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This paper deals with finite groups. J. L. Brenner and James Wiegold defined a finite group G as lying in $\Gamma_1^{(2)}$ if G is nonabelian and for every $1 \neq x \in G$, either x is an involution and $G = \langle x, y \rangle$ for some $y \in G$ or x is not an involution and there is an involution $z \in G$ with $G = \langle x, z \rangle$. In this paper we expand the work of J. L. Brenner and James Wiegold, and that of Martin J. Evans in the investigation of which finite groups lie in $\Gamma_1^{(2)}$.

1. Introduction.

Definition 1.1. An element x in a group is called a *mate* of an element $y \in G$ if $G = \langle x, y \rangle$.

Definition 1.2. We say that a finite group G lies in $\Gamma_1^{(2)}$ if G is nonabelian and for every $1 \neq x \in G$, either x is an involution and x has a mate $y \in G$ or x is not an involution and x has an involution $z \in G$ as a mate (see [**3**]).

Brenner and Wiegold [**3**] proved that $PSL(2, q)$ lies in $\Gamma_1^{(2)}$ except when $q = 9$, and that $PSL(n, q)$ does not lie in $\Gamma_1^{(2)}$ for $n \geq 3$ unless $n = 3$ and $q = 2$, or 4.

Evans [**6**] proved that if $G = \mathrm{Sz}(2^{2n+1})$ is a Suzuki group, then G lies in $\Gamma_1^{(2)}$; moreover if G is a simple Chevalley group over a finite field K of odd characteristic and G lies in $\Gamma_1^{(2)}$, then $G \cong PSL(2, K)$.

Using the fact that each of the groups $S_p(2n, K)$, $P\Omega_{2n}^{+}(K)$, and $P\Omega_{2n}^{-}(K)$ act irreducibly on $V = K^{2n}$, and $PSU_m(K)$ acts irreducibly on $V = K^m$, we find an element x of order greater than 2 that acts trivially on a "large enough" subspace, so that two conjugates of x can not act irreducibly on V, if dimension of V is large. We get in [**7**] that over a field K of characteristic 2, $P\Omega_{2n}^{-}(K)$, for $n \geq 4$ or $|K| > 2$, and $PSU_m(K)$ for $n > 3$, $S_p(2n, K)$ for $n \geq 3$, and $P\Omega_{2n}^{+}(K)$ for $n \geq 5$, or $|K| > 2$, do not lie in $\Gamma_1^{(2)}$. Similarly we show in [**7**] that Σ_n for $n \geq 5$ and A_n for $n > 5$ do not lie in $\Gamma_1^{(2)}$.

In this paper we classify all those solvable groups that lie in $\Gamma_1^{(2)}$, and we show that a finite non-simple non-solvable group lies in $\Gamma_1^{(2)}$ if it is isomorphic to the semi-direct product of N and $\langle x\rangle$ where x is an involution and N is a simple nonabelian group. Many simple groups are excluded from being candidates for the N above. We also continue in the investigation of which simple groups lie in $\Gamma_1^{(2)}$.

2. A Preliminary look at $\Gamma_1^{(2)}$.

This section deals with general facts about groups which lie in $\Gamma_1^{(2)}$. We state Lemmas 2.1-2.6 but since they are obvious we omit the proofs.

Lemma 2.1. *If G is a group and $G = \langle x, y\rangle$ where y is an involution and $x \neq 1$, then $H = \langle x, x^y\rangle$ is a normal subgroup of G.*

Lemma 2.2. *If G is a nonabelian simple group and $G \leq M \leq \mathrm{Aut}(G)$, and $M = \langle x, y\rangle$ where $1 \neq x \in G$, $y \in M$ and is an involution, then $G = \langle x, x^y\rangle$.*

Corollary 2.3. *If G is simple and G lies in $\Gamma_1^{(2)}$, then every conjugacy class of G other than the classes of elements of order 1 or 2 contains a pair of conjugate elements which generate G.*

Lemma 2.4. *If x and y are conjugate in G and y has a mate, then x has a mate.*

Lemma 2.5. *If G lies in $\Gamma_1^{(2)}$, then $Z(G) = 1$.*

Lemma 2.6. *If G lies in $\Gamma_1^{(2)}$ and N is a nontrivial normal sungroup of G, then $G^* = G/N$ is cyclic.*

Definition 2.7. A finite nonabelian group is called a Γ-*group* if $G = NP$, where N is an elementary abelian normal 2-subgroup, and P is a cyclic group of prime order acting irreducibly on N.

The last condition says that no proper subgroup of N is P-invariant. Since $C_N(P)$ is P-invariant, $C_N(P) = 1$ or $C_N(P) = N$. Since G is nonabelian $C_N(P) = 1$.

Remark 2.8. If G is a Γ-group, then $p = |P|$ is an odd prime.

Remark 2.9. Given an odd prime p, there is a Γ-group G of order divisible by p. It is the subgroup of $\mathrm{Aff}(1, 2^n)$, where n is the unique positive integer such that $n|(p-1)$, and $p|(2^n-1)$ but p does not divide 2^m-1 for $0 < m < n$.

Lemma 2.10. *A Γ-group $G = NP$ (N and P as in Definition 2.7) has the following properties:*

1) *G lies in $\Gamma_1^{(2)}$.*
2) *N is the commutator subgroup of G.*

Proof. Let $y \in G - \{1\}$ and $P = \langle x \rangle$.

If $y \in N - \{1\}$, then $\langle y \rangle^{\langle x \rangle}$ is a non-trivial P-invariant subgroup of N and $\langle y \rangle^{\langle x \rangle} = N$. Therefore, $G = \langle x, y \rangle$. If $y \in G - N$, then y is N-conjugate to a power of x, so that any $\langle y \rangle$-invariant subgroup of N is also P-invariant. Thus, $G = \langle y, n \rangle$ for any $n \in N - \{1\}$, and G lies in $\Gamma_1^{(2)}$. Since P is an abelian group, $G' \leq N$ and $G' = N$ because G' is P-invariant and nontrivial. □

Notation. If $y \in G$, then $O(y)$ is the order of y.

Theorem 2.11. *If G lies in $\Gamma_1^{(2)}$, then G has a proper normal subgroup of odd index if and only if it is a Γ-group.*

Proof. By Lemma 2.10 any Γ-group lies in $\Gamma_1^{(2)}$, and has a proper normal subgroup of odd index.

Assume that G has a proper normal subgroup K of odd index. Since $|G/K|$ is odd and $|G|$ is even, K is nontrivial; therefore, Lemma 2.6 gives that G/K is cyclic. Let M^* be a subgroup of G/K of odd prime index p. If $M \leq G$ such that $M/K = M^*$, then M is normal in G and $|G/M| = p$ is an odd prime.

If x is an involution in G, then x is in M because $O(xM) = 1$. If y is an element of $M - \{1\}$ with $O(y) \neq 2$, then y has a mate t of order 2. Thus, t is in M. But $G = \langle y, t \rangle$ gives that $G = M$, a contradiction. Therefore, $|M| = 2^n$ and M is an elementary abelian 2-group. Since, $|G/K|$ is odd and $K \leq M, M = K$.

Let x be an element of order p in G, and set $P = \langle x \rangle$. Then G is the semidirect product of M by P, because $M \cap P = \{1\}$ and $G = MP$. If $z \in G - M$, then $O(zM) \neq 1$ and so $O(zM) = p$. Thus, $p | O(z)$ so we see that $O(z) = 2^k p$ for some k. If $t = z^p$, then $t \in M \cap C_G(z)$; therefore t is in $Z(G)$ and by Lemma 2.5, $t = 1$. Therefore, $O(z) = p$.

Let N be a minimal P-invariant subgroup of M, and d an element of $N - \{1\}$. Now, d has a mate y. Since $G = \langle d, y \rangle$, y is not an element of M. By the preceding paragraph $O(y) = p$. Hence, $\langle y \rangle$ is M-conjugate to P, by Sylow's Theorem. Thus, $\langle y \rangle^m = P$ for some $m \in M$, and because M is abelian $d^m = d$. It follows that

$$G = \langle d, y \rangle = \langle d^m, y^m \rangle = \langle d, P \rangle = \langle N, P \rangle \ \text{(because } d \in N\text{)}.$$

Hence, $G = NP$ and $|G| = |N| \cdot |P|$. This implies that $N = M$. Therefore P acts irreducible on M and G is a Γ-group. □

Theorem 2.12. *If G lies in $\Gamma_1^{(2)}$ and $G \neq G'$, then either*
1) G is a Γ-group,
or
2) G is isomorphic to a semi-direct product of G' by $\langle y \rangle$, where y is an involution. If, in this case, G' is abelian, then G is isomorphic to D_{2p}, where p is an odd prime.

Proof. If $G^* = G/G'$, then G^* is cyclic by Lemma 2.6.

If G^* is not a 2-group, then there exists a normal subgroup N of G with $G' \leq N < G$ and $|G/N|$ is odd. Since $|G/N| > 1$ is odd, G is a Γ-group and $G' = N$, by Lemma 2.10.

If G^* is a 2-group, note that G is not a 2-group, for $Z(G) = 1$. Therefore there exists an $x \in G'$ of odd prime order p. The element x has a mate y of order 2. Since $G = \langle x, y \rangle$, $G^* = \langle xG', yG' \rangle = \langle yG' \rangle$; therefore $|G^*| = 2$, $G' \cap \langle y \rangle = 1$, and G is isomorphic to a semi-direct product of G' and $\langle y \rangle$.

If G' is abelian, then $z^y = z$ for any $z \in G'$ if and only if $z = 1$, because $Z(G) = 1$. Hence, y acts fixed point free on G' and, since y is an involution, G' is of odd order and for all $z \in G' - \{1\}$ $z^y = z^{-1}$. Since $G = \langle x, y \rangle$, $G' = \langle x \rangle$ and $|G'| = p$. Therefore $G \cong D_{2p}$, for $|G| = 2p$ and $G = \langle x, y \rangle$, where $x^p = y^2 = 1$, and $x^y = x^{-1}$. □

Lemma 2.13. *Let G lie in $\Gamma_1^{(2)}$. If G is isomorphic to a semi-direct product of N by $\langle x \rangle$, where x is an involution, then x acts on N as an outer automorphism of N.*

Proof. Assume x acts on N as conjugation by an element $y \in N$, then xy^{-1} acts trivially on N. Therefore, $1 \neq xy^{-1}$ is in the center of G, contradicting Lemma 2.5. □

Definition 2.14. A group is a *proper semi-direct product* of N by P if G is a semi-direct product of N by P and $NC_G(N) \neq G$.

Note. The situation in Lemma 2.13 gives us an example of a proper semi-direct product, since $NC_G(N) = N \neq G$.

In [**6**] Evans shows that if a simple Chevalley group over a field of odd characteristic lies in $\Gamma_1^{(2)}$, then it is isomorphic to $PSL(2, K)$. Below we will present a generalization of this result.

Definition 2.15. 1) Δ denotes a root system of a Lie algebra.
2) Δ^+ denotes the set of positive roots in Δ.
3) Δ^+ denotes the set of negative roots in Δ.

4) Π denotes the fundamental system of roots of Δ.
5) $x_r(k)$ denotes the generator $\exp(f\mathrm{ad}e_r)$ of a Chevalley group.
6) $X_s = \langle x_x(k) | k \in K\rangle$.
7) $U = \langle x_r(k) | r \in \Delta^+, k \in K\rangle$.

Recall that each root $r \in \Delta$ can be written as $r = \sum k_\alpha \alpha\,(\alpha \in \Pi)$ with integral coefficients k_α all nonnegative or all nonpositive [**10**, 10.1].

Definition 2.16. The height of $r \in \Delta$ (relative to Π) is $\mathrm{ht}(r) = \sum k_\alpha$.

From the Steinberg decomposition of automorphisms of a finite simple Chevalley group M [**4**, Theorem 12.5.1], we get that if $\tau \in \mathrm{Aut}(M)$, then $\tau = idgf$, where i is an inner automorphism, d is a diagonal automorphism, g is a graph automorphism, and f a field automorphism. From the Bruhat decomposition [**4**, Chapter 8] we get that if $i \in \mathrm{inn}(M) \cong M$ since M is a simple group, then $i = u_1h_1nu_2h_2$, where $u_1, u_2 \in U$, $h_1, h_2 \in H$ and $n_t \in N$. H and N denote the diagonal and monomial subgroups of M respectively.

So by combining the above remarks we get that every $\tau \in \mathrm{Aut}(M)$ can be written as

$$\tau = u_1h_1nu_2h_2dgf.$$

Theorem 2.17. *Let M be a non-nilpotent subgroup of a finite simple Chevalley group G over a finite field K, and $M = \langle X_s, X_s^y\rangle$, where $y \in \mathrm{Aut}(G)$ and s is a root of greatest height in Δ^+, then there is a homomorphism from $SL(2,K)$ onto M.*

Proof. Let g be a graph automorphism, then since $g(r) \in \Delta^+$ for all $r \in \Delta^+$, $g|_U$ and $g|_{Z(U)}$ are automorphisms.

Since $y \in \mathrm{Aut}(M)$, it follows from the remarks above that

$$y = u_1h_1n_tu_2h_2dgf.$$

Let $s' = g^{-1}(s)$, then by the choice of s we get that X_s and $X_{s'}$ are central in U [**4**, Theorem 5.3.3], and that H [**4**, page 100], all diagonal automorphisms, and all field automorphisms normalize X_s and $X_{s'}$. So we get that

$$\begin{aligned} M &= \left\langle X_s, (X_s)^{u_1h_1n_tu_2h_2dgf}\right\rangle = \left\langle X_s, (X_s)^{n_tu_2h_2dgf}\right\rangle \cong \\ &\cong \left(\left\langle X_s, (X_s)^{n_tu_2h_2dgf}\right\rangle\right)^{f^{-1}g^{-1}d^{-1}h_2^{-1}u_2^{-1}} = \\ &= \left\langle (X_s)^{f^{-1}g^{-1}d^{-1}h_2^{-1}u_2^{-1}}, (X_s)^{n_t}\right\rangle = \langle X_{s'}, (X_s)^{n_t}\rangle. \end{aligned}$$

Let w_t denote the image of n_t in the Weyl group under the natural homomorphism from N to $N/H \cong W$. Since $X_s^{n_t} = X_{w_t(s)}$, $M \cong \langle X_{s'}, X_{w_t(s)} \rangle$.

For all $r, s \in \Delta$ which are linearly independent there exists a $w \in W$ such that $w(r), w(s) \in \Delta^+$. Therefore $\langle X_r, X_s \rangle$ is isomorphic to a subgroup of U which is nilpotent. Since M is not nilpotent, we get that $w_t(s) = -s'$. So we see that $M \cong \langle X_{s'}, X_{-s'} \rangle$ and by [**4**, Theorem 6.3.1] there is a homomorphism from $SL(2, K)$ onto M. □

Theorem 2.18. *Let M be a finite simple Chevalley group over a finite field K of odd characteristic p, and $M \leq G \leq \mathrm{Aut}(M)$. If G lies in $\Gamma_1^{(2)}$, then $M \cong PSL(2, K)$.*

Proof. Since G lies in $\Gamma_1^{(2)}$ and $O(x_s(1)) = p \neq 2$ where s is a root of greatest height in Δ^+, $x_s(1)$ has a mate y of order 2. By Lemma 2.2 $M = \langle x_s(1), x_s(1)^y \rangle$, therefore $M = \langle X_s, X_s^y \rangle$. So by Lemma 2.16 we see that there is a homomorphism from $SL(2, K)$ onto M, and since M is simple $M \cong PSL(2, K)$. □

3. Non-Simple Groups that lie in $\Gamma_1^{(2)}$.

An easy consequence of Lemma 2.6 and Theorem 2.12 is:

Corollary 3.1. *A solvable group G lies in $\Gamma_1^{(2)}$ if and only if $G \cong D_{2p}$, where p is an odd prime, or G is a Γ-group.*

Theorem 3.2. *If G is a non-solvable group lying in $\Gamma_1^{(2)}$, then either G is a nonabelian simple group or G is isomorphic to a proper semi-direct product of N by $\langle x \rangle$, where x is an involution and N is a simple group.*

Proof. Let M be a maximal normal subgroup of G and let $G^* = G/M$.

Case 1: If $M = \{1\}$, then G is simple.

Case 2: If $M \neq \{1\}$, then G^* is cyclic, by Lemma 2.6, and $G' \neq G$. Since, any Γ-group is solvable, by Theorem 2.12 and Lemma 2.13, G is isomorphic to a proper semi-direct product of G' by $\langle x \rangle$, where x is an involution.

If $\{1\} \neq N < G'$ and N is normal in G, then G/N is cyclic, by Lemma 2.6. But G/N is a nonabelian group; a contradiction. Therefore G' is a nonabelian characteristically simple group (since G is not a solvable group); that is, $G' \cong K_1 \times \ldots \times K_n$ where $K_i \cong K_j$ are nonabelian simple groups. For $k \in K_1$ a nontrivial element of odd order, let y be its mate of order 2. By Lemma 2.1, $H = \langle k, k^y \rangle$ is a nontrivial normal subgroup of G, and $H \leq G'$, thus $H = G'$. Since, K_1 is a normal subgroup of G' and $G'/K_1 = \langle k^y K_1 \rangle \cong K_2 \times \ldots \times K_n$ is abelian, we have, $n = 1$, and so $G' = K-1$ is a simple group. Thus by Lemma 2.13 G is isomorphic to the proper semi-direct product of G', a simple nonabelian group, by $\langle x \rangle$ where x is an involution. □

4. Non-Simple Non-Solvable Groups.

In this section we look more at the structure of non-simple non-solvable groups which lie in $\Gamma_1^{(2)}$.

Theorem 4.1. *If G is isomorphic to the semi-direct product of A_n and $\langle x\rangle$ where x is an involution and $n > 5$, then G does not lie in $\Gamma_1^{(2)}$.*

Proof. Suppose that $n \neq 6$, then by **[11]** $\mathrm{Aut}\,(A_n) = \Sigma_n$. Assume that G lies in $\Gamma_1^{(2)}$, then by Theorem 3.2 $G \cong \Sigma_n$. It is easy to see that in Σ_n $(n > 4)$ the 3-cycles do not have an involution as a mate, so G does not lie in $\Gamma_1^{(2)}$. □

Note. Note that there are two nonisomorphism classes of proper semi-direct products of A_6 and Z_2: one is Σ_6 which does not lie in $\Gamma_1^{(2)}$, while the other involves the exceptional automorphism of A_6.

Theorem 4.2. *If G is isomorphic to the semi-direct product of A_6 and $\langle x\rangle$ where x is an exceptional automorphism of A_6 of order 2, then G lies in $\Gamma_1^{(2)}$.*

Proof. Let θ send the following transpositions in Σ_6 to products of three 2-cycles:

$$\begin{aligned}(12)&\longrightarrow(23)(15)(46)\\(23)&\longrightarrow(12)(34)(56)\\(34)&\longrightarrow(23)(16)(45)\\(45)&\longrightarrow(34)(15)(26)\\(56)&\longrightarrow(23)(14)(56).\end{aligned}$$

Since the five 2-cycles above generate Σ_6 and θ acts on them as a homomorphism, and the five products of three 2-cycles above also generate Σ_6, θ is an automorphism of Σ_6. Since θ does not preserve the cycle structure θ is an exceptional automorphism of Σ_6 and A_6. It is easy to see that $\theta^2 = 1$.

Let $G = \langle A_6, \theta\rangle$ we will show that G lies in $\Gamma_1^{(2)}$. Set $a = (123) = (23)(12)$, $b = a^\theta = (136)(254)$ (multiply left to right). Let $H = \langle a, b\rangle \leq \langle a, \theta\rangle = \langle b, \theta\rangle$, $ab = (15426)$, and $ab^{-1} = (1452)(36) \in H$, so $|H|$ is divisible by 9,5, and 4. Thus $H = A_6$, and $G = \langle a, \theta\rangle = \langle b, \theta\rangle$. Thus, elements of order 3 in A_6 have mates of order 2 in G.

One can check that $(12345) \in C_G(\theta)$. There are in A_6 two maximal subgroups that contain (12345), M the stabilizer of 6, and N a transitive A_5. All elements of order 3 in M are 3-cycles, while N contains no 3-cycles. θ maps M into N, and since, $M = \langle (12345), (12)(34)\rangle$,

$$N = \langle (12345), (12)^\theta(34)^\theta\rangle = \langle (12345), (14)(56)\rangle\,.$$

Since $(34)(56)(14)(56) = (143)$, $(34)(56) \notin N$, and clearly $(34)(56) \notin M$. Thus we see that

$$\langle (12345), (34)(56) \rangle = A_6,$$

and since $(12345) \in C_G(\theta)$, $G = \langle (12345)^{i\theta}, (34)(56) \rangle$ for $i = 1$ and 2. Note that $(12345)\theta$ and $(12345)^2\theta$ are not conjugate in G. Thus, elements of order 2 in A_6 have mates in G.

Set $A = ab = (15426)$, and $B = ba = (25436)$, the θ sends $A \leftrightarrow b$. Set $H = \langle A, B \rangle \leq \langle A, \theta \rangle$, then $(AB^3)^2 = [(16)(2435)]^2 = (23)(45) \in H$. Since $(12345)^{(12536)} = (15426) = A$ and $(34)(56)^{(12536)} = (23)(45)$, thus, $\langle A, \theta \rangle \geq \langle A, (23)(45) \rangle = A_6$. So, $\langle (15426)^i, \theta \rangle = G$ for $i = 1$ and 2. Therefore, elements of order 5 in A_6 have mates of order 2 in G.

By similar reasoning $AB^3 = (16)(2435)$ has θ as a mate. Thus, elements of order 4 in A_6 have mates or order 2 in G. So all elements in A_6 have mates as required.

Since θ sends $ba^{-1} \leftrightarrow ab^{-1}$, θ inverts $ab^{-1} = (1452)(36)$. The element $(24)(36)$ also inverts $(1452)(36)$. So, $\eta = (24)(36) \in C_G(ab^{-1})$. $\eta^2 = (24)(36)(12)(45) = (1254)(36) = (ab^{-1})^{-1}$, since $\theta(24)(36)\theta = (12)(45)$. It follows that η has order 8 and $\langle \eta, \theta \rangle \cong D_8$. Since $|\langle \eta, \theta \rangle| = 16$, this group is a Sylow 2-subgroup of G, and all involutions outside A_6 are conjugate to θ. Let $x \in G - A_6$, note that $x^2 \in A_6$. If the order of x^2 is odd, then x is conjugate to θ, or $(12345)^{i\theta}$ for $i = 1$ or 2 (since no elements of order 3 in A_6 commute with θ). If x^2 has even order, then x is contained in a Sylow 2-subgroup of G. Since $x \notin A_6$, x is conjugate to η or to η^3. We have seen that $\eta^2 = (1254)(36)$, so $\eta^2(12)(34) = (25364) = (65432)^2$. Since $(65432)^{(16)(25)(34)} = (12345)$ and $(12)(34)^{(16)(25)(34)} = (34)(56)$, we get $\langle \eta^i, (12)(34) \rangle = G$ for $i = 1$ or 3. □

Proposition 4.3. *Let M be a finite simple Chevalley group over a finite field K of characteristic p, and let G be isomorphic to the semi-direct product of M and $\langle x \rangle$ where x is an involution. If G lies in $\Gamma_1^{(2)}$, then $M \cong PSL(2, K)$.*

Proof. This is a direct consequence of Theorem 2.18. □

Note. Note that since $A_5 \cong PSL(2,5)$, it is not true that for every $M = PSL(2, K)$ where K is a finite field of odd characteristic p, there exists a proper semi-direct product G of M and $\langle x \rangle$ where x is an involution such that G lies in $\Gamma_1^{(2)}$.

In [**7**, Chapter 8] we eliminate many more simple groups from the possibilities for N in Theorem 3.2.

5. E_8 Case.

The Lie algebra L over a field K has a Cartan decomposition [**4**, Ch.7] $L = H \oplus \sum_{r\in\Delta} L_r$, where H is generated by h_r for $r \in \Pi$, and L_r is a one-dimesional vector space generated by an element e_r for $r \in \Delta$. For a simple Lie algebra L over K, Chevalley defined a group $L(K)$ which is simple except for a few exceptional cases. $L(K)$ is uniquely determined by its action on $\{h_q, e_r\}_{r\in\Delta, q\in\Pi}$.

The elements of Π will be denoted by $\alpha_1, \dots, \alpha_1$. The method of this section gives us some insight into the action of $L(K)$ on L and proves that $L(K)$ does not lie in $\Gamma_1^{(2)}$ when $\Delta = E_8$. In this section we are using the standard notation for roots in E_8.

Definition 5.1. Let f be a function from $\Pi \cup \Pi^-$ to the integers defined by $f(\alpha_i) = f(-\alpha_i) = i - 1$.

Definition 5.2. $\Psi = \{\tau \in \Delta | (\tau, \alpha_1) = 0\}$, where (,) is as in [**10**, page 39].

Lemma 5.3. *If $\Delta = E_8$, and $\tau \in \Psi$, then $x_{\alpha_1} e_\tau = e_\tau$, and $x_{-\alpha_1} e_\tau = e_\tau$ (where $x_r = x_r(1)$).*

Proof. Since all roots in E_8 have the same length, this implies that if $\tau \in \Psi$, $\tau \neq -\alpha_1$ and $\tau - \alpha_1$ and $\tau + \alpha_1 \notin \Delta$. Thus by the formula on [**4**, page 61] we see that $x_{\pm\alpha_1} e_\tau = e_\tau$. □

Let $L = H \oplus \sum_{t\in\Delta} L_t$ be the Cartan decomposition of L [**4**, Ch.3]. L can be written as $L = \sum_{t\in\Pi} H_t \oplus \sum_{t\in\Delta} L_t$ where $H_t = \langle h_t \rangle$, and $L_t = \langle e_t \rangle$. For $\Delta = E_8$, let $T = H_{\alpha_1} \oplus H_{\alpha_2} \oplus H_{\alpha_3} \oplus \sum_{t\in\Delta-\Psi} L_t$ and,

$$L' = \sum_{t\in\Pi-\{\alpha_2,\alpha_3\}} H_t \oplus \sum_{t\in\Psi} L_t.$$

Note. Note that $L = L' \oplus T$, and x_{α_1} and $x_{-\alpha_1}$ act trivially on L'.

Lemma 5.4. *Given L, L' and T as before, then there exists no subset $0 \neq S \leq L' \cap H$ invariant under the action of $E_8(K)$ where K is a finite field of characteristic 2.*

Proof. Assume that such an S exists, then let $x \in S - \{0\}$. We can write x as $x = a_1 h_{\tau_1} + \dots + a_n h_{\tau_n}$ where each $a_i \neq 0$, each $\tau_i \in \Pi$, and $f(\tau_1) < \dots < f(\tau_n)$.

Note that in all cases, $f(\tau_1) > 2$ by the way we constructed L'. Thus we can choose a root $t \in \Pi$ with $f(t) = f(\tau_1) - 1$ such that $\langle t, \tau_1 \rangle =$

$2(t,\tau_1)/(t,t) = 1(\bmod 2)$, and $\langle t,\tau_i\rangle = 0$ for $i = 2,\dots,n$. So we get that $x_t h_{\tau_1} = h_{\tau_1} + e_\tau$, and $x_t h_{\tau_i} = h_{\tau_i}$ for $i = 2,\dots,n$. Therefore $x_t x \notin S$, contradicting that S is $L(K)$-invariant. Therefore there exists no subset $0 \neq S \leq L' \cap H$ invariant under the action of $L(K)$. □

Lemma 5.5. *There is no invariant subspace of L' under the action of $E_8(K)$ where K is a finite field of characteristic 2.*

Proof. In view of Lemma 5.4 it will be sufficient to prove that any $E_8(K)$-invariant nonzero subspace S of L' is a subspace of H.

Assume that S is not a subspace of H, then we can write any element $v \in S - H$ as $v = a_{s_1}e_{s_1} + a_{s_2}e_{s_2} + \dots + h$, where $h \in H$, and $s_i \in \Psi$.

Since all the roots in E_8 have the same length, then by [**10**, 10.4 Lemma C] all the roots are conjugate under W, the Weyl group of E_8. Thus there exists an element $w \in W$, with $w(s_1) = \alpha_1$. Note that since w is an automorphism $w(s_i) \neq \alpha_1$ for $i > 1$, and that $w = w_{r_1} \dots w_{r_t}$ (here w_{r_i} is the reflection in the hyperplane orthogonal to the root r_i). Let N be the monomial subgroup of $E_8(K)$, and n_r as in [**4**, Proposition 6.4.2]. By [**4**, Theorem 7.2.2] there is a homomorphism from N onto W under which $n_r \longrightarrow w_r$ for all $r \in \Delta$. n_r acts invariantly on H, and $n_r e_s = \pm e_{w_r(s)}$. Since we are in characteristic 2, $n_r e_s = e_{w_r(s)}$. Let $n = n_{r_1} \dots n_{r_t} \in N$, then $n \longrightarrow w$ by the homomorphism $N \longrightarrow W$. So $ne_{s_1} = e_{w(s_1)} = e_{\alpha_1}$, and $nh \in H$. Hence nv contains a term e_{α_1}, contradicting that $v \in S$. Thus S is a subspace of H. □

Theorem 5.6. *If $G = E_8(K)$ where K is a finite field of characteristic 2, then G does not lie in $\Gamma_1^{(2)}$.*

Proof. Using the description of roots in the system E_8 in terms of orthogonal vectors given in [**2**, page 268] we get that the number of positive roots in Ψ is

$$1 + 6 + 6 + \binom{6}{2} + \binom{6}{2} + \binom{6}{3} = 63.$$

So we get that $\operatorname{cod}(L') = 117$. Both x_{α_1} and $x_{-\alpha_1}$ act trivially on L'; therefore $x_{\alpha_1}x_{-\alpha_1}$ also acts trivially on L'. Assume that G lies in $\Gamma_1^{(2)}$, then since $O(x_{\alpha_1}x_{-\alpha_1}) \neq 1$ or 2, $x_{\alpha_1}x_{-\alpha_1}$ has a mate of order 2. Let $S = L' \cap (L')^y$; since $G = \langle x_{\alpha_1}x_{-\alpha_1}, y\rangle$, we see that G acts invariantly on S. Since

$$\operatorname{cod}(S) \leq 2\operatorname{cod}(L') = 234 < 248,$$

$S \neq 0$ contradicting Lemma 5.5. □

6. Twisted Groups in odd Characteristic.

Lemma 6.1. *If q is odd and $G = {}^2A_l(q^2)$ for $l \geq 2$, or ${}^2D_l(q^2)$ for $l \geq 4$, or ${}^2E_6(q^2)$, then $|G| > |SL_2(q^2)|$. Also $|{}^3D_4(q^3)| > |SL_2(q^3)|$ for q odd.*

Proof. For $q = p^n$ where p is an odd prime we get:

1) $|SL_2(q^2)| = q^2(q^4-1) < (1/3)q^3(q^2-1)(q^3+1) \leq |{}^2A_l(q^2)|$, $l \geq 2$.
2) $|SL_2(q^2)| = q^2(q^4-1) < (1/4)q^{12}(q^4-1) < |{}^2D_l(q^2)|$, $l \geq 4$.
3) $|SL_2(q^2)| = q^2(q^4-1) < (1/3)q^{36}(q^5+1) < |{}^2E_6(q^2)|$.
4) $|SL_2(q^3)| = q^3(q^6-1) < q^{12}(q^6-1) < |{}^3D_4(q^3)|$.

□

Consider a Chevalley group $G^* = L(K)$ and ρ a non-trivial symmetry of the Dynkin diagram for L. Recall that if K has a certain order depending on L, then we can choose an automorphism σ (a product of a field automorphism and a graph automorphism determined by ρ) of G^* such that the twisted group $G = {}^iL(K)$ is a subgroup of G^* stabilized by σ, and U^1 is the subgroup of U centralized by σ (recall that $U^1 \leq G$). We will call G^* the corresponding Chevalley group of G. Note that $\sigma(x_r(t)) = x_{r'}(\gamma_r t')$, where $\gamma_r = \pm 1$ and $t' = f(t)$, f a certain field isomorphism, and r' arises from the symmetry of the Dynkin diagram [**4**, 12.2]. Since all the roots in the system associated with the groups of Lemma 6.1 have same length, the above action is an isometry [**4**, Prop. 12.2.2], and there exists a unique root of maximal length s [**10**, 10.4 Lemma A], and thus $s' = s$ (note that this is not the case for ${}^2G_2(3^{2m+1})$ [**4**, 12.4]).

Lemma 6.2. *If G is a simple twisted group, and G^* is the corresponding Chevalley group of G, then* $\mathrm{Aut}(G) \leq \mathrm{Aut}\,(G^*)$.

Proof. From [**8**, page 303 and 5, Table 5] $\mathrm{Aut}(G)$ is a product of an inner, a diagonal, and a field automorphism. □

Theorem 6.3. *If $q = p^n$ where p is an odd prime and G is a simple group of type ${}^2A_l(q^2)$, $l \geq 2$, or ${}^2D_l(q^2)$, $l \geq 4$, or ${}^2E_6(q^2)$, or ${}^3D_4(q^3)$, then G does not appear as a composition factor of any group in $\Gamma_1^{(2)}$.*

Proof. Let $K = GF(q^2)$ if $G = {}^2A_l(q^2), l \geq 2$, or ${}^2D_l(q^2), l \geq 4$, or ${}^2E_6(q^2)$, and $K = GF(q^3)$ if $G = {}^3D_4(q^3)$. Let s be the root of maximal height. If $\gamma_s = 1$, let $x = x_s(1)$. If $\gamma_s = -1$ (this can happen in the case that

$K = GF(q^2)$), then the automorphism f associated with the group has order 2 and there exists an element $t \in K - \{0\}$ such that $t' = -t$. So, let $x = x_s(t)$. In each case $\sigma(x) = x$ and $x \in Z(U) \cap U^1$ [**4**, Def. 13.4.2] so $x \in Z(U^1)$ and $O(x) = p$.

If G appears as a composition factor of any group in $\Gamma_1^{(2)}$, then x has a mate $y \in \mathrm{Aut}(G) \leq \mathrm{Aut}(G^*)$ of order 2, and $G = \langle x, x^y \rangle \leq \langle X_s, (X_s)^y \rangle = M$ (note that M is a subgroup of G^*, not of G) from Lemma 2.2. Since M is a nonnilpotent subgroup of G^* by Lemma 2.17, there is a homomorphism from $SL(2, K)$ onto M, but by Lemma 6.1 $|G| > |M|$, a contradiction. $\square$

7. Centralizers and $\Gamma_1^{(2)}$.

In this section we will use the result that if two subgroups have order greater that the square root of the order of the group, then they have a nontrivial intersection [**9**, Sec. 2.5], to investigate whether some groups lie in $\Gamma_1^{(2)}$.

Lemma 7.1. *If G is a finite simple group with an element of order not* 1 *or* 2 *that has a centralizer of order larger $(|G|)^{1/2}$, then G does not appear as a composition factor of any group in $\Gamma_1^{(2)}$.*

Proof. Assume that G appears as a composition factor of a group H in $\Gamma_1^{(2)}$. Then, by previous results, G is a normal subgroup of H with index at most two. Let $x \in G$ such that $O(x) \neq 1$ or 2 and $|C_G(x)| > (|G|)^{1/2}$. Since H lies in $\Gamma_1^{(2)}$, x has a mate $y \in H$ of order 2. By Lemma 2.2 $G = \langle x, x^y \rangle$, and since $|C_G(x^y)| = |C_G(x)|$ from the above we get that $Z(G) \neq 1$, contradicting the simplicity of G. $\square$

Theorem 7.2. *The groups J_2, Suz, Co_1, Ly, Fi_{22}, Fi_{23}, Fi'_{24}, $D_4(2)$, and $^2D_4(2)$ do not appear as a composition factor of any group in $\Gamma_1^{(2)}$.*

Proof. From [**5**] we see that each of the above groups has an element of order 3 with a centralizer of order greater than the square root of the order of the group. $\square$

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