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IRREDUCIBLE BIMODULES ASSOCIATED WITH CROSSED PRODUCT ALGEBRAS II

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Crossed product construction of bimodules is generalized to incorporate compact groups and its categorical structure is clarified.

1. Introduction.

The theory of bimodules is becoming more and more important in recent studies of operator algebras, particularly in Jones index theory (see [**A**], [**CK**], [**EK**],[**I1**], [**I2**], [**L1**], [**L2**], [**O1**], and [**O2**] for example).

In our previous paper ([**KY**]), we have introduced crossed product construction of bimodules based on finite groups which provides examples of integer index as well as the way to compute their invariants (paragroups) in terms of purely finite-dimensional representation theory of relevant finite groups.

As to examples of integer index, there is the construction of Wassermann based on fixed point algebras of compact group actions ([**Wa**]). See [**PW**] for the resent status of this construction.

The purpose of the present paper is to generalize the crossed product construction to incorporate the case of compact groups and clarify its categorical structure purely in terms of representation theory of compact groups. As an unexpected bonus of this generalization, we can reproduce the fixed point algebra construction of Wassermann in the form of $N \rtimes G - N \rtimes G$ bimodules. In particular, the known description of higher relative commutants as well as the associated graph invariants of Wassermann's construction can be neatly handled from the categorical point of view.

Our construction also supplies examples of irreducible bimodules with infinite index, which are still controllable in the sense that we can completely describe their fusion structure, i.e., the way of decompositions of tensor products.

As to the technical part of our construction, we are forced to impose a strong condition of outerness on relevant automorphic actions. This is much stronger than the notion of minimality from the appearance, but it is not clear at present whether there are any differences between them.

Though being restricted to integer index, the present construction would provide a variety of examples according to the reader's requirements. In the present paper, we describe just a few examples of index 6 and 8 as an illustration of our construction. Particularly we have obtained an irreducible bimodule of index 6 with graphs of Cayley type.

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Notation. We shall make free use of the following notations in this paper.

Γ: a second countable locally compact group.

G, H, K: compact subgroups in Γ.

$L^2(N)$: the standard space of a von Neumann algebra N.

$C_0(\Gamma)$: the commutative C^*-algebra of continuous functions on Γ vanishing at infinity.

$C(\Gamma)$: the commutative C^*-algebra of bounded continuous functions on Γ.

2. Vector bundles.

Let Γ be a locally compact second countable group and denote by $C_0(\Gamma)$ the commutative C^*-algebra of continuous functions on Γ vanishing at infinity. Let V be a $C_0(\Gamma)$-module, i.e., V is a Hilbert space on which $C_0(\Gamma)$ is represented as a C^*-algebra. By the representation theory of commutative C^*-algebras (see **[Ta]** for example), we can find a measure μ on Γ and μ-measurable field of Hilbert spaces $\{V_\gamma\}_{\gamma\in\Gamma}$ such that

$$V \cong \int_\Gamma^\oplus \mu(d\gamma) V_\gamma.$$

Here the action of $C_0(\Gamma)$ on V is identified with the multiplication operation in the right hand side. Note that such a realization is unique up to isomorphisms of measurable fields.

Take two compact subgroups H and K in Γ. A $C_0(\Gamma)$-module V is called an $H-K$ **bundle** over Γ if V admits an $H-K$ action (i.e., a commuting pair of left H- and right K-unitary representations) satisfying

$$h(fv) = (h.f)(hv), \quad (fv)k = (f.k)(vk), \qquad \text{for } f \in C_0(\Gamma),\ v \in V,\ h \in H,\ k \in K.$$

Here $h.f$ and $f.k$ denote left and right translations of f:

$$(h.f)(\gamma) = f(h^{-1}\gamma), \quad (f.k)(\gamma) = f(\gamma k^{-1}).$$

In the following, an $H-K$ bundle is expressed as $V =_H V_K$ (the $C_0(\Gamma)$-module structure is implicitly assumed).

In terms of direct integral realization of V, the above $H-K$ action is expressed as follows: Firstly the measure (class) μ must be quasi-invariant under the $H-K$ translation (translations in Γ by H from left and by K from right). Since H and K are assumed to be compact, we can adjust the measure μ so that it is invariant under the $H-K$ translation. Secondly the $H-K$ translation is lifted to an $H-K$ action on the measurable field $\{V_\gamma\}$: For $h \in H$ and $k \in K$, we have a measurable field of unitary maps $\{u_\gamma(h,k) : V_\gamma \to V_{h\gamma k}\}$ satisfying

$$u_{h\gamma k}(h',k')u_\gamma(h,k) = u_\gamma(h'h,kk') \text{ for almost all } \gamma \in \Gamma.$$

Decomposing Γ into $H-K$ orbits (this decomposition is well behaved because H and K are compact) and then applying Mackey's imprimitivity machine to each orbit, we can adjust the measurable field $\{V_\gamma\}$ so that the above cocycle condition is satisfied pointwise (without 'almost all' qualification). In other words, the measurable field $\{V_\gamma\}$ is identified (on each orbits) with induced vector bundles from stabilizing subgroups. (On basic facts in induced representations, we refer to [**Ki**].)

Now we go over to the categorical structure of $H-K$ bundles. For $V = {}_H V_K$, the adjoint $(K-H)$ bundle $V^* = {}_K V_H$ is defined in the following way: V^* is the dual vector space of V as a Hilbert space and $C_0(\Gamma)$- and $K-H$ actions are introduced by

$$fv^* = (f^*v)^*, \quad kv^*h = (h^{-1}vk^{-1})^*.$$

Here, for $v \in V$, $v^* \in V^*$ denotes the linear functional determined by $v^*(v') = (v|v')$ and, for $f \in C_0(\Gamma)$, $f^* \in C_0(\Gamma)$ is defined by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Note that if $\{V_\gamma\}$ with an $H-K$ invariant measure μ gives a decomposition of V, then V^* is decomposed according to the family $\{(V_\gamma)^*\}$ with the $K-H$ invariant measure $\mu^*(d\gamma) = \mu((d\gamma)^{-1})$.

Let G be another compact subgroup of Γ and let $V = {}_G V_H$, $W = {}_H W_K$. Then a $G-K$ bundle $V \otimes_H W$ is defined in the following manner: In the (usual) tensor product Hilbert space $V \otimes W$, define a unitary representation σ of H by

$$\sigma(h)(v \otimes w) = (vh^{-1}) \otimes (hw).$$

Set $V \otimes_H W = \{u \in V \otimes W;\ \sigma(h)u = u \text{ for all } h \in H\}$. For $v \in V$ and $w \in W$, we denote by $v \otimes_H w$ the projection of $v \otimes w$ to the closed linear subspace $V \otimes_H W$ of $V \otimes W$. Note that $vh \otimes_H w = v \otimes_H hw$ for $v \in V$, $w \in W$, and $h \in H$. The explicit form of the inner product is given by

$$(v \otimes_H w | v' \otimes_H w') = \int_H dh (vh|v')(w|hw').$$

Here dh denotes the normalized Haar measure of H.

The $G-K$ action on $V \otimes_H W$ is defined by multiplications from outside: $g(v \otimes_H w)k = (gv) \otimes_H (wk)$. To describe the $C_0(\Gamma)$-action, we first remark that, as the multiplier algebra of $C_0(\Gamma \times \Gamma) = C_0(\Gamma) \otimes C_0(\Gamma)$, $C(\Gamma \otimes \Gamma)$ acts on $V \otimes W$ (extending the obvious action of $C_0(\Gamma) \otimes C_0(\Gamma)$). Define a *-homomorphism $\sigma : C_0(\Gamma) \to C(\Gamma \otimes \Gamma)$ by $\sigma(f)(g_1, g_2) = f(g_1 g_2)$. As a combination of these, a $C_0(\Gamma)$-action is introduced in $V \otimes W$ by $f(v \otimes w) = (\sigma f)(v \otimes w)$. Since $V \otimes_H W$ is invariant under this action, we get the desired action of $C_0(\Gamma)$ on $V \otimes_H W$.

Finally set

$$\mathrm{Hom}(V, W) = \{T : V \to W; T(fv) = fT(v),\ T(hvk) = hT(v)k\}.$$

Taking $H-K$ bundles over Γ (for various H and K) as objects, we obtain a categorical *-algebra $\mathcal{C}(\Gamma)$.

Remark. The categorical structure of $\mathcal{C}(\Gamma)$ is completely described in terms of representation theoretical information on compact subgroups of Γ.

For later use, we shall describe the decomposition of $V \otimes_H W$ in terms of those for V and W. Let

$$V = \int_\Gamma^\oplus \mu(d\gamma) V_\gamma, \quad W = \int_\Gamma^\oplus \nu(d\gamma) W_\gamma$$

be direct integral decompositions with μ (resp. ν) a $G-H$ (resp. $H-K$) invariant measure in Γ. Take a $G-K$ invariant measure m in the image measure class of $\mu \times \nu$ under the map $\Gamma \times \Gamma \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in \Gamma$. Then $\mu \times \nu$ is decomposed as $\int m(d\gamma)\omega_\gamma$. Here $\{\omega_\gamma\}$ is an m-measurable family of measures in $\Gamma \times \Gamma$ with ω_γ supported by $\{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma; \gamma_1\gamma_2 = \gamma\}$. From the H-invariance of μ and ν, we deduce that ω_γ is invariant under the action of H on $\Gamma \times \Gamma$ defined by $h(\gamma_1, \gamma_2) = (\gamma_1 h^{-1}, h\gamma_2)$. Then the fibres of $V \otimes W$ with respect to the action of $C_0(\Gamma)$ on $V \otimes W$ (through $\sigma : C_0(\Gamma) \to C(\Gamma) \otimes C(\Gamma)$) and the measure m is given by

$$(V \otimes W)_\gamma = \int_{\Gamma \times \Gamma}^\oplus d\omega_\gamma(\gamma_1, \gamma_2) V_{\gamma_1} \otimes W_{\gamma_2}.$$

Note that an element of $(V \otimes W)_\gamma$ is given by a measurable section of the measurable field $\{V_{\gamma_1} \otimes W_{\gamma_2}\}_{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma}$ of Hilbert spaces.

Since the H-action σ considered in the definition of $V \otimes_H W$ is given by the fibre-wise action σ_γ, we can identify the fibre $(V \otimes_H W)_\gamma$ of $V \otimes_H W$ at $\gamma \in \Gamma$ with

$$\begin{aligned}\{f \in (V \otimes W)_\gamma; f(\gamma_1 h^{-1}, h\gamma_2) \\ = \sigma_\gamma(h)(f(\gamma_1, \gamma_2)) \forall h \in H, \forall (\gamma_1, \gamma_2) \in \text{ the support of } \omega_\gamma\}.\end{aligned}$$

In particular, applying this argument to the case $H = K$, we have

Lemma 1. *Let G be a compact group and $\nabla^1, \dots, \nabla^n$ be $G-G$ bundles over G. Let $\{V_g^i\}_{g\in G}$ be the disintegration of V^i with respect to the (normalized) Haar measure of G. Then the fibre of $V^1 \otimes_G \cdots \otimes_G V^n$ at $g \in G$ is given by the set of L^2-sections, say f, of the measurable field $\{V_{g_1}^1 \otimes \cdots \otimes V_{g_n}^n\}_{(g_1,\dots,g_n)\in G^n(g)}$ of Hilbert spaces over $G^n(g) = \{(g_1, \dots, g_n) \in G^n; g_1 \cdots g_n = g\}$ which is multiply G-equivariant:*

$$f(g_1h_1^{-1}, h_1g_2h_2^{-1}, \dots, h_{n-2}g_{n-1}h_{n-1}^{-1}, h_{n-1}g_n) = \sigma^{n-1}(h_1, \dots, h_{n-1})f(g_1, \dots, g_n).$$

Here

$$\sigma^{n-1}(h_1, \dots, h_{n-1}) : V_{a_1}^1 \otimes \cdots \otimes V_{a_n}^n \to V_{a_1h_1^{-1}} \otimes V^2_{h_1a_2h_2^{-1}} \otimes V^n_{h_{n-1}a_n}$$

is defined by

$$\sigma^{n-1}(h_1, \dots, h_{n-1})(v_1 \otimes \cdots \otimes v_n) = v_1h_1 \otimes h_1^{-1}v_2h_2^{-1} \otimes \cdots \otimes h_{n-1}v_n.$$

The situation is symbolically expressed as

$$(V^1 \otimes_G \cdots \otimes_G V^n)_g = \left[\int^{\oplus}_{g_1\cdots g_n = g} V_{g_1}^1 \otimes \cdots \otimes V_{g_n}^n\right]^{G\times\cdots\times G}.$$

Corollary 2.

(i) *The disintegration of the $G-G$ bundle ${}_GL^2(G)_G$ of the regular representation of G is given by the trivial bundle $G \times \mathbb{C} = \{\mathbb{C}_g\}_{g\in G}$ (here g in G is identified with the element $(g, 1)$ in the fibre of $G \times \mathbb{C}$).*

(ii) *For a $G-G$ bundle $V = \int_G^{\oplus} dg\, V_g$ over G, the canonical isomorphism ${}_GV_G \cong_G L^2(G) \otimes_G V_G$ is induced from the fibre-wise isomorphism defined by*

$$V_g \ni v \mapsto \tilde{v} \in (L^2(G) \otimes_G V)_g$$

with

$$\tilde{v}(g_1, g_2) = g_1 \otimes g_1^{-1}v_g \in \mathbb{C}_{g_1} \otimes V_{g_2}.$$

3. Crossed products.

Let α be an automorphic action of Γ on a von Neumann algebra N. For a (compact) subgroup G of Γ, the crossed product $N \rtimes G$ should be understood with respect to the restriction of α to G. Given an $H-K$ bundle V over Γ, define an $N \rtimes H - N \rtimes K$ bimodule $\widehat{V}$ in the following way: As a Hilbert space, we set $\widehat{V} = L^2(N) \otimes V$.

Lemma 3. *Take an $H - K$ invariant measure μ from the spectral measure class of the $C_0(\Gamma)$-action on V and let $\{V_\gamma\}$ be a measurable field realization of V with respect to μ. Define a unitary map $I_V : L^2(N) \otimes V \to V \otimes L^2(N)$ by*

$$I_V(\xi \otimes v) = \int_\Gamma^\otimes \mu(d\gamma) v(\gamma) \otimes \alpha_{\gamma^{-1}}(\xi).$$

Here $v = \int_\Gamma^\otimes \mu(d\gamma)v(\gamma)$ and α_γ on $L^2(N)$ denotes the canonical unitary implementation of $\alpha_\gamma \in \mathrm{Aut}(N)$.

Then I_V does not depend on the choice of μ and $\{V_\gamma\}$.

On $L^2(N) \otimes V$, define representations (or left actions) of N and H by

$$y(\xi \otimes v) = (y\xi) \otimes v, \;\; h(\xi \otimes v) = \alpha_h(\xi) \otimes hv.$$

These form a covariant representation of the automorphic action $\alpha|_H$ of H on N. Since the representation of N is clearly normal, we obtain the normal representation the W^*-crossed product $N \rtimes H$ on $\widehat{V}$ by the following lemma:

Lemma 4. *Let G be a compact group and α be an automorphic action of G on a factor N. Take a covariant representation (π, u) of (N, G) in a Hilbert space X such that π is normal as a representation of N. Then the covariant representation (π, u) is (uniquely) extended to a normal representation of the W^*-crossed product $N \rtimes G$.*

Proof. ∵) Recall that the crossed product algebra $N \rtimes G$ is identified with the von Neumann algebra generated from the covariant representation of N and G on $L^2(G) \otimes X = L^2(G; X)$ defined by

$$(y\xi)(g) = \alpha_{g^{-1}}(y)\xi(g), \;\; (g\xi)(g') = \xi(g^{-1}g').$$

Transporting these actions through the unitary map $I : L^2(G) \otimes X \to X \otimes L^2(G)$ defined by $(I\xi)(g) = u(g)\xi(g)$, we see that $N \rtimes G$ is generated by $N \otimes 1$ and $\{u(g) \otimes \lambda(g);\; g \in G\}$ in $X \otimes L^2(G)$. Here λ refers to the left regular representation of G. Since $L^2(G)$ contains the trivial representation of G (G being assumed to be compact), the covariant representation (π, u) in X is identified with a subrepresentation of $(N \otimes 1, u \otimes \lambda)$ in $X \otimes L^2(G)$. Thus, as a corner of $N \rtimes G$, this covariant representation is extended to a normal representation of $N \rtimes G$ on X. □

Similarly, we can define a right (normal) representation of $N \rtimes K$ on $V \otimes L^2(N)$ and then, transporting by I_V, a right representation of $N \rtimes K$

on $\widehat{V}$. It is immediate to check that these two actions on $\widehat{V}$ commute with each other, i.e., $\widehat{V}$ is an $N \rtimes H - N \rtimes K$ bimodule.

Let V and W be two $H - K$ bundles over Γ. For $T \in \mathrm{Hom}(V, W)$, we associate an intertwiner $\widehat{T} \in \mathrm{Hom}(\widehat{V}, \widehat{W})$ by

$$\widehat{T}(\xi \otimes v) = \xi \otimes T(v).$$

Proposition 5. *The correspondences $V \mapsto \widehat{V}$ and $T \mapsto \widehat{T}$ give a functor from $\mathcal{C}(\Gamma)$ into the category of bimodules which preserves adjoints, direct sums, and tensor products:*
(i) $\widehat{V^*} = \widehat{V}^*$.
(ii) $\widehat{V \oplus W} = \widehat{V} \oplus \widehat{W}$.
(iii) $\widehat{V \otimes W} = \widehat{V} \otimes \widehat{W}$.
This functor is refered to as **crossed product functor** *in this paper.*

Proof. ∵) (i) and (ii) are immediate. For (iii), the argument in [**Y2**] works here without essential changes. □

Remark. We can find a primitive form of the crossed product construction of bimodules in [**CJ**].

4. Outer actions.

An automorphic action α of Γ on a factor N is called **very outer** if, for any Radon-measure μ on Γ, we have

$$(*) \qquad \mathrm{End}({}_N L^2(N) \otimes L^2(\Gamma, \mu)_N) = 1 \otimes L^\infty(\Gamma, \mu).$$

Here the $N - N$ action on $L^2(N) \otimes L^2(\Gamma, \mu) = L^2(\Gamma, \mu;\ L^2(N))$ is defined by

$$(y\xi)(\gamma) = y\xi(\gamma),\ (\xi y)(\gamma) = \xi(\gamma)\alpha_\gamma(y),\ y \in Y,\ \gamma \in \Gamma.$$

Note that, for discrete groups, this condition is equivalent to the usual outerness of actions.

The outerness of this kind is closely related with the minimality of action, i.e., $N' \cap (N \rtimes \Gamma) = \mathbb{C}$. For example, if the above condition is satisfied for the Haar measure of Γ, then the relative commutant $N' \cap (N \rtimes \Gamma)$ is trivial which is, in turn, equivalent to the minimality of the action as long as Γ being compact.

Lemma 6. *Let π be a faithful representation of a compact group G in a finite dimensional vector space V. Then the action of G on the* AFD II_1

factor R induced from the infinite tensor product of $\mathrm{Ad}\,\pi$ *on* $\mathrm{End}(V)$ *is very outer.*

Proof. $\because$) Let S be the subalgebra of R generated by finite permutations in $V \otimes V \otimes \cdots$. Clearly the product type action of G fixes the generators of S and hence any elements in S.Since the relative commutant $S' \cap R$ is known to be trivial (see [**Wa**] and also cf. [**GHJ**, p. 231]), we have

$$\mathrm{End}({}_RL^2(R) \otimes L^2(G,\mu)_S) = 1 \otimes \mathcal{B}(L^2(G,\mu)).$$

Thus any $T \in \mathrm{End}({}_RL^2(R) \otimes L^2(G,\mu)_R)$ can be regarded as an operator in $\mathcal{B}(L^2(G,\mu))$. Moreover T commutes with the right action of $\mathrm{End}(V) \otimes \cdots \otimes \mathrm{End}(V) \subset R$, i.e.,

$$T((x\tau^{1/2} \otimes f)y) = T(x\tau^{1/2} \otimes f)y \text{ for } x, y \in \mathrm{End}(V) \otimes \cdots \otimes \mathrm{End}(V).$$

Here $\tau^{1/2}$ denotes the GNS-vector associated with the normalized trace τ of R. Taking the partial inner product with $\tau^{1/2}$ (the L^2-version of slice map), we have

$$T : \tau(x\alpha_g(y))f(g) \mapsto \tau(x\alpha_g(y))T(f)(g).$$

Since functions $g \mapsto \tau(x\alpha_g(y))$ on G for various $x, y \in \mathrm{End}(V) \otimes \cdots \otimes \mathrm{End}(V)$ provide the *-algebra generated by the coefficient functions of the representation V, the faithfulness assumption of G on V implies that these functions form a uniformly dense *-subalgebra in $C(G)$ (see, for example, [**Ch**]). Thus T needs to commute with multiplication operators in $L^2(G,\mu)$, i.e., T itself is the multiplication operator by a function in $L^\infty(G,\mu)$. In this way, we have shown that $\mathrm{End}({}_RL^2(R) \otimes L^2(G,\mu)_R) \subset L^\infty(G,\mu)$. The reverse inclusion is trivial. □

Question. Does minimality imply very outerness for compact groups?

Remark. The uniqueness of minimal actions of compact groups on the AFD II_1 factor has been announced by A. Ocneanu and Popa-Wassermann. This combined with the above lemma supplies an affirmative answer to the above question for actions on AFD II_1 factors.

Theorem 7. *Let Γ be a locally compact second countable group and α be a very outer automorphic action of Γ on a factor N. Then the crossed product functor is an isomorphism, i.e.,*

$$\mathrm{Hom}(V, W) \cong \mathrm{Hom}(\widehat{V}, \widehat{W}).$$

Proof. $\because$) Since $\mathrm{Hom}(\widehat{V}, \widehat{W})$ is a reduction of $\mathrm{End}(\widehat{V} \oplus \widehat{W})$, we may assume that $V = W$. Considering the direct integral decomposition of V over Γ

and then taking an orthonormal basis for the relevant measurable field, we can find a mutually orthogonal family $\{\mu_i\}_{l\geq 1}$ of measures in Γ and a family $\{E_i\}_{i\geq 1}$ of Hilbert spaces such that ${}_N\widehat{V}_N$ is equivalent to

$$\sum_{i\geq 1} {}_N L^2(N) \otimes L^2(\Gamma, \mu_i)_N \otimes E_i.$$

Then, applying the very outerness condition to this bimodule, we see that

$$\operatorname{End}\left(\sum_{i\geq 1} {}_N L^2(N) \otimes L^2(\Gamma, \mu_i)_N \otimes E_i\right)$$
$$= \sum_{i\geq 1} 1_{L^2(N)} \otimes L^\infty(\Gamma, \mu_i) \otimes \mathcal{B}(E_i).$$

Transporting this last relation into the starting Hilbert space $\widehat{V}$, we get the following relation:

$$\operatorname{End}({}_N L^2(N) \otimes V_N) = 1 \otimes \operatorname{End}({}_{C_0(\Gamma)}V).$$

Now taking the $H - K$ action into account, we finally obtain the claimed identification:

$$\operatorname{End}({}_{N\rtimes H}\widehat{V}_{N\rtimes K}) = 1_{L^2(N)} \otimes \operatorname{End}({}_H V_K).$$

□

As discussed in **[KY]**, **[Y2]**, this categorical equivalence enables us to compute the invariants of inclusion relations associated with bimodules of crossed product type.

Remark. Suppose that we are given a full subcategory $\mathcal{C}'$ of $\mathcal{C}(\Gamma)$. Then the set $\mathcal{M}$ of measure classes in Γ which are obtained decomposing objects V in $\mathcal{C}'$ is closed under convolution and the categorical isomorphism remains to hold for the subcategory $\mathcal{C}'$ if the condition $(*)$ in the definition of very outerness is satisfied for μ in $\mathcal{M}$.

In particular, letting $\Gamma = G$ (a compact group), the totality of $G - G$ bundles supported by G gives a full subcategory $\mathcal{C}$ of $\mathcal{C}(G)$ and $\mathcal{M}$ for $\mathcal{C}$ consists of a single measure class, i.e., the Haar measure class of G.

Thus the crossed product construction for bundles in $\mathcal{C}$ gives an isomorphism if the action is minimal. This observation will be used below with relation to the Wassermann's construction.

5. Criterion of irreducibility.

For an $H - K$ bundle V, its support $\mathbf{s}(V)$ is defined to be the support of the spectral measure μ for the $C_0(\Gamma)$-action on V:

$$\Gamma \setminus \mathbf{s}(V) = \bigcup\{O;\ O \text{ is an open subset of } \Gamma \text{ such that } \mu(O) = 0\}.$$

Note that $\mathbf{s}(V)$ is an $H-K$ invariant closed subset of Γ. The following property of support is easy to check: (i) $\mathbf{s}(V^*) = \mathbf{s}(V)$, (ii) $\mathbf{s}(V \oplus W) = \mathbf{s}(V) \cup \mathbf{s}(W)$, and (iii) $\mathbf{s}(V \otimes W) = \mathbf{s}(V)\mathbf{s}(W)$.

Proposition 8. *$\widehat{V}$ is irreducible if $\mathbf{s}(V)$ consists of a single $H-K$ orbit, say $H\gamma K$, and its stabilizer $H \times_\gamma K$ at γ acts on the fibre V_γ irreducibly*

Proof. $\because$) This follows from Theorem 7 and Mackey's imprimitivity theorem (cf. the Glimm's theorem on the regularity of transformation groups). □

Remark. By Mackey's imprimitivity theorem, the fibre V_γ in (i) is canonically (and uniquely) determined.

6. Index formula.

For an $H-K$ bundle V, set

$$^{H}V = \{v \in V;\ hv = v,\ \forall h \in H\}, V^K = \{v \in V;\ vk = v,\ \forall k \in K\}.$$

The following formula generalizes the one in [**KY**], [**Y2**].

Theorem 9. (Index formula). *Let α be a very outer automorphic action of Γ on a factor N. For an $H-K$ bundle V over Γ with $\widehat{V}$ the associated $N \rtimes H - N \rtimes K$ bimodule, the index of $\widehat{V}$ (= the minimal index of the inclusion relation of factors associated with $\widehat{V}$) is given by*

$$\dim(^{H}V)\dim(V^K).$$

Remark. By the above index formula, one sees that the most case of crossed product construction has infinite index. Even in that case, we can still describe the structure of their higher relative commutants in terms of vector bundles.

Although we can deduce the above formula based on the invariance principle for the fixed point algebra construction (cf. the discussion in the part of Examples), we shall present a much more direct proof here. To this end, we need the following observation which would be interesting by itself:

Lemma 10. *Let ${}_AX_B$ be an irreducible bimodule with A and B factors. Assume that we can find isometries I in $\mathrm{Hom}({}_AL^2(A)_A, {}_AX \otimes^B X^*_A)$ and J in $\mathrm{Hom}({}_BL^2(B)_B, {}_BX^* \otimes^A X_B)$. Then the index for X (i.e., the index of the inclusion relation $A \subset B'$) is finite and given by*

$$\|(I^* \otimes 1_X)(1_X \otimes J)\|^{-2}.$$

Proof. $\because$) By [**Y1**, Theorem 3.6] and the irreducibility of X, we can find (normal) conditional expectations $E : B' \to A$ and $F : A' \to B$, and complex numbers u, v of modulus 1 satisfying

$$I(\varphi^{1/2}) = u(\varphi \circ E)^{1/2}, \ J(\psi^{1/2}) = v(\psi \circ F)^{1/2}$$

for $\varphi \in A_*^+$ and $\psi \in B_*^+$. Here $\varphi^{1/2}$ and so on refer to the canonical implementing vectors in standard spaces.

As a result, the index of X is finite (cf. [**Y3**, Corollary 2.8]) and [**Y4**, Lemma 4.1(i)] shows that

$$(I^* \otimes 1_X)(1_X \otimes J) = \bar{u}vd(X)^{-1}1_X.$$

(Here $d(X)$ denotes the square root of the minimal index of the inclusion $A \subset B'$.) □

Remark.

(i) By the irreducibility of X, $(I^* \otimes 1_X)(1_X \otimes J)$ is always a scalar operator.

(ii) If I and J preserve positive parts in standard spaces (this is possible if one takes positive parts in the polar decomposition of I and J), then we have $(I^* \otimes 1_X)(1_X \otimes J) = d(X)^{-1}1_X$.

(iii) For an irreducible X with finite index, we can always find such I, J as 'square roots' of conditional expectations. Thus the above formula can be used as a definition of index.

Let G be a compact group and V_e be a unitary representation (space) of G. Put $V = L^2(G) \otimes V_e$ on which $C(G)$ acts by multiplication. Moreover V is made into a $G - G$ bundle over G by

$$g(\xi \otimes v) = (g\xi) \otimes v, \ (\xi \otimes v)g = \xi g \otimes g^{-1}v.$$

Lemma 11. *Let G be a compact group and σ be a finite-dimensio-nal (unitary) representation of G on a Hilbert space V_e. Let $V = L^2(G) \otimes V_e$ be a $G - G$ bundle over G described above. Then the index of V (= the index of $\widehat{V}$) is given by* $(\dim V_e)^2$.

Proof. $\because$) By the additivity of the square root of index and the additivity of $V_e \mapsto \dim V_e$, we may assume that V_e is irreducible. Take an orthonormal basis $\{v_i\}$ of V_e. To each $g \in G$, assign a vector i_g in $(V \otimes_G V^*)_g$ by

$$i_g(g_1, g_2) = d^{-1/2} \sum_i g_1 v_i \otimes v_i^* g_2 \in V_{g_1} \otimes (V^*)_{g_2} \text{ with } g_1 g_2 = g.$$

Here we put $d = \dim V_e$. Since the summation $\sum_i v_i \otimes v_i^*$ is independent of the choice of orthogonal basis, we see that

$$G \times \mathbb{C} \ni (g, z) \mapsto z i_g \in (V \otimes_G V^*)_g$$

gives a $G-G$ bundle map which turns out to be isometric by the calculation:

$$\begin{aligned}(\dim V_e)\|i_g\|^2 &= \int_G dg_1 \left(\sum_i g_1 v_i \otimes v_i^* g g_1^{-1} \middle| \sum_i g_1 v_j \otimes v_j^* g g_1^{-1}\right)\\ &= \int_G dg_1 \sum_{i,j} (g_1 v_i | g_1 v_j)(v_i^* g g_1^{-1})\\ &= \dim V_e.\end{aligned}$$

Integrating the family $\{i_g\}_{g\in G}$ we obtain an isometry I in $\mathrm{Hom}({}_G L^2(G)_G, {}_G V \otimes_G V_G^*)$.

Similarly we construct an isometry $J :_G L^2(G)_G \to_G V^* \otimes_G V_G$ with the help of $j_g \in (V^* \otimes_G V)_g$ defined by

$$j_g(g_1, g_2) = \sum_i g_1 v_i^* \otimes v_i g_2 \in (V^*)_{g_1} \otimes V_{g_2} \text{ with } g_1 g_2 = g.$$

To apply Lemma 11, we need to compute $(I^* \otimes 1_V)(1_V \otimes J)$. By Corollary 2, $I \otimes 1_V$ and $1_V \otimes J$ are given by the following bundle maps $(I \otimes 1_V)_g$ and $(1_V \otimes J)_g$ from V_g into $(V \otimes_G V^* \otimes_G V)_g$:

$$\begin{aligned}(I \otimes 1_V)_g(v) = {}`(g_1, g_2, g_3)\\ \mapsto d^{-1/2} \sum_i g_1 v_i \otimes v_i^* g_2 \otimes (g_1 g_2)^{-1} v \in V_{g_1} \otimes (V^*)_{g_2} \otimes V_{g_3},'\end{aligned}$$

$$\begin{aligned}(1_V \otimes J)_g(v) = {}`(g_1, g_2, g_3)\\ \mapsto d^{-1/2} \sum_j v g_1 \otimes g_2 v_i^* \otimes v_i (g_1 g_2)^{-1} v \in V_{g_1} \otimes (V^*)_{g_2} \otimes V_{g_3}.'\end{aligned}$$

(Note that (g_1, g_2, g_3) is subject to the condition $g_1 g_2 g_3 = g$.) On each fibre we have

$$
\begin{aligned}
&((I \otimes 1_V)_g v | (1_V \otimes J)_g v') \\
&= d^{-1} \int_{G\times G} dg_1 dg_2 \Big(\sum_i g_1 v_i \otimes v_i^* g_2 \otimes \\
&(g_1 g_2)^{-1} v \Big| \sum_j v' g_1 \otimes g_2 v_j^* \otimes v_j (g_1 g_2)^{-1} \Big) \\
&= d^{-1} \sum_{i,j} \int dg_1 dg_2 (g_1 v_i | v' g_1)(v_i^* g_2 | g_2 v_j^*)((g_1 g_2)^{-1} v | v_j (g_1 g_2)^{-1}) \\
&= d^{-1} \sum_{i,j} \int dg_1 dg_2 (v_i | \sigma(g_1)^{-1} v')(\sigma(g_2) v_j | v_i)(v | \sigma(g_1 g_2) v_j) \\
&= d^{-1} \sum_{j} \int dg_1 dg_2 (\sigma(g_2) v_j | \sigma(g_1)^{-1} v')(v | \sigma(g_1 g_2) v_j) \\
&= d^{-1} \int dg_1 dg_2 (\sigma(g_1 g_2)^{-1} v | \sigma(g_1 g_2)^{-1} v') \\
&= d^{-1} (v | v')
\end{aligned}
$$

for $v, v' \in V_g$.

Thus $(I^* \otimes 1_V)(1_V \otimes J) = (\dim V_e)^{-1} 1_V$, which shows that the index of the $N \rtimes G - N \rtimes G$ bimodule $\widehat{V}$ is equal to $(\dim V_e)^2$. □

Proof of the Index formula. First we assume that $\widehat{V}$ is irreducible and finite index. Let $H\gamma K$ be the support of V. Consider the bimodule ${}_K L^2(K)\gamma^{-1}_{\gamma K \gamma^{-1}}$ associated to the trivial $K - \gamma K \gamma^{-1}$ bundle supported by $K\gamma^{-1}$. Then the actions of K and $\gamma K \gamma^{-1}$ generate commutants of each other and hence $V \otimes_K L^2(K)\gamma^{-1}$ takes the same index with V. Since $V \otimes_K (K)\gamma^{-1}$ is supported by $H\gamma K \gamma^{-1}$, we may assume that $\gamma = 1$ for the proof of the index formula.

By Frobenius reciprocity (see [**Y4**, Corollary 1.6] for example), $\widehat{V} \otimes_{N \rtimes K} \widehat{V}^*$ needs to contain the trivial bundle $L^2(N \rtimes H)$ with multiplicity 1. By Theorem 7, this implies that we should have a non-trivial intertwiner T in $\mathrm{Hom}({}_H L^2(H)_H, {}_H V \otimes_K V_H^*)$. Since T is a decomposable operator, we see that $H \subset \Gamma$ is not negligible with respect to the measure (class) associated with $V \otimes_K V^*$.

Since the measure (class) in question is the image of the Haar measure of $H \times K \times H$ nder the map $(h_1, k, h_2) \mapsto h_1 k h_2$ (recall that V is supported by HK) and since the inverse image of H under this map is given by $H \times (H \cap K) \times H$, H is not negligible iff $H \cap K$ is open in K. By the symmetry of arguments, $H \cap K$ is open in H. Thus $H_0 = K_0$, where the suffix 0 refers to connected components.

Put $G = H_0 = K_0$ and consider the restricted bundle ${}_G V_G$. The bundle

${}_G V_G$ is decomposed as ${}_G V_G = \oplus_{iG} V_G^i$ according to the double coset decomposition $G \backslash HK/G$ of the support HK of V. Here i represents each $G-G$ coset in HK. Applying the previous tensoring argument to shift the supports if necessary, we can use the index formula in Lemma 11 to each $G-G$ bundle ${}_G V_G^i$, obtaining

$$d({}_G V_G^i) = \dim V_e.$$

Here V_e denotes the fibre of V at the unit $e \in \Gamma$ while $d(\cdot)$ refers to the square root of minimal indices. Since G is normal in H and K, the number of $G-G$ cosets in HK is calculated as

$$|H/H \cap K||H/G|.$$

Whence we get

$$d({}_G V_G) = |H/H \cap K||H/G| \dim V_e.$$

Since $d({}_G L^2(H)_H) = |H/G|^{1/2}$ and $d({}_K L^2(K)_G) = |K/G|^{1/2}$ (cf. the argument in [**Y2**]), we have

$$\begin{aligned} d({}_G V_G) &= d({}_G L^2(H) \otimes_H V \otimes_K L^2(K)_G) \\ &= d({}_G L^2(H)_H) d({}_H V_K) d({}_K L^2(K)_G) \\ &= |H/G|^{1/2} |K/G|^{1/2} d({}_H V_K). \end{aligned}$$

Combining this with the above formula shows that

$$d({}_H V_K) = |K/H \cap K|^{1/2} |H/H \cap K|^{1/2} \dim V_e.$$

Since $\dim^H V = |K/H \cap K| \dim V_e$ and $\dim V^K = |H/H \cap K| \dim V_e$, we obtain the index formula for irreducible $\widehat{V}$.

If $\widehat{V}$ is assumed to be of finite index but not necessarily irreducible, then $\widehat{V}$ is decomposed into a direct sum of irreducible bimodules, which, in turn, gives rise to the decomposition of V into irreducible components. Since $(\dim({}^H V) \dim(V^K))^{1/2}$ is additive with respect to the direct sum operation on V, the index formula holds in this case as well. Note that the discussion of this kind remains valid for infinite index case as long as $H_0 = K_0$.

Finally assume that the index of $\widehat{V}$ is finite and $H_0 \neq K_0$. Then the subgroup $H \cap K$ is not open in either K or H, which implies that $\dim({}^H V) = +\infty$ or $\dim(V^K) = +\infty$. Thus the index formula holds also in this case. $\square$

Corollary 12. *The index of $\widehat{V}$ is finite if and only if $\dim({}^H V) < +\infty$ and $\dim(V^K) < +\infty$.*

Furthermore $\widehat{V}$ is locally finite in the sense of [**Y3**] *if ${}_H V_K$ is supported at most countably many $H-K$ orbits in Γ.*

7. Examples.

Let us begin with the relation between the present construction and the Wassermann's fixed point algebra construction.

Let G be a compact group with a very outer automorphic action α on a factor N (cf. the remark after Theorem 7). Take a left G-module V_e. Then we obtain the crossed product bimodule $\widehat{V}$ on the one hand and the inclusion of fixed point algebras $N^G \subset (N \otimes \mathcal{B}(V_e))^G$ on the other hand.

Now consider a bimodule $H =_{N^G} (L^2(N) \otimes V_e)_{N \rtimes G}$: The N^G-action is just the left multiplication on $L^2(N)$ and the $N \rtimes G$-action is the one accompanied with the (right) covariant representation defined by

$$(\xi \otimes v)y = (\xi y) \otimes v, \quad (\xi \otimes v)g = \alpha_{g^{-1}}(\xi) \otimes g^{-1}v.$$

It is immediate to see that the left inclusion relation $N^G \subset \mathrm{End}(H_{N \rtimes G})$ of H is given by $N^G \subset (N \otimes \mathcal{B}(V_e))^G$, while the right inclusion relation of H is isomorphic to that of $\widehat{V}$, i.e., $N \rtimes G \subset \mathrm{End}({}_{N \rtimes G}\widehat{V})$ ($N \rtimes G$ is identified with its image in $\mathcal{B}(\widehat{V})$ because $N \rtimes G$ is a factor). Since an inclusion of factors and the opposite inclusion of its commutants are in the dual relation (cf. [**Y4**, Corollary 2.3]), the structure of higher relative commutants of $N^G \subset (N \otimes \mathcal{B}(V_e))^G$ is completely determined by the representation theory of G (cf. [**Yi**]).

Our crossed product bimodules take integers as index by Theorem 9. Since the classification of index ≤ 4 is now available (see [**P2**] and references cited there), we are interested in examples of index ≥ 5.

First consider the case of index 5. Given a pair of a (finite) group G and its subgroup H with $|G/H| = 5$, the inclusion relation $N \rtimes H \subset N \rtimes G$ gives an example of index 5 (as a matter of fact, the name 'index' is taken from this kind of examples) whose invariant (or paragroup) can be calculated by the method discussed in [**KY**]. These are examples of finite depth.

Since the fixed point algebra construction for compact but not finite groups provides examples of infinite depth, it would be natural to seek for such examples in the crossed product construction. The answer is, however, negative:

Proposition 13. *In the crossed product construction, all the irreducible examples of prime index are equivalent to the group-subgroup case. In particular, there does not appear examples of infinite depth.*

Proof. ∵) Let V be an irreducible $H - K$ bundle of index 5. From the above proof of the index formula, we may assume that V is supported by HK. Let σ be the irreducible representation of $H \cap K$ associated with V (see

Proposition 8). Then the index formula gives

$$(\dim \sigma)^2 |H/H \cap K||K/H \cap K| = p$$

which shows that $\dim \sigma = 1$ and $H \subset K$ or $K \subset H$. Since the situation is symmetric, we may assume that $H \subset K$ with $|K/H| = p$. Let V_1 be the $H - H$ bundle associated with σ and V_2 be the trivial $H - K$ bundle supported by K. Then $V = V_1 \otimes_H V_2$. Since the index of V_1 is 1, the inclusion relation associated to $\widehat{V}$ is same with that of $\widehat{V}_2$. In this way, the bundle V is reduced to the case $\sigma = 1$ (the trivial representation), i.e., the associated inclusion relation is isomorphic to $N \rtimes H \subset N \rtimes K$. □

Remark. A bimodule of index 5 or 7 is necessarily irreducible. This follows from the additivity of the square roots of induces and the restricted values for index ≤ 4.

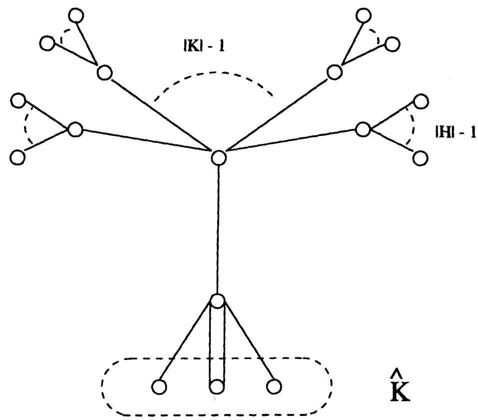
As to the case of non-prime index, there are a lot of examples of infinite depth. For instance, an example of index 6 is constructed as follows: Take the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as an ambient group Γ and consider a $\mathbb{Z}_2 - \mathbb{Z}_3$ bundle V supported by $\mathbb{Z}_2 \cdot \mathbb{Z}_3$. Since the support of $V \otimes V^* \otimes \cdots \otimes V$ (or V^*) increases to the whole Γ, there appear infinitely many inequivalent irreducible components and hence gives an example of infinite depth with index 6.

More generally, we can prove the following:

Proposition 14. *Let H and K be two finite groups and let $\Gamma = H * K$ be the free product of H and K (so H and K are identified with subgroups of Γ). Let V be the trivial $H - K$ bundle supported by $HK \subset \Gamma$. Then the left graph of V (i.e., the graph obtained from V by tensoring V or V^* repeatedly from left) is described as follows: $V^* \otimes_H V$ consists of the regular representation of K supported by K and mutually inequivalent irreducible bundles supported by KhK with $1 \neq h \in H$. In the second step of the tensoring, the irreducible components of the regular representation of K go back to V while the bundle supported with KhK splits into irreducible components supported by $HkhK$ with $k \in K$, i.e., there appear new stuffs parametrized by the supports $HkhK$ with $1 \neq k \in K$. Repeating the argument of this kind, we obtain a kind of Cayley graph with ornaments.*

Question. Are there any examples of infinite depth with prime index?

Now we shall give illustrating examples of the present construction. The idea is as follows: Let $T = \mathbb{T}^2$ be the two-dimensional toral group and take $\mathbb{T} \rtimes GL(2, \mathbb{Z})$ as the ambient group. Here $GL(2, \mathbb{Z})$ denotes the group of 2×2-matrices of integer entry. As acting groups, we take ones of the form



$G = \mathbb{T}^2 \rtimes G'$ with G' a finite subgroup of $GL(2,\mathbb{Z})$. First we are concerned with what kind of finite groups will appear in $GL(2,\mathbb{Z})$. The following can be checked with the help of the McKay correspondence (cf. **[GHJ]** for the McKay correspondence):

Lemma 15. *A finite subgroup of $GL(2,\mathbb{Z})$ is isomorphic to one of the cyclic groups of order 2, 3, 4, 6, or the dihedral groups of order 4, 6, 8, 12.*

To be specific, we consider $H = \mathbb{T}^2 \rtimes H'$ and $k = \mathbb{T}^2 \rtimes K'$. Here H' and K' are finite cyclic groups. For such a choice, stabilizers of $H - K$ orbits again take the same form and their irreducible representations are described via Mackey's orbit method (note that 2-cohomology groups vanish for cyclic groups):

Proposition 16. *Let L be a locally compact group and T be a normal subgroup of L such that the quotient L/T is a cyclic group. Let $\sigma \in \widehat{T}$ and denote by $L(\sigma)$ the stabilizer of L at σ. Set $L^*(\sigma) = \{\rho \in \widehat{L(\sigma)}; \rho|_T = \sigma\}$.*

(i) *By point-wise multiplication, the character group $\widehat{L(\sigma)/T}$ acts on $L^*(\sigma)$ freely and transitively.*

(ii) *For each $\rho \in L^*(\sigma)$, the induced representation $\mathrm{ind}_{L(\sigma)}^{L} \rho$ is irreducible.*

(iii) *For $\rho_i \in L^*(\sigma_i)$ $(i = 1,2)$, $\mathrm{ind}\,\rho_1$ and $\mathrm{ind}\,\rho_2$ are equivalent if $\exists g \in L$, $\sigma_2 = g\sigma_1$ and $\rho_2 = g\rho_1$.*

(iv) *The map $\rho \mapsto \mathrm{ind}\,\rho$ induces a bijection $\mathrm{ind} : L^*(\widehat{T})/L \to \widehat{G}$.*

(v) *For $\rho \in L^*(\sigma)$, $(\mathrm{ind}\,\rho)|_T = \oplus_{\sigma' \in L.\sigma}\sigma'$.*

Notation. For $g \in G$ and a representation σ of $H \cap gKg^{-1}$, we denote by $V^g(\sigma)$ the fibre of the associated $H - K$ bundle at g and the bundle itself is denoted by $HV^g(\sigma)K$.

We shall illustrate how to compute the graph invariants by a more specific example: Let $H' \cong \mathbb{Z}_2$ and $K' \cong \mathbb{Z}_3$ such that these generate the group isomorphic to S_3. For example we can assume that H' and K' are generated by

$$h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$k = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

respectively. The relevant double coset decompositions are then described in the following way:

$$\begin{aligned} H \setminus G/H &= H \sqcup HkH \\ K \setminus G/K &= K \sqcup KhK \\ H \setminus G/K &= HK. \end{aligned}$$

Lemma 17. *Let* $\chi, \eta \in \widehat{T}$, $\sigma \in \widehat{H}$, *and* $\rho \in \widehat{K}$.

(i) $HV^1(\chi)K \otimes_K KV^1(\rho)K = HV^1(\chi \otimes \rho|_T)K$ *and*

$$HV^1(\sigma)H \otimes_H HV^1(\chi)K = HV^1(\sigma|_T \otimes \chi)K.$$

(ii) $HV^k(\chi)H \otimes_H HV^1(\eta)K = HV^1(\chi \otimes k\eta)K \oplus HV^1(h\chi \otimes hkh^{-1}\eta)K.$

(iii) $HV(\chi)K \otimes_K KV^h(\sigma)K = HV(h^{-1}\chi \otimes h^{-1}\sigma|_T)K.$

(iv) $HV(\chi)K \otimes_K KV(\eta)H$
$$= HV(\mathrm{ind}_T^H \chi \otimes \eta)H \oplus HV^k(\chi \otimes k\eta)H \oplus HV^k(h\chi \otimes kh\eta)H.$$

(v) $KV(\chi)H \otimes_H HV(\eta)K$
$$KV(\mathrm{ind}_T^K(\chi \otimes \eta))K \oplus KV^h(\mathrm{ind}_T^K(\chi \otimes h\eta))K.$$

Proof. $\because$) (ii) Since $H \cap kHk^{-1} = T$ and $H \cap K = T$, H' and K' can be used as representatives of $H/H \cap kHk^{-1}$ and $H \cap K \setminus K$. Then (the integrated form of) the tensor bundle $HV^k(\chi)H \otimes_H HV(\eta)K$ is given by

$$\bigoplus_{h', h'' \in H',\, k' \in K'} \int_T^{\oplus} dt h_1 V^k(\chi) \otimes_H h'tV(\eta)k'.$$

Hence the fibre at $1 \in G$ is given by

$$\begin{aligned} &\bigoplus_{h'kh''k'=1} h'V^k(\chi) \otimes_H h''V(\eta)k' \\ &\qquad = \bigoplus_{h' \in H'} h'V^k(\chi) \otimes_H h'^{-1}V(\eta)h'k^{-1}h'^{-1} \\ &\qquad = V^k(\chi) \otimes_H V(\eta)k^{-1} \oplus hV^k(\chi) \otimes_H h^{-1}V(\eta)hk^{-1}h^{-1}. \end{aligned}$$

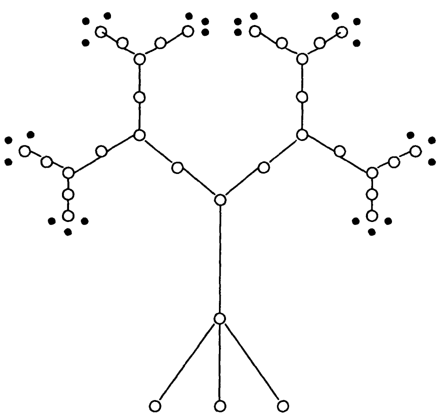
(Note that $h'h'' = h'k^{-1}h'^{-1}k'^{-1} \in H \cap K$.)

The action of T (= the stabilizer of $H - K$ action at $1 \in G$) is calculated as follows:

$$\begin{aligned} tv^k \otimes_H vk^{-1}t^{-1} &= tv^k k^{-1}t^{-1}k \otimes_H k^{-1}tkvk^{-1}t^{-1}kk^{-1} \\ &= \chi(t)v^k \otimes_H \eta(k^{-1}tk)v. \end{aligned}$$

Similarly for others. $\square$

Applying the above branching rule to $HV(\chi)K$ with $h\chi \neq \chi$ (when $h_\chi = \chi$, the graphs are those for $N \subset N \rtimes S_6$), we obtain the graphs:



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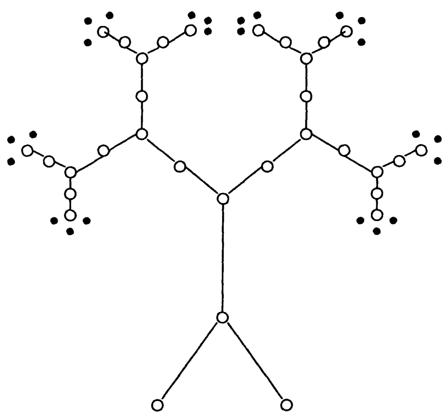

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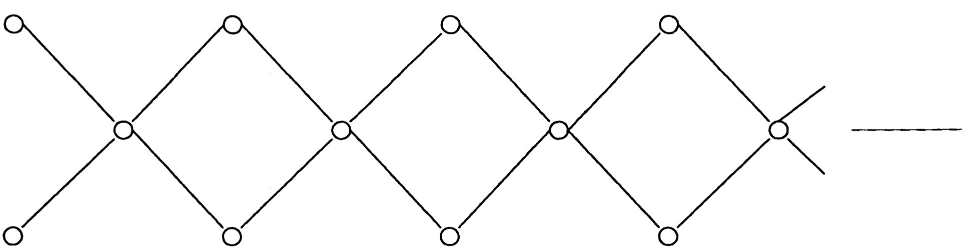
Similar computations hold for other finite subgroups. For example, take $K' \cong \mathbb{Z}_4$ and $H' \subset K'$ with $H' \cong \mathbb{Z}_2$ and consider $HV(\operatorname{ind}_T^H \chi)K$. If χ is not invariant under H, then the graph invariants for $HV(\operatorname{ind}_T^H \chi)K$ are given by



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