \), then either \( G \) is a nonabelian simple group or \( G \) is isomorphic to a proper semi-direct product of \( N \) by \( \langle x \rangle \), where \( x \) is an involution and \( N \) is a simple group.

**Proof.** Let \( M \) be a maximal normal subgroup of \( G \) and let \( G^* = G/M \).

**Case 1:** If \( M = \{1\} \), then \( G \) is simple.

**Case 2:** If \( M \neq \{1\} \), then \( G^* \) is cyclic, by Lemma 2.6, and \( G' \neq G \). Since, any \( \Gamma \)-group is solvable, by Theorem 2.12 and Lemma 2.13, \( G \) is isomorphic to a proper semi-direct product of \( G' \) by \( \langle x \rangle \), where \( x \) is an involution.

If \( \{1\} \neq N < G' \) and \( N \) is normal in \( G \), then \( G/N \) is cyclic, by Lemma 2.6. But \( G/N \) is a nonabelian group; a contradiction. Therefore \( G' \) is a nonabelian characteristically simple group (since \( G \) is not a solvable group); that is, \( G' \cong K_1 \times \ldots \times K_n \) where \( K_i \cong K_j \) are nonabelian simple groups. For \( k \in K_1 \) a nontrivial element of odd order, let \( y \) be its mate of order 2. By Lemma 2.1, \( H = \langle k, k^y \rangle \) is a nontrivial normal subgroup of \( G \), and \( H \leq G' \), thus \( H = G' \). Since, \( K_1 \) is a normal subgroup of \( G' \) and \( G'/K_1 = \langle k^yK_1 \rangle \cong K_2 \times \ldots \times K_n \) is abelian, we have, \( n = 1 \), and so \( G' = K - 1 \) is a simple group. Thus by Lemma 2.13 \( G \) is isomorphic to the proper semi-direct product of \( G' \), a simple nonabelian group, by \( \langle x \rangle \) where \( x \) is an involution. \( \square \)

In this section we look more at the structure of non-simple non-solvable groups which lie in $\Gamma_1^{(2)}$.

**Theorem 4.1.** If $G$ is isomorphic to the semi-direct product of $A_n$ and $\langle x \rangle$ where $x$ is an involution and $n > 5$, then $G$ does not lie in $\Gamma_1^{(2)}$.

**Proof.** Suppose that $n \neq 6$, then by [11] $\text{Aut}(A_n) = \Sigma_n$. Assume that $G$ lies in $\Gamma_1^{(2)}$, then by Theorem 3.2 $G \cong \Sigma_n$. It is easy to see that in $\Sigma_n$ ($n > 4$) the 3-cycles do not have an involution as a mate, so $G$ does not lie in $\Gamma_1^{(2)}$. \hfill \Box

**Note.** Note that there are two nonisomorphism classes of proper semi-direct products of $A_6$ and $Z_2$: one is $\Sigma_6$ which does not lie in $\Gamma_1^{(2)}$, while the other involves the exceptional automorphism of $A_6$.

**Theorem 4.2.** If $G$ is isomorphic to the semi-direct product of $A_6$ and $\langle x \rangle$ where $x$ is an exceptional automorphism of $A_6$ of order 2, then $G$ lies in $\Gamma_1^{(2)}$.

**Proof.** Let $\theta$ send the following transpositions in $\Sigma_6$ to products of three 2-cycles:

- $(12) \rightarrow (23)(15)(46)$
- $(23) \rightarrow (12)(34)(56)$
- $(34) \rightarrow (23)(16)(45)$
- $(45) \rightarrow (34)(15)(26)$
- $(56) \rightarrow (23)(14)(56)$.

Since the five 2-cycles above generate $\Sigma_6$ and $\theta$ acts on them as a homomorphism, and the five products of three 2-cycles above also generate $\Sigma_6$, $\theta$ is an automorphism of $\Sigma_6$. Since $\theta$ does not preserve the cycle structure $\theta$ is an exceptional automorphism of $\Sigma_6$ and $A_6$. It is easy to see that $\theta^2 = 1$.

Let $G = \langle A_6, \theta \rangle$ we will show that $G$ lies in $\Gamma_1^{(2)}$. Set $a = (123) = (23)(12)$, $b = a^\theta = (136)(254)$ (multiply left to right). Let $H = \langle a, b \rangle \leq \langle a, \theta \rangle = \langle b, \theta \rangle$, $ab = (15426)$, and $ab^{-1} = (1452)(36) \in H$, so $|H|$ is divisible by 9, 5, and 4. Thus $H = A_6$, and $G = \langle a, \theta \rangle \cong \langle b, \theta \rangle$. Thus, elements of order 3 in $A_6$ have mates of order 2 in $G$.

One can check that $(12345) \in C_G(\theta)$. There are in $A_6$ two maximal subgroups that contain $(12345)$, $M$ the stabilizer of 6, and $N$ a transitive $A_5$. All elements of order 3 in $M$ are 3-cycles, while $N$ contains no 3-cycles. $\theta$ maps $M$ into $N$, and since, $M = \langle (12345), (12)(34) \rangle$,

$$N = \langle (12345), (12)^\theta(34)^\theta \rangle = \langle (12345), (14)(56) \rangle.$$
Since \((34)(56)(14)(56) = (143)\), \((34)(56) \not\in N\), and clearly \((34)(56) \not\in M\). Thus we see that
\[
((12345), (34)(56)) = A_6,
\]
and since \((12345) \in C_G(\theta)\), \(G = \langle (12345)^i, (34)(56) \rangle\) for \(i = 1\) and \(2\). Note that \((12345)\theta\) and \((12345)^2\theta\) are not conjugate in \(G\). Thus, elements of order 2 in \(A_6\) have mates in \(G\).

Set \(A = ab = (15426)\), and \(B = ba = (25436)\), the \(\theta\) sends \(A \leftrightarrow b\). Set \(H = \langle A, B \rangle \leq \langle A, \theta \rangle\), then \((AB^3)^2 = [(16)(2435)]^2 = (23)(45) \in H\). Since \((12345)^{(12536)} = (15426) = A\) and \((34)(56)^{(12536)} = (23)(45)\), thus, \(\langle A, \theta \rangle \geq \langle A, (23)(45) \rangle = A_6\). So, \(\langle (15426)^i, \theta \rangle = G\) for \(i = 1\) and \(2\). Therefore, elements of order 5 in \(A_6\) have mates of order 2 in \(G\).

By similar reasoning \(AB^3 = (16)(2435)\) has \(\theta\) as a mate. Thus, elements of order 4 in \(A_6\) have mates of order 2 in \(G\). So all elements in \(A_6\) have mates as required.

Since \(\theta\) sends \(ab^{-1} \leftrightarrow ab^{-1} = (1452)(36)\). The element \((24)(36)\) also inverts \((1452)(36)\). So, \(\eta = (24)(36) \in C_G(ab^{-1})\). \(\eta^2 = (24)(36)(12)(45) = (1254)(36) = (ab^{-1})^{-1}\), since \(\theta(24)(36)\theta = (12)(45)\). It follows that \(\eta\) has order 8 and \(\langle \eta, \theta \rangle \cong D_8\). Since \(|\langle \eta, \theta \rangle| = 16\), this group is a Sylow 2-subgroup of \(G\), and all involutions outside \(A_6\) are conjugate to \(\theta\). Let \(x \in G - A_6\), note that \(x^2 \in A_6\). If the order of \(x^2\) is odd, then \(x\) is conjugate to \(\theta\), or \((12345)^i\) for \(i = 1\) or \(2\) (since no elements of order 3 in \(A_6\) commute with \(\theta\)). If \(x^2\) has even order, then \(x\) is contained in a Sylow 2-subgroup of \(G\). Since \(x \not\in A_6\), \(x\) is conjugate to \(\eta\) or to \(\eta^3\). We have seen that \(\eta^2 = (1254)(36)\), so \(\eta^2(12)(34) = (25364) = (65432)^2\). Since \((65432)^{(16)(25)(34)} = (12345)\) and \((12)(34)^{(16)(25)(34)} = (34)(56)\), we get \(\langle \eta^i, (12)(34) \rangle = G\) for \(i = 1\) or \(3\).

**Proposition 4.3.** Let \(M\) be a finite simple Chevalley group over a finite field \(K\) of characteristic \(p\), and let \(G\) be isomorphic to the semi-direct product of \(M\) and \(\langle x \rangle\) where \(x\) is an involution. If \(G\) lies in \(\Gamma_1^{(2)}\), then \(M \cong PSL(2, K)\).

**Proof.** This is a direct consequence of Theorem 2.18.

**Note.** Note that since \(A_5 \cong PSL(2, 5)\), it is not true that for every \(M = PSL(2, K)\) where \(K\) is a finite field of odd characteristic \(p\), there exists a proper semi-direct product \(G\) of \(M\) and \(\langle x \rangle\) where \(x\) is an involution such that \(G\) lies in \(\Gamma_1^{(2)}\).

In [7, Chapter 8] we eliminate many more simple groups from the possibilities for \(N\) in Theorem 3.2.
5. $E_8$ Case.

The Lie algebra $L$ over a field $K$ has a Cartan decomposition [4, Ch.7] $L = H \oplus \sum_{r \in \Delta} L_r$, where $H$ is generated by $h_r$ for $r \in \Pi$, and $L_r$ is a one-dimensional vector space generated by an element $e_r$ for $r \in \Delta$. For a simple Lie algebra $L$ over $K$, Chevalley defined a group $L(K)$ which is simple except for a few exceptional cases. $L(K)$ is uniquely determined by its action on $\{h_r, e_r\}_{r \in \Delta, q \in \Pi}$.

The elements of $\Pi$ will be denoted by $\alpha_1, \ldots, \alpha_1$. The method of this section gives us some insight into the action of $L(K)$ on $L$ and proves that $L(K)$ does not lie in $\Gamma^{(2)}$ when $\Delta = E_8$. In this section we are using the standard notation for roots in $E_8$.

**Definition 5.1.** Let $f$ be a function from $\Pi \cup \Pi^-$ to the integers defined by $f(\alpha_i) = f(-\alpha_i) = i - 1$.

**Definition 5.2.** $\Psi = \{\tau \in \Delta| (\tau, \alpha_1) = 0\}$, where $(\, , \,)$ is as in [10, page 39].

**Lemma 5.3.** If $\Delta = E_8$, and $\tau \in \Psi$, then $x_{\alpha_1} e_\tau = e_\tau$, and $x_{-\alpha_1} e_\tau = e_\tau$ (where $x_\tau = x_\tau(1)$).

**Proof.** Since all roots in $E_8$ have the same length, this implies that if $\tau \in \Psi$, $\tau \neq -\alpha_1$ and $\tau - \alpha_1$ and $\tau + \alpha_1 \not\in \Delta$. Thus by the formula on [4, page 61] we see that $x_{\pm \alpha_1} e_\tau = e_\tau$. \hfill $\Box$

Let $L = H \oplus \sum_{t \in \Delta} L_t$ be the Cartan decomposition of $L$ [4, Ch.3]. $L$ can be written as $L = \sum_{t \in \Pi} H_t \oplus \sum_{t \in \Delta} L_t$ where $H_t = \langle h_t \rangle$, and $L_t = \langle e_t \rangle$. For $\Delta = E_8$, let $T = H_{\alpha_1} \oplus H_{\alpha_2} \oplus H_{\alpha_3} \oplus \sum_{t \in \Delta - \Psi} L_t$ and,

$$L' = \sum_{t \in \Pi - \{\alpha_2, \alpha_3\}} H_t \oplus \sum_{t \in \Psi} L_t.$$

**Note.** Note that $L = L' \oplus T$, and $x_{\alpha_1}$ and $x_{-\alpha_1}$ act trivially on $L'$.

**Lemma 5.4.** Given $L$, $L'$ and $T$ as before, then there exists no subset $0 \neq S \subseteq L' \cap H$ invariant under the action of $E_8(K)$ where $K$ is a finite field of characteristic 2.

**Proof.** Assume that such an $S$ exists, then let $x \in S - \{0\}$. We can write $x$ as $x = a_1 h_{\tau_1} + \ldots + a_n h_{\tau_n}$ where each $a_i \neq 0$, each $\tau_i \in \Pi$, and $f(\tau_1) < \ldots < f(\tau_n)$.

Note that in all cases, $f(\tau_1) > 2$ by the way we constructed $L'$. Thus we can choose a root $t \in \Pi$ with $f(t) = f(\tau_1) - 1$ such that $\langle t, \tau_1 \rangle = -1$ and $f((t, \tau_1) = 2$. Therefore,

$$x_{\tau_1} L_t = x_{\tau_1} h_t \oplus x_{\tau_1} e_t,$$

and $x_{\tau_1} e_t.$
2(t,τ_i)/(t,t) = 1(mod 2), and (t,τ_j) = 0 for i = 2,...,n. So we get that
x_i h_{r_i} = h_{r_i} + e_{r_i}, and x_i h_{r_i} = h_{r_i} for i = 2,...,n. Therefore x_i x \not\in S,
contradicting that S is L(K)-invariant. Therefore there exists no subset
0 \neq S \leq L' \cap H invariant under the action of L(K).

\[\square\]

Lemma 5.5. There is no invariant subspace of L' under the action of E_8(K)
where K is a finite field of characteristic 2.

Proof. In view of Lemma 5.4 it will be sufficient to prove that any E_8(K)-
invariant nonzero subspace S of L' is a subspace of H.

Assume that S is not a subspace of H, then we can write any element
v \in S - H as v = a_{s_1} e_{s_1} + a_{s_2} e_{s_2} + ... + h, where h \in H, and s_i \in \Psi.

Since all the roots in E_8 have the same length, then by [10, 10.4 Lemma C]
all the roots are conjugate under W, the Weyl group of E_8. Thus there exists
an element w \in W, with w(s_1) = \alpha_1. Note that since w is an automorphism
w(s_i) \neq \alpha_1 for i > 1, and that w = w_{r_1} ... w_{r_i} (here w_{r_i} is the reflection in
the hyperplane orthogonal to the root r_i). Let N be the monomial subgroup
of E_8(K), and n_r as in [4, Proposition 6.4.2]. By [4, Theorem 7.2.2] there
is a homomorphism from N onto W under which n_r \rightarrow w_r for all r \in \Delta. n_r
acts invariantly on H, and n_r e_s = \pm e_{w_r(s)}. Since we are in characteristic 2,
n_r e_s = e_{w_r(s)}. Let n = n_{r_1} ... n_{r_i} \in N, then n \rightarrow w by the homomorphism
N \rightarrow W. So n e_{s_1} = e_{w(s_1)} = e_{\alpha_1}, and nh \in H. Hence nv contains a term
e_{\alpha_1}, contradicting that v \in S. Thus S is a subspace of H.

\[\square\]

Theorem 5.6. If G = E_8(K) where K is a finite field of characteristic 2,
then G does not lie in \Gamma^{(2)}_1.

Proof. Using the description of roots in the system E_8 in terms of orthogonal
vectors given in [2, page 268] we get that the number of positive roots in \Psi is

\[1 + 6 + 6 + \binom{6}{2} + \binom{6}{2} + \binom{6}{3} = 63.\]

So we get that cod (L') = 117. Both x_\alpha_1 and x_{-\alpha_1} act trivially on L';
therefore x_\alpha_1 x_{-\alpha_1} also acts trivially on L'. Assume that G lies in \Gamma^{(2)}_1, then
since O (x_\alpha_1 x_{-\alpha_1}) \neq 1 or 2, x_\alpha_1 x_{-\alpha_1} has a mate of order 2. Let S = L' \cap (L')^y;
since G = \langle x_\alpha_1, x_{-\alpha_1}, y \rangle, we see that G acts invariantly on S. Since

\[\text{cod}(S) \leq 2 \text{cod} (L') = 234 < 248,\]

S \neq 0 contradicting Lemma 5.5.

\[\square\]
6. Twisted Groups in odd Characteristic.

Lemma 6.1. If $q$ is odd and $G = {}^2A_t(q^2)$ for $l \geq 2$, or ${}^2D_t(q^2)$ for $l \geq 4$, or ${}^2E_6(q^2)$, then $|G| > |SL_2(q^2)|$. Also $|{}^3D_4(q^3)| > |SL_2(q^3)|$ for $q$ odd.

Proof. For $q = p^n$ where $p$ is an odd prime we get:

1) $|SL_2(q^2)| = q^2(q^4 - 1) < (1/3)q^2(q^2 - 1)(q^3 + 1) \leq |{}^2A_t(q^2)|$, $l \geq 2$.
2) $|SL_2(q^2)| = q^2(q^4 - 1) < (1/4)q^4(q^4 - 1) < |{}^2D_t(q^2)|$, $l \geq 4$.
3) $|SL_2(q^2)| = q^2(q^6 - 1) < (1/3)q^6(q^6 + 1) < |{}^2E_6(q^2)|$.
4) $|SL_2(q^3)| = q^2(q^6 - 1) < q^6(q^6 - 1) < |{}^3D_4(q^3)|$.

Consider a Chevalley group $G^* = L(K)$ and $\rho$ a non-trivial symmetry of the Dynkin diagram for $L$. Recall that if $K$ has a certain order depending on $L$, then we can choose an automorphism $\sigma$ (a product of a field automorphism and a graph automorphism determined by $\rho$) of $G^*$ such that the twisted group $G = {}^1L(K)$ is a subgroup of $G^*$ stabilized by $\sigma$, and $U^1$ is the subgroup of $U$ centralized by $\sigma$ (recall that $U^1 \leq G$). We will call $G^*$ the corresponding Chevalley group of $G$. Note that $\sigma(x_r(t)) = x_{r'}(\gamma_r t')$, where $\gamma_r = \pm 1$ and $t' = f(t)$, $f$ a certain field isomorphism, and $r'$ arises from the symmetry of the Dynkin diagram [4, 12.2]. Since all the roots in the system associated with the groups of Lemma 6.1 have same length, the above action is an isometry [4, Prop. 12.2.2], and there exists a unique root of maximal length $s$ [10, 10.4 Lemma A], and thus $s' = s$ (note that this is not the case for ${}^2G_2(3^{2m+1})$ [4, 12.4]).

Lemma 6.2. If $G$ is a simple twisted group, and $G^*$ is the corresponding Chevalley group of $G$, then $\text{Aut}(G) \leq \text{Aut}(G^*)$.

Proof. From [8, page 303 and 5, Table 5] $\text{Aut}(G)$ is a product of an inner, a diagonal, and a field automorphism. □

Theorem 6.3. If $q = p^n$ where $p$ is an odd prime and $G$ is a simple group of type ${}^2A_t(q^2)$, $l \geq 2$, or ${}^2D_t(q^2)$, $l \geq 4$, or ${}^2E_6(q^2)$, or ${}^3D_4(q^3)$, then $G$ does not appear as a composition factor of any group in $\Gamma^{(2)}$.

Proof. Let $K = GF(q^2)$ if $G = {}^2A_t(q^2), l \geq 2$, or ${}^2D_t(q^2), l \geq 4$, or ${}^2E_6(q^2)$, and $K = GF(q^3)$ if $G = {}^3D_4(q^3)$. Let $s$ be the root of maximal height. If $\gamma_s = 1$, let $x = x_s(1)$. If $\gamma_s = -1$ (this can happen in the case that
$K = GF(q^2)$, then the automorphism $f$ associated with the group has order 2 and there exists an element $t \in K \setminus \{0\}$ such that $t' = -t$. So, let $x = x_s(t)$. In each case $\sigma(x) = x$ and $x \in Z(U) \cap U^1$ [4, Def. 13.4.2] so $x \in Z(U^1)$ and $O(x) = p$.

If $G$ appears as a composition factor of any group in $\Gamma_1^{(2)}$, then $x$ has a mate $y \in \text{Aut}(G) \leq \text{Aut}(G^*)$ of order 2, and $G = \langle x, x^y \rangle \leq \langle X_s, (X_s)^y \rangle = M$ (note that $M$ is a subgroup of $G^*$, not of $G$) from Lemma 2.2. Since $M$ is a nonnilpotent subgroup of $G^*$ by Lemma 2.17, there is a homomorphism from $SL(2, K)$ onto $M$, but by Lemma 6.1 $|G| > |M|$, a contradiction.

7. Centralizers and $\Gamma_1^{(2)}$.

In this section we will use the result that if two subgroups have order greater that the square root of the order of the group, then they have a nontrivial intersection [9, Sec. 2.5], to investigate whether some groups lie in $\Gamma_1^{(2)}$.

Lemma 7.1. If $G$ is a finite simple group with an element of order not 1 or 2 that has a centralizer of order larger $(|G|)^{1/2}$, then $G$ does not appear as a composition factor of any group in $\Gamma_1^{(2)}$.

Proof. Assume that $G$ appears as a composition factor of a group $H$ in $\Gamma_1^{(2)}$. Then, by previous results, $G$ is a normal subgroup of $H$ with index at most two. Let $x \in G$ such that $O(x) \neq 1$ or 2 and $|C_G(x)| > (|G|)^{1/2}$. Since $H$ lies in $\Gamma_1^{(2)}$, $x$ has a mate $y \in H$ of order 2. By Lemma 2.2 $G = \langle x, x^y \rangle$, and since $|C_G(x^y)| = |C_G(x)|$ from the above we get that $Z(G) \neq 1$, contradicting the simplicity of $G$. \hfill \Box

Theorem 7.2. The groups $J_2$, Suz, $Co_1$, Ly, $Fi_{22}$, $Fi_{23}$, $Fi'_{24}$, $D_4(2)$, and $^2D_4(2)$ do not appear as a composition factor of any group in $\Gamma_1^{(2)}$.

Proof. From [5] we see that each of the above groups has an element of order 3 with a centralizer of order greater than the square root of the order of the group. \hfill \Box

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